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An application of modular spaces to approximation problems. IV

Abstract. The paper gives a generalization of a theorem on approximation of a measurable function $x \geqslant 0$ by F_p-pseudomodulars, where $0 < p \leqslant 1$ (see [5]), for the case where x is approximated by (F_n, ψ_n, ω_n)-pseudomodulars. In particular, a measurable function $x \geqslant 0$ is approximated by singular integrals of the form $\varrho_n(t, x) = \varphi_n^{-1}\{\int_a^b K_n(u)\varphi_n(x(u+t))du\}$, where $\varphi_n(u) = \sum_{k=1}^{k_n} a_k^n u^{p_k^n}$, with $a_k^n > 0$, $0 < p_k^n < 1$. (In [5] the case $\varphi_n(u) = u^p$ is discussed for every $n = 1, 2, \ldots, 0 < p < 1$.)

Let F be an F-operation in $R_+ = \langle 0, \infty)$, i.e., let F be a mapping F: $R_+ \times R_+ \to R_+$ satisfying the following conditions (see [1]–[3]):

(a) $F(u, v) = F(v, u)$,
(b) $F(u, F(v, w)) = F(F(u, v), w)$,
(c) $F(u, 0) = u$, $F(0, v) = v$,
(d) F is non-decreasing in each variable separately,
(e) F is continuous.

EXAMPLES. (a) For an increasing φ-function φ (see [1] and [2]) let

$$F_\varphi(u, v) = \varphi^{-1}(\varphi(u)+\varphi(v)) \quad \text{for } u, v \geqslant 0,$$

where φ^{-1} is the function inverse to φ for $u \geqslant 0$; F_φ is an F-operation in R_+. In particular, for $\varphi(u) = u^p$, $p > 0$, $u \geqslant 0$,

$$F_p(u, v) = (u^p+v^p)^{1/p} \quad \text{for } u, v \geqslant 0$$

is an F-operation.

(b) $F_\infty(u, v) = \max(u, v)$ for $u, v \geqslant 0$ is an F-operation in R_+.

If F is an F-operation, then:

(a) $F(u_1, v_1) \leqslant F(u_2, v_2)$ for $0 \leqslant u_1 \leqslant u_2$, $0 \leqslant v_1 \leqslant v_2$,
(b) $F(u, v) \geqslant 0$, $F(0, 0) = 0$.

It is known (see [5]) that for $0 < p_1 \leqslant p_2 < \infty$ we have

$$F_\infty(u, v) \leqslant F_{p_2}(u, v) \leqslant F_{p_1}(u, v) \quad \text{for } u, v \geqslant 0.$$

If

$$\varphi(u) = \sum_{i=1}^{\infty} \alpha_i \varphi_i(u) \quad \text{for } u \geqslant 0,$$

where α_i, $i = 1, 2, \ldots$, are positive constants, φ_i is an increasing φ-function, $i = 1, 2, \ldots$, the series $\sum_{i=1}^{\infty} \alpha_i \varphi_i(u)$ is convergent for every $u \geqslant 0$, then the F-operation

$$F_\Sigma(u, v) = \varphi^{-1}\big(\varphi(u) + \varphi(v)\big) \quad \text{for } u, v \geqslant 0$$

satisfies the inequality

$$(*) \qquad F_\Sigma(u, v) \leqslant \sup_i F_{\varphi_i}(u, v) \quad \text{for } u, v \geqslant 0.$$

Proof. For $u, v \geqslant 0$ and $\sup_i F_{\varphi_i}(u, v) < \infty$ we have

$$(*) \Leftrightarrow \sum_{i=1}^{\infty} \alpha_i \big(\varphi_i(u) + \varphi_i(v)\big) \leqslant \varphi\big(\sup_i F_{\varphi_i}(u, v)\big).$$

Since

$$\sum_{j=1}^{\infty} \alpha_j \varphi_j\big(\sup_i F_{\varphi_i}(u, v)\big) \geqslant \sum_{j=1}^{\infty} \alpha_j \big(\varphi_j(u) + \varphi_j(v)\big),$$

we get $(*)$.

In particular, for

$$\varphi_n(u) = \sum_{i=1}^{m_n} \alpha_i^n \varphi_i^n(u), \qquad F_\Sigma^n(u, v) = \varphi_n^{-1}\Big(\sum_{i=1}^{m_n} \alpha_i^n \big(\varphi_i^n(u) + \varphi_i^n(v)\big)\Big),$$

where $n = 1, 2, \ldots$, $\alpha_i^n > 0$ for $i = 1, \ldots, m_n$; $u, v \geqslant 0$, we have the following statements:

(a) If $\varphi_i^n(u) = u^{p_i^n}$, $p_i^n > 0$ for $i = 1, 2, \ldots, m_n$, then for every $u, v \geqslant 0$

$$F_\Sigma^n(u, v) \leqslant F_{p_0^n}(u, v),$$

where $p_0^n = \min_i p_i^n$.

