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On minimal projections generated by isometries of Banach spaces

Let B be a real Banach space. Let A be a linear isometry in B which has a fixed point. The present paper is inspiried on a fact that is known to some degree but not precisely discused in any source, i.e., if the space B is represented as a direct (topologic) sum of $\mathrm{Im}(I-A)$ and $\mathrm{Ker}(I-A)$ [where I is the identity map of B], then the projection P onto $\mathrm{Im}(I-A)$ along $\mathrm{Ker}(I-A)$ is the minimal projection(1) (cf. below, Theorem 1.1).

In connection with this fact two problems arise. The first problem concerns a possibility of the representation of a space B as a direct topologic sum of $\mathrm{Im}(I-A)$ and $\mathrm{Ker}(I-A)$.

In Section 1 we give a necessary and sufficient conditions of this representation for any linear operation in B (cf. [8], [9]).

The second problem concerns the estimation of a norm of a minimal projection $\tilde{P}$. The answer is given in Section 2 with the help of the concave function g (which has no name for the present) and a Chebyshev radius, which were before used by the estimation of a norm of a minimal projection onto hyperspaces [6].

As the examples showing the obtained results we take the vector-valued Orlicz space $B = L_{M_1}([0, 1]; L_{M_2}(0, 1))$, the Banach space with symmetric norm (in particular, l_1, c_0) and the space $C(S^1)$ of the all continuous functions on the circle S^1.

Note that in the last two examples, our results can also be obtained in a more complicated way with the help of the theory of operators acting on compact topological groups (cf. for example [11], [12]).

1. A condition of the representation of B as $B = \mathrm{Im}(I-A)\oplus\mathrm{Ker}(I-A)$. As usually the Banach space B is called (*topologic*) *direct sum* of the

(1) The projection $\hat{P}$ (i.e., a linear bounded idempotent operator) from B onto a complemented subspace D is called *minimal* if $\|P\| \geqslant \|\hat{P}\|$ for any projection $P\colon B \to D$. A subspace is always assumed to be closed. The terms and also the notation of the classical spaces l_p, L^p, $C([a, b])$, which are encountered in this paper follow the books [2], [5], [17].

subspaces D and K if each $x \in B$ can be uniquely expressed as $x = y+z$, where $y \in D$, $z \in K$, and the linear operator P: $B \to D$, $P(x) = y$ is bounded. We shall write $B = D \oplus K$. The relative projection constant of a complemented subspace D in a Banach space B is the number $\lambda(B, D) = \inf\{\|P\|$: P projects B onto $D\}$.

Let A be a linear bounded map: $B \to B$. Let $B^A = \mathrm{Ker}(I-A)$, $B_A = \mathrm{Im}\,(I-A)$, let θ be the zero element in B, let N be the set of natural numbers. If a set $D \subset B$ and f is a map on B, then by $f|_D$ we denote the restriction of f to the set D.

THEOREM 1.1. *Let A be a linear isometric operator of the Banach space B onto itself, and $B = \mathrm{Im}(I-A) \oplus \mathrm{Ker}(I-A)$. Let $\tilde{P}$ be the projection from B onto $\mathrm{Im}(I-A)$ annihilated on $\mathrm{Ker}(I-A)$. Then $\tilde{P}$ is a minimal projection and it can be defined by*

$$(1) \qquad \tilde{P} = \lim_{n \to \infty} \frac{1}{n} \sum_{k=0}^{n-1} A^{-k} P A^k,$$

where P is a projection from B onto $\mathrm{Im}(I-A)$; $A^0 = I$.

Proof. Let P be a projection on subspace B_A. Let $\tilde{P}$ be the map defined by (1).

We shall show that $\tilde{P}$ is defined correctly for each $x \in B$ and it is a projection from B onto B_A along B^A. Indeed, if $z \in B_A$, then there exists a $x \in B$ such that $z = x - Ax$. Then $A^k(z) = (I-A)(A^k z) \in B_A$ for all $k \in N$. Hence $P(A^k(z)) = A^k(z)$ for all $k \in N$ and $\tilde{P}(z) = z$.

If $z \in B^A$, then $A^k(z) = z$ for all $k \in N$. Since $P(z) \in B_A$ and $P(z) = x' - Ax'$ for a $x' \in B$, we have

$$\tilde{P}(z) = \lim_{n \to \infty} \frac{1}{n} \sum_{k=0}^{n-1} A^{-k}(x' - Ax') = \lim_{n \to \infty} \frac{1}{n}\big(A^{-n+1}(x') - A(x')\big) = 0,$$

because $\|A^{-1}\| = 1$.

By triangle inequality, we obtain $\|\tilde{P}\| \leqslant \|P\|$. Therefore, by linearity and boundedness of the map $\tilde{P}$ defined by (1), $\tilde{P}$ is the minimal projection from B onto B_A along B^A.

Remark 1.1. The proof of this theorem is essentially a proof of some ergodic statistic theorem (cf. [5]). Note that some ergodic statistic theorems were used in fact before, in connection with the investigation of a minimal projection (cf. [1], [13], [16]).

If $\mathrm{Im}(I-A)$ is a reflexive subspace, then the existence of minimal projection follows from the Isbell–Semadeni results ([10]), i.e., if a complemented subspace D of a Banach space B is isometrically isomorphic to a space Z^*, then there exists a minimal projection from B onto D.

For using of Theorem 1.1 we ought to have a condition for a decomposition of B as a direct sum $B = B_A \oplus B^A$. The next proposition gives the conditions of this decomposition (if A is a linear continuous map) in terms of a convergence of averaging operators $A(n)$:

$$A(n) = \frac{1}{n} \sum_{k=0}^{n-1} A^k \quad (n = 1, 2, \ldots),\ A^0 = I. \tag{2}$$

EXAMPLE 1.1 (Fürstenberg [7]). Let $C(S^1)$ be the space of all continuous functions on the circle S^1. Then for every linear isometric surjective operator $A\colon C(S^1) \to C(S^1)$ and for every $x \in S^1$ there exists $\lim_{n\to\infty} (A(n))x$, where $A(n)$ is defined by (2).

PROPOSITION 1.1. *Suppose that a linear continuous operator A in a Banach space B is such that B_A is closed and*

$$\lim_{n\to\infty} \frac{\|A^n\|}{n} = 0. \tag{3}$$

In order that B be a direct sum $B_A \oplus B^A$ it is necessary and sufficient that

(4) *there exists* $\lim_{n\to\infty} (A(n))(x)$ *for any* $x \in B$.

Proof. *Necessity*. Notice that if $x \in B^A$, then $(A(n))(x) = x$, and therefore $\lim_{n\to\infty} (A(n))(x) = x$. If $x \in B_A$, then there exists $y \in B$ such that $x = y - Ay$, i.e., $(A(n))(x) = (1/n)(y - A^n(y))$. Hence, by condition (3), $\lim_{n\to\infty} (A(n))x = \theta$. Thus, for each $x \in B$, (4) is true.

The proof of *sufficiency* follows directly from [5] (Chapter VIII, § 5.2).

Remark 1.2. (a) If A is an isometry, i.e., a linear isometric operator in B, then from the proof of necessity we have $B_A \cap B^A = \{\theta\}$. In general case the last equation is not true. For example, if $B = \operatorname{span}\{e^x, xe^x\} \subset C([0, 1])$ and A is the differentiation operator in B, then $B_A = B^A = \operatorname{span}\{e^x\}$.

(b) If A is an isometry of B onto itself and B is a reflexive space, then $B = \bar{B}_A \oplus B^A$ (cf. [5]), where $\bar{B}_A$ is the closure of $B_A = (I - A)B$.

(c) For the convergence of the sequence $(A(n))_1^\infty$ it is not sufficient that A can be an isometry of B onto itself. Indeed, let $\mathfrak{A} = (a_{ij})$, $1 \leqslant j,\ i \leqslant 2$, be a matrix of order two, $a_{ij} \in N$, $1 \leqslant i, j \leqslant 2$, $\det \mathfrak{A} = 1$, $|\operatorname{tr} \mathfrak{A}| > 2$. (For example,

$$\mathfrak{A} = \begin{pmatrix} 2 & 5 \\ 1 & 3 \end{pmatrix}.)$$

Let T be a standard automorphism of a torus $V = S^1 \times S^1$ corresponding to $\mathfrak{A}$, i.e., for a point $v = (z, w) \in V$, $z = e^{2\pi i x}$, $w = e^{2\pi i y}$, $x, y \in R$, we have