(b) If $\varphi_i^n(u) = \log_{p_i^n}(1+u)$, $p_i^n > 1$, $i = 1, 2, \ldots, m_n$, then for every $u, v \geqslant 0$

$$F_\Sigma^n(u, v) \leqslant u + v + uv.$$

If we set

$$\bar{F}(u, v) = \begin{cases} F(u, v) & \text{if } u < \infty,\ v < \infty, \\ \infty & \text{if } u = \infty \text{ or } v = \infty, \end{cases}$$

where F is an F-operation, then we can extend F to a function $\bar{F}$: $\bar{\boldsymbol{R}}_+ \times \bar{\boldsymbol{R}}_+ \to \bar{\boldsymbol{R}}_+$, where $\bar{\boldsymbol{R}}_+ = \langle 0, \infty \rangle$.

Let X be a real linear space and let F be an F-operation. Suppose that ω is an increasing continuous function of $u \geqslant 0$ such that $\omega(0) = 0$, $\omega(1) = 1$, $\omega(u) \to \infty$ as $u \to \infty$ and $\omega(uv) \geqslant \omega(u)\omega(v)$ for $u, v \geqslant 0$.

A functional ϱ: $X \to \langle 0, \infty \rangle$ is called an *(F, ω, ψ)-pseudomodular* if for every $x, y \in X$:

(a) $\varrho(0) = 0$,
(b) $\varrho(-x) = \varrho(x)$,
(c) $\varrho(\alpha x + \beta y) \leqslant \bar{F}(\psi(\alpha)\varrho(x),\ \psi(\beta)\varrho(y))$,

where $\alpha, \beta \geqslant 0$ and $\omega(\alpha) + \omega(\beta) \leqslant 1$, ψ: $\langle 0, 1 \rangle \to \langle 0, 1 \rangle$, $\psi(t) \geqslant t$ for $t \in \langle 0, 1 \rangle$. In the case where ϱ satisfies conditions (b), (c) and the condition

(a′) $\varrho(x) = 0 \Leftrightarrow x = 0$,

in the place of (a), ϱ is called an *(F, ω, ψ)-modular* (see [4] and [7]). Let

$$X_\varrho = \{x \in X: \lim_{\lambda \to 0+} \varrho(\lambda x) = 0\},$$

where ϱ is an (F, ω, ψ)-pseudomodular. X_ϱ is called an *(F, ω, ψ)-modular space*. In the following, we shall assume that X_ϱ contains elements $\neq 0$, i.e., there exists an $x \in X$ such that $x \neq 0$ and $\varrho(\lambda x) \to 0$ as $\lambda \to 0+$.

If $F = F_1$, $\omega(u) = u$, $\psi(t) \equiv 1$, then X_ϱ is a modular space (see [2]). For $F = F_1$, $\omega(u) = u$, $\psi(t) = t$, X_ϱ is a modular space generated by a convex pseudomodular ϱ.

If $\omega(u) = u$, $\psi(t) \equiv 1$, then X_ϱ is an F-modular space (see [3]) which is generated by an F-pseudomodular ϱ. In the case $\psi(t) \equiv 1$, X_ϱ is an (F, ω)-modular space (see [1], [2]). If $F = F_1$, $\omega(u) = u^s$, $\psi(t) = t^s$, $0 < s \leqslant 1$, then X_ϱ is an (F_1, u^s, t^s)-modular space generated by an s-convex pseudomodular ϱ. Then X_ϱ is also an (F_∞, u^s)-modular space.

Let (Ω, Σ, μ) denote a measure space with a finite measure μ defined on Σ, a σ-algebra of subsets of Ω, $\Omega \neq \emptyset$, $\varrho_n(t, x)$: $\Omega \times \mathscr{X} \to \langle 0, \infty \rangle$ for $n = 1, 2, \ldots$ and let $x \in \mathscr{X}$ – the space of functions x: $\Omega \to \langle -\infty, \infty \rangle$ which are Σ-measurable and almost everywhere finite with the convention that $x = y$ iff $x(t) = y(t)$ almost everywhere.

Let us assume that:

(i) $\varrho_n(t, x)$ is an (F^n, ω_n, ψ_n)-pseudomodular in $\mathscr{X}$ for all $t \in \Omega$ and for every $n = 1, 2, \ldots$, where F^n are arbitrary F-operations.

(ii) $\varrho_n(t, x)$ is Σ-measurable and almost everywhere finite with respect to t for every $x \in \mathscr{X}$ and every $n = 1, 2, \ldots$

(iii) If for $n = 1, 2, \ldots$ $\varrho_n(t, x) = 0$ for almost all t, then $x = 0$.