$$Tv = (e^{2\pi i(a_{11}x + a_{12}y)}, e^{2\pi i(a_{21}x + a_{22}y)}).$$

Let A be an isometry of the space $C(V)$ of all real continuous functions on V, generated by T (i.e., $A\varphi(v) = \varphi(Tv)$, where $\varphi \in C(V)$, $v \in V$). Then by a result of H. Fürstenberg from [7] (Theorem 3.3) in $C(V)$ there exists a function f for which the sequence $\{(A(n))f\}_1^\infty$ is not convergent, though A is a surjection. By Proposition 1.1,

$$C(V) \neq (C(V))_A \oplus (C(V))^A.$$

PROPOSITION 1.2. *Let A: $B \to B$ be a linear continuous operator in B. Then the following statements are equivalent*:

(i) $B = B_A \oplus B^A$;

(ii) B_A *is a subspace of* B *and* $(I-A)|_{B_A}$ *is a one-to-one operator of* B_A *onto* B_A;

(iii) *for each* $x \in B$ *there exists* $x' \in B$ *such that* $(I-A)x = (I-A)^2 x'$ *and for each sequence* $(x_k)_1^\infty$ *such that* $\lim_{k\to\infty} (I-A)^2 x_k = \theta$, *there holds* $\lim_{k\to\infty} (I-A)x_k = \theta$.

Proof. (ii) $\Rightarrow$ (i). We shall prove that for every element $x \in B$ there exists $y \in B^A$ and $z \in B_A$ such that $x = y + z$.

Let $x \in B$. Since $(I-A)|_{B_A}$ is a surjective map onto B_A, there exists $z \in B_A$ such that $x - Ax = z - Az$. Let $y = x - z$; then $y - Ay = (x - Ax) - (z - Az) = \theta$. Hence $y \in B^A$. Next, assume that $x = y + z$ and $x = y' + z'$, where $y, y' \in B^A$, $z, z' \in B_A$. Then $z - z' = y - y' \in B^A$. Hence, $(I-A)(z-z') = \theta$. Since the operator $(I-A)|_{B_A}$ is one-to-one, we have $z = z'$ and $y = y'$.

Next note that by the Banach Open Mapping Principle (cf. [2], p. 33) the operator $((I-A)|_{B_A})^{-1}$: $B_A \to B_A$ is a continuous linear operator. Therefore for each sequence $(z_k)_1^\infty \subset B_A$ such that $(I-A)z_k \underset{k\to\infty}{\to} \theta$, we get $\lim_{k\to\infty} z_k = \theta$.

We define here the operator P: $B \to B_A$ by the formula $Px = z$, where $z \in B_A$ is such that $x = y + z$ $(y \in B^A)$.

Next, we prove that P is a continuous operator. Indeed, let $(x_k)_1^\infty$ be a sequence such that $x_k \underset{k\to\infty}{\to} \theta$, and $(y_k)_1^\infty, (z_k)_1^\infty$ are such that $x_k = y_k + z_k$ $(y_k \in B^A,\ z_k \equiv B_A,\ k = 1, 2, \ldots)$. Then

$$\|(I-A)x_k\| \leqslant ((1 + \|A\|)\|x_k\|) \underset{k\to\infty}{\to} 0.$$

Hence,

$$(I-A)x_k \underset{k\to\infty}{\to} \theta.$$

By

$$(I-A)x_k = y_k+z_k-y_k-Az_k = (I-A)z_k,$$

we obtain as above $z_k = P(x_k) \underset{k\to\infty}{\to} \theta$.

Since P is evidently a linear operator, P is a projection from B onto B_A.

(i) $\Rightarrow$ (iii). Let $x \in B$, $x = y+z$, where $y \in B^A$, $z \in B_A$. Then $z = (I-A)x'$, where $x' \in B$. Hence $(I-A)x = (I-A)(y+z) = (I-A)z = (I-A)^2 x'$. Next it is easy to verify that $(I-A)|_{B_A}$ is a one-to-one surjective operator. Finally, we note that by the Banach Open Mapping Principle the operator $((I-A)|_{B_A})^{-1}$ is a continuous linear operator on B_A.

The implication (iii