For $x \in \mathscr{X}$ let us write

$$\varrho_{ns}(x) = \int_{\Omega} \varrho_n(t, x)\, d\mu, \qquad \varrho^s(x) = \sum_{n=1}^{\infty} \frac{1}{2^n} \cdot \frac{\varrho_{ns}(x)}{1+\varrho_{ns}(x)}$$

and

$$X_{\varrho^s} = \{x \in \mathscr{X}\colon\ \varrho^s(\lambda x) \to 0 \text{ for } \lambda \to 0+\}.$$

In the sequel we suppose that the following condition is satisfied, in addition to conditions (i)–(iii):

(iv) If $x, y \in \mathscr{X}$, $x(t) \leqslant y(t)$ almost everywhere in Ω, then for $n = 1, 2, \ldots$ we have $\varrho_n(t, x) \leqslant \varrho_n(t, y)$ almost everywhere in Ω.

We say that a sequence (ϱ_n) preserves constants if:

(a) constant functions belong to X_{ϱ^s},

(b) $\varrho_n(t, c) = c$ for every $t \in \Omega$, $c \geqslant 0$ and $n = 1, 2, \ldots$,

(c) $\varrho_n\big(t, x - x(t)\big)$ is a Σ-measurable function of t for every $x \in \mathscr{X}$, $x \geqslant 0$.

THEOREM 1. *If a sequence (ϱ_n) preserves constants, $x \in \mathscr{X}$, $x \geqslant 0$, then for every $\lambda > 0$, $\alpha, \beta > 0$, $\omega_n(\alpha) + \omega_n(\beta) \leqslant 1$, we have*

$$\begin{aligned} &\varrho^s\{\lambda |x(\cdot) - \varrho_n(\cdot, x)|\} \\ &\leqslant \varrho^s\Bigg\{\lambda \max\Bigg[\frac{1}{\psi_n(\alpha)} H^n\Bigg(\psi_n(\alpha)\varrho_n(\cdot, x), \\ &\qquad \psi_n(\beta)\varrho_n\Big(\cdot, \frac{\omega_n(\alpha)-\alpha}{\beta} x(\cdot) - \alpha\frac{x - x(\cdot)}{\beta}\Big)\Bigg), \\ &\qquad H^n\Big(\frac{\psi_n(\alpha)}{\alpha} x(\cdot), \psi_n(\beta)\varrho_n\Big(\cdot, \frac{x - x(\cdot)}{\beta}\Big)\Big)\Bigg] + \\ &\qquad + \lambda \max\Bigg[\frac{\psi_n(\alpha) - \omega_n(\alpha)}{\psi_n(\alpha)}, \frac{\psi_n(\alpha) - \alpha}{\alpha}\Bigg] x(\cdot)\Bigg\} \end{aligned}$$

for $n = 1, 2, \ldots$, where $H^n(u, v) = F^n(u, v) - u$.

Proof. Let $x \in \mathscr{X}$, $x \geqslant 0$, $\lambda > 0$, $\alpha, \beta > 0$, $\omega_n(\alpha) + \omega_n(\beta) \leqslant 1$, $n = 1, 2, \ldots$

Since the sequence (ϱ_n) preserves constants, we have for $t \in A = \{t \in \Omega\colon x(t) \text{ is finite}\}$

$$\varrho_n\big(t, x(t)\big) = x(t).$$

Therefore, for $t \in A$,

$$\varrho_n(t, x) - x(t) \leqslant \overline{F^n}\Big\{\psi_n(\alpha)\frac{x(t)}{\alpha}, \psi_n(\beta)\varrho_n\Big(t, \frac{x - x(t)}{\beta}\Big)\Big\} - x(t).$$

Let $B_{n,cx} = \{t \in \Omega: \varrho_n(t, cx)$ is finite, c is non-negative constant$\}$. Then for $t \in A \cap B_{n,x/(\alpha\beta)}$ we have

$$\varrho_n(t, x) - x(t) \leqslant F^n\left\{\psi_n(\alpha)\frac{x(t)}{\alpha}, \psi_n(\beta)\varrho_n\left(t, \frac{x-x(t)}{\beta}\right)\right\} - \\ -\psi_n(\alpha)\frac{x(t)}{\alpha} + \frac{\psi_n(\alpha)-\alpha}{\alpha}x(t).$$

For $t \in A \cap B_{n,x}$ we have

$$x(t) - \varrho_n(t, x) = \frac{1}{\psi_n(\alpha)}[\omega_n(\alpha)x(t) - \psi_n(\alpha)\varrho_n(t, x)] + \frac{\psi_n(\alpha)-\omega_n(\alpha)}{\psi_n(\alpha)}x(t),$$

and for $t \in A \cap B_{n,x} \cap B_{n,x/\beta}$ we obtain

$$\omega_n(\alpha)x(t) \leqslant F^n\left\{\psi_n(\alpha)\varrho_n(t, x), \psi_n(\beta)\varrho_n\left(t, \alpha\frac{x(t)-x}{\beta} + \frac{\omega_n(\alpha)-\alpha}{\beta}x(t)\right)\right\}$$

and for $t \in A \cap B_{n,x/\beta}$

$$\varrho_n\left(t, \alpha\frac{x(t)-x}{\beta} + \frac{\omega_n(\alpha)-\alpha}{\beta}x(t)\right) \leqslant F^n\left(\psi_n(\alpha)\varrho_n\left(t, \frac{x}{\beta}\right), \psi_n(\beta)\frac{\omega_n(\alpha)}{\beta^2}x(t)\right).$$

Thus for $t \in A \cap B_{n,x} \cap B_{n,x/\beta}$ we have

$$x(t) - \varrho_n(t, x) \leqslant \frac{1}{\psi_n(\alpha)}F^n\left(\psi_n(\alpha)\varrho_n(t, x), \psi_n(\beta)\varrho_n\left(t, \alpha\frac{x(t)-x}{\beta} + \right.\right. \\ \left.\left. + \frac{\omega_n(\alpha)-\alpha}{\beta}x(t)\right)\right) - \varrho_n(t, x) + \frac{\psi_n(\alpha)-\omega_n(\alpha)}{\psi_n(\alpha)}x(t).$$

Hence, for $t \in A \cap B_{n,x} \cap B_{n,x/(\alpha\beta)} \cap B_{n,x/\beta}$ and for $n = 1, 2, \ldots$ we obtain

$$|x(t) - \varrho_n(t, x)| \leqslant \max\left\{H^n\left(\psi_n(\alpha)\frac{x(t)}{\alpha}, \psi_n(\beta)\varrho_n\left(t, \frac{x-x(t)}{\beta}\right)\right) + \right. \\ + \frac{\psi_n(\alpha)-\alpha}{\alpha}x(t), \frac{1}{\psi_n(\alpha)}H^n\left(\psi_n(\alpha)\varrho_n(t, x), \psi_n(\beta)\varrho_n\left(t, \alpha\frac{x(t)-x}{\beta} + \right.\right. \\ \left.\left.\left. + \frac{\omega_n(\alpha)-\alpha}{\beta}x(t)\right)\right) + \frac{\psi_n(\alpha)-\omega_n(\alpha)}{\psi_n(\alpha)}x(t)\right\},$$

where $H^n(u, v) = F^n(u, v) - u$. Applying condition (iv), we obtain the assertion.

In particular, if $H^n(u, v) = v$, $\psi_n(\alpha) = \psi_n(\beta) \equiv 1$, $\omega_n(\alpha) = \alpha$ for $n = 1, 2, \ldots,$ we have the inequality

$$\varrho^s\{\lambda|x(\cdot) - \varrho_n(\cdot, x)|\} \leqslant \varrho^s\left\{2\lambda\frac{x-x(\cdot)}{\beta}\right\} + \varrho^s\left\{2\lambda\frac{\beta}{\alpha}x(\cdot)\right\}$$

for $\alpha, \beta > 0$, $\alpha+\beta = 1$, $\lambda > 0$ (see [1], [2]).

In the following, we investigate certain particular cases:

1.1. *Let* $\varrho_n(t, x)$ *be the* (F^n, ω_n, ψ_n)*-pseudomodular in* $\mathscr{X}$ *defined for* $t \in \Omega$, $n = 1, 2, \ldots$, *by*

$$F^n(u, v) = F^n_\Sigma(u, v) = \varphi_n^{-1}\Big(\sum_{i=1}^{k_n} \alpha_i^n (u^{p_i^n} + v^{p_i^n})\Big),$$

$\varphi_n(u) = \sum_{i=1}^{k_n} \alpha_i^n u^{p_i^n}$, α_i^n *being positive constants,* $p_i^n > 0$, $i = 1, \ldots, k_n$, $n = 1, 2, \ldots$, $\omega_n(u) = u$, $\psi_n(t) \equiv 1$. *Moreover, let* $\varrho_n(\cdot, x) \in L(\Omega, \Sigma, \mu)$ *for* $n = 1, 2, \ldots$

If $p_0 = \inf_n p_0^n \in (0, 1)$, *where* $p_0^n = \min_i p_i^n$, *then the space* X_{ϱ^s}, *generated by the sequence* (ϱ_n), *is an* F_{p_0}*-modular space.*

Proof.

(a) $x = 0 \Leftrightarrow \varrho^s(x) = 0$.

(b) $\varrho^s(-x) = \varrho^s(x)$ for $x \in \mathscr{X}$.

(c) For $x, y \in \mathscr{X}$, $\alpha, \beta \geqslant 0$, $\alpha + \beta = 1$, $S(u) = u/(1+u)$ for $u \geqslant 0$ we have

$$\varrho^s(\alpha x + \beta y) \leqslant \sum_{n=1}^{\infty} \frac{1}{2^n} S\Big(\int_\Omega F^n_\Sigma(\varrho_n(t, x), \varrho_n(t, y))\, d\mu\Big).$$

It is known (see [5]) that for $x, y \in L^r(\Omega, \Sigma, \mu)$ and for the F-operation F_p, $0 < p \leqslant 1$, the inequality

$$(\overset{*}{*}) \qquad \Big\{\int_\Omega [F_p(|x(t)|, |y(t)|)]^r d\mu\Big\}^{1/r} \leqslant F_p\Big\{\Big[\int_\Omega |x(t)|^r d\mu\Big]^{1/r}, \Big[\int_\Omega |y(t)|^r d\mu\Big]^{1/r}\Big\}$$

holds for $r \geqslant p$.

If $0 < p \leqslant 1$, $u_n, v_n \geqslant 0$, the F-operation F_p satisfies the condition

$$({}_*{}^*{}_*) \qquad \sum_{n=1}^{\infty} a_n F_p(u_n, v_n) \leqslant \bar{F}_p\Big(\sum_{n=1}^{\infty} a_n u_n, \sum_{n=1}^{\infty} a_n v_n\Big),$$

where a_n, $n = 1, 2, \ldots$, are non-negative constants (see [5]).

Applying $(*)$, $(\overset{*}{*})$ and $({}_*{}^*{}_*)$ and the inequality $S(F_p(u, v)) \leqslant F_p(S(u), S(v))$ for $p \in (0, 1\rangle$, we obtain

$$\begin{aligned}\varrho^s(\alpha x + \beta y) &\leqslant \sum_{n=1}^{\infty} \frac{1}{2^n} S\Big(\int_\Omega F_{p_0^n}(\varrho_n(t, x), \varrho_n(t, y))\, d\mu\Big)\\ &\leqslant \sum_{n=1}^{\infty} \frac{1}{2^n} S\big(F_{p_0}(\varrho_{ns}(x), \varrho_{ns}(y))\big)\\ &\leqslant \sum_{n=1}^{\infty} \frac{1}{2^n} F_{p_0}\big(S(\varrho_{ns}(x)), S(\varrho_{ns}(y))\big) \leqslant F_{p_0}(\varrho^s(x), \varrho^s(y)).\end{aligned}$$

Therefore X_{ϱ^s} is an F_{p_0}-modular space.

Using Theorem 1 and proceeding as in the proof of Corollary in [5], we obtain

THEOREM 2. *If a sequence* (ϱ_n) *preserves constants, then for every* $\lambda > 0$, $\varepsilon > 0$, $x \in X_{\varrho^s}$, $x \geqslant 0$, *there exists* $\beta \in (0, 1)$ *such that for every* $n = 1, 2, \ldots$

$$(\varrho^s\{\lambda[x(\cdot)-\varrho_n(\cdot, x)]\})^{p_0} < \left(\varrho^s\left\{2\lambda \max\left[H_{p_0}\left(\frac{x(\cdot)}{1-\beta}, \varrho_n\left(\cdot, \frac{x-x(\cdot)}{\beta}\right)\right), H_{p_0}\left(\varrho_n(\cdot, x), \varrho_n\left(\cdot, \frac{x-x(\cdot)}{\beta}\right)\right)\right]\right\}\right)^{p_0} + \varepsilon,$$

where $H_{p_0}(u, v) = F_{p_0}(u, v) - u$.

A sequence (ϱ_n) is called *singular at a point* $x \in X_{\varrho^s}$, $x \geqslant 0$, if for every $a' > 0$, $b' > 1$, $m = 1, 2, \ldots$ we have

$$J_n^m(x) = \int_\Omega \varrho_m\left\{t, a' \max\left[H^n\left(\frac{b'}{b'-1}x(\cdot), \varrho_n(\cdot, b'(x-x(\cdot)))\right), H^n\left(\varrho_n(\cdot, x), \varrho_n(\cdot, b'(x-x(\cdot)))\right)\right]\right\} d\mu \to 0$$

as $n \to \infty$.

From Theorem 2 follows the following corollary.

COROLLARY 1. *If a sequence* (ϱ_n) *preserves constants and* (ϱ_n) *is singular at a point* $x \in X_{\varrho^s}$, $x \geqslant 0$, *then for every* $\lambda > 0$

$$\varrho^s\{\lambda[x(\cdot)-\varrho_n(\cdot, x)]\} \to 0$$

as $n \to \infty$.

1.2. *Let* $\Omega = \langle a, b\rangle$, *where* $a = k(b-a)$ *and* k *is an integer, let* Σ *be the* σ*-algebra of Lebesgue measurable sets in* $\langle a, b\rangle$ *and* μ *be the Lebesgue measure. Let* $\mathscr{X}$ *denote the set of* Σ*-measurable and almost everywhere finite functions in* $\langle a, b)$, *extended by periodicity, with period* $b-a$, *outside* $\langle a, b)$. *Let* K_n, $n = 1, 2, \ldots$, *be* Σ*-measurable functions positive almost everywhere in* $\langle a, b\rangle$ *and such that*

$$\int_a^b K_n(u)\,du = 1 \quad \text{for } n = 1, 2, \ldots$$

Let

$$\text{(A)} \qquad \varrho_n(t, x) = \varphi_n^{-1}\left\{\int_a^b K_n(u)\,\varphi_n(|x(u+t)|)\,du\right\},$$

where $n = 1, 2, \ldots$, $t \in \langle a, b\rangle$, $x \in L(\Omega, \Sigma, \mu)$, $\varphi_n(u) = \sum_{i=1}^{k_n} \alpha_i^n u^{p_i^n}$, $\alpha_i^n > 0$, $p_i^n \in (0, 1)$ *for* $n = 1, 2, \ldots$; $i = 1, \ldots, k_n$; $p_0^n = \min_i p_i^n$, $p_0 = \inf_n p_0^n > 0$.

For $n = 1, 2, \ldots$ *and* $t \in \langle a, b\rangle$ *the inequality*

$$(\ast\ast)\qquad \varrho_n(t, x) \leqslant \sum_{i=1}^{k_n} \Big(\int_a^b K_n(u)|x(u+t)|^{p_i^n}du\Big)^{1/p_i^n}$$

holds.

Proof. Since

$$\int_a^b K_n(u)\Big[\sum_{i=1}^{k_n} \alpha_i^n |x(u+t)|^{p_i^n}\Big]du \leqslant \varphi_n\Big(\max_i \Big\{\int_a^b K_n(u)|x(u+t)|^{p_i^n}du\Big\}^{1/p_i^n}\Big),$$

we have

$$\varrho_n(t, x) \leqslant \max_i \Big\{\int_a^b K_n(u)|x(u+t)|^{p_i^n}du\Big\}^{1/p_i^n}$$

for $n = 1, 2, \ldots$; $t \in \langle a, b\rangle$. Hence $(\ast\ast)$ follows.

The sequence (ϱ_n), where $\varrho_n(t, x)$ is defined by formula (A), satisfy conditions (i)–(iv) and $\varrho_n(\cdot, x) \in L(\Omega, \Sigma, \mu)$ for $n = 1, 2, \ldots, x \in \mathcal{X}$; (ϱ_n) preserves constants.

We say that (K_n) is a singular kernel if

$$\lim_{n\to\infty} \int_{a+\delta}^{b-\delta} K_n(u)\,du = 0$$

for every $\delta \in (0, (b-a)/2)$.

THEOREM 3. *If*:

(a) $x \geqslant 0$, $x \in L^{1+\gamma}(\langle a, b\rangle, \Sigma, \mu) \cap \mathcal{X}$, *where* $0 < \gamma < 1$,

(b) *the sequence* (ϱ_n) *is defined by formula* (A), *where*

$$\sup_n k_n < \infty, \quad p_i^n \in \Big\langle \frac{1}{l+1}, \frac{2}{l+1}\Big\rangle, \quad p_0 \in \Big\langle \frac{1}{l+1}, \frac{1}{l}\Big\rangle,$$

$$l = \max\Big\{m = 1, 2, \ldots:\ \gamma > \frac{m-1}{m+1}\Big\},$$

and (K_n) *is a singular kernel, then for every* $\lambda > 0$ *we have*

$$\varrho^s\{\lambda[x - \varrho_n(\cdot, x)]\} \to 0$$

as $n \to \infty$, *where* ϱ^s *is an* F_{p_0}*-modular.*

Proof. Let $x \in L^{1+\gamma}(\Omega, \Sigma, \mu) \cap \mathcal{X}$, $x \geqslant 0$, where $0 < \gamma < 1$, and let (ϱ_n) satisfy condition (b), $a' > 0$, $b' > 1$, $m = 1, 2, \ldots$

Let us denote

$$y(\cdot) = a' \max\Big[H_{p_0}\Big(\frac{b'}{b'-1}x(\cdot), \varrho_n\big(\cdot, b'(x - x(\cdot))\big)\Big),$$
$$H_{p_0}\Big(\varrho_n(\cdot, x), \varrho_n\big(\cdot, b'(x - x(\cdot))\big)\Big)\Big].$$

Then, applying $(\ast\ast)$ we have

$$J_n^m(x) = \int_\Omega \varrho_m(t, y)\,dt \leqslant \sum_{i=1}^{k_m} \int_\Omega \varrho_{mi}(t, y)\,dt,$$

where

$$\varrho_{mi}(t, y) = \Big\{\int_a^b K_m(u)\,[y(u+t)]^{p_i^m}\,du\Big\}^{1/p_i^m}.$$

Let

$$u_1 = \frac{b'}{b'-1}x(u+t), \qquad u_2 = \varphi_n^{-1}\Big\{\int_a^b K_n(v)\,\varphi_n\big(x(v+u+t)\big)\,dv\Big\},$$

$$v_0 = \varphi_n^{-1}\Big\{\int_a^b K_n(v)\,\varphi_n\big(b'\,|x(v+u+t)-x(u+t)|\big)\,dv\Big\}.$$

Since $1/(l+1) \leqslant p_0 < 1/l$,

$$J_n^m(x) \leqslant a' \sum_{i=1}^{k_m} 2^{1/p_i^m-1}\Big\{(l+1)^{1/p_i^m-1}\int_a^b\Big[\int_a^b K_m(u)\,v_0^{p_i^m}\,du\Big]^{1/p_i^m}dt +$$

$$+(l+1)^{1/p_i^m-1}\sum_{k=1}^{l}\binom{l+1}{k}\int_a^b\Big[\int_a^b K_m(u)\,u_1^{kp_i^m/(l+1)}\,v_0^{(l,k)p_i^m}\,du\Big]^{1/p_i^m}dt +$$

$$+\int_a^b\left[\int_a^b K_m(u)\left(\sum_{k=1}^{l}\binom{l+1}{k}^{p_i^m} u_2^{kp_i^m/(l+1)}\,v_0^{(l,k)p_i^m}\right)du\right]^{1/p_i^m}dt\Big\}$$

$$= a'\sum_{i=1}^{k_m} 2^{1/p_i^m-1}\Big\{(l+1)^{1/p_i^m-1}I_1 + (l+1)^{1/p_i^m-1}\sum_{k=1}^{l}\binom{l+1}{k}I_2 + I_3\Big\},$$

where $(l, k) = (l+1-k)/(l+1)$.

In the sequel we shall apply the generalized Minkowski inequality

$$\text{(M)}\qquad \int_a^b\Big(\int_c^d |F(u, v)|\,dv\Big)^q du \leqslant \Big[\int_c^d\Big(\int_a^b |F(u, v)|^q\,du\Big)^{1/q}dv\Big]^q$$

for measurable function F in the rectangle $\langle a, b\rangle \times \langle c, d\rangle$, $q \geqslant 1$ (see [6]).

Using inequality $(\ast\ast)$ and (M), we have

$$I_1 \leqslant b'\,k_n^{1/p_i^m-1}\sum_{j=1}^{k_n}\Big\{\int_a^b K_n(v)\Big[\int_a^b |x(v+s)-x(s)|\,ds\Big]^{p_j^n}dv\Big\}^{1/p_j^n}$$

and

$$I_2 \leqslant b'(b'-1)^{-k/(l+1)}\,M\sum_{j=1}^{k_n}\Big\{\int_a^b K_n(v)\Big[\int_a^b |x(v+s)-x(s)|^{(l,k)P}\,ds\Big]^{p_j^n/((l,k)P)}dv\Big\}^{(l,k)/p_j^n},$$

where $P = (l+1)(\omega+1)p_j^n$,

$$\frac{1+\gamma}{(1+\gamma+\gamma l)\,p_j^n}-1<\omega<\frac{1+\gamma}{lp_j^n}-1 \quad \text{for } \frac{l-1}{l+1}<\gamma<1,$$

because for $x \in L^{1+\gamma}$ there exists a constant M such that for $k = 1, \ldots, l$ and $Q = P/(P-1)$

$$\Big(\int_a^b (x(s))^{kQ/(l+1)}\,ds\Big)^{1/Q} \leqslant M.$$

Applying the generalized Minkowski inequality, we obtain

$$I_3 \leqslant l^{1/p_i^m-1}\sum_{k=1}^{l}\binom{l+1}{k}\Big[\int_a^b K_m(u)\Big(\int_a^b u_2^{k/(l+1)} v_0^{(l,k)}\,dt\Big)^{p_i^m} du\Big]^{1/p_i^m}$$

and, using $(\substack{*\ *\\ *\ *})$, we have

$$\int_a^b u_2^{k/(l+1)} v_0^{(l,k)}\,dt$$

$$\leqslant b'^{(l,k)} M k_n \sum_{r=1}^{k_n}\Big\{\int_a^b K_n(v)\Big[\int_a^b |x(v+s)-x(s)|^{(l,k)P}\,ds\Big]^{p_r^n/((l,k)P)} dv\Big\}^{(l,k)/p_r^n},$$

where

$$\frac{1+\gamma}{(1+\gamma+\gamma l)\,p_j^n}-1<\omega<\min\Big(\frac{1+\gamma}{lp_j^n}, \frac{1}{(l+1)\,p_j^n-1}\Big)-1.$$

For $\delta \in (0, (b-a)/2)$ and $j = 1, \ldots, k_n$; $k = 1, \ldots, l$, we obtain

$$\Big\{\int_a^b K_n(v)\Big[\int_a^b |x(v+s)-x(s)|\,ds\Big]^{p_j^n} dv\Big\}^{1/p_j^n}$$

$$\leqslant 2^{1/p_0}\Big\{[(b-a)^{w(\gamma)-1/(1+\gamma)}\,\omega_{1+\gamma}(\delta;\ x)]^{(l,k)}+$$

$$+2^{(\gamma,l,k,p_j^n)}\Big[\int_{a+\delta}^{b-\delta} K_n(v)\,dv\Big]^{(l,k)/p_j^n}[(b-a)^{w(\gamma)-1-\gamma}\|x\|_{1+\gamma}]^{(l,k)}\Big\},$$

where

$$(\gamma,\ l,\ k,\ p_j^n) = (l,\ k)\Big(\frac{1}{p_j^n}+\frac{\gamma(1+\gamma+\gamma l)}{1+\gamma}\Big)-1,$$

$$\|x\|_p = \Big\{\int_a^b |x(t)|^p\,dt\Big\}^{1/p},$$

$$\omega_p(\delta;\ x) = \sup_{|h|\leqslant\delta}\Big\{\int_a^b |x(t+h)-x(t)|^p\,dt\Big\}^{1/p},$$

$$w(\gamma) = \begin{cases} \dfrac{1}{1+\gamma}, & \text{when } b-a \in (0,\ 1\rangle,\\[2mm] \dfrac{1+\gamma+\gamma l}{1+\gamma}, & \text{when } b-a>1. \end{cases}$$

Therefore, for $m = 1, 2, \ldots$, we have

$$J_n^m(x) \leqslant a' A(m) \Big\{ B(b', l, p_0) k_n^{1/p_0} \omega_1(\delta; x) + B(b', l, p_0) k_n^{1/p_0 - 1} \|x\|_1 \sum_{j=1}^{k_n} \Big(\int_{a+\delta}^{b-\delta} K_n(v)\, dv \Big)^{1/p_j^n} + C(b', l, M, p_0) \sum_{k=1}^{l} \binom{l+1}{k} D(b', k, l) \times \times \Big[2k_n E(\gamma, k, l, p_0) (\omega_{1+\gamma}(\delta; x))^{1/(l+1)} + F(\gamma, k, l, p_0) (\|x\|_{1+\gamma})^{(l,k)} \sum_{j=1}^{k_n} \Big(\int_{a+\delta}^{b-\delta} K_n(v)\, dv \Big)^{(l,k)/p_j^n} \Big] \Big\},$$

where B, C, D, E, F are bounded functions.

Since $x \in L^{1+\gamma}(\Omega, \Sigma, \mu)$, (K_n) is a singular kernel and $\sup_n k_n < \infty$, we have for every $m = 1, 2, \ldots$ $J_n^m(x) \to 0$ as $n \to \infty$.

It follows from Corollary 1 that for every $\lambda > 0$

$$\varrho^s \{ \lambda [x - \varrho_n(\cdot, x)] \} \to 0 \quad \text{as } n \to \infty .$$

COROLLARY 2. *Suppose that:*

(a) $x \geqslant 0$, $x \in L^{1+\gamma}(\Omega, \Sigma, \mu) \cap \mathscr{X}$, *where* $0 < \gamma < 1$,

(b) *the sequence* (ϱ_n) *is of the form*

$$\varrho_n(t, x) = \Big\{ \int_a^b K_n(u) [x(u+t)]^{p_n} du \Big\}^{1/p_n},$$

where $n = 1, 2, \ldots$; $t \in \langle a, b \rangle$, $p_n \in \langle 1/(l+1), 1/l)$ *for* $n = 1, 2, \ldots$,

$$l = \max \Big\{ m = 1, 2, \ldots : \gamma > \frac{m-1}{m+1} \Big\},$$

and (K_n) *is a singular kernel.*

Under these assumptions:

(a) *if* $p_n \downarrow p_0$, *then for every* $\lambda > 0$

$$\varrho^s \{ \lambda [x - \varrho_n(\cdot, x)] \} \to 0 \quad \text{as } n \to \infty,$$

where ϱ^s *is an* F_{p_0}*-modular in* $\mathscr{X}$,

(b) *if* $p_n \uparrow p_0$, *then for every* $\lambda > 0$

$$\varrho^s \{ \lambda [x - \varrho_n(\cdot, x)] \} \to 0 \quad \text{as } n \to \infty,$$

where ϱ^s *is an* F_{p_1}*-modular in* $\mathscr{X}$.

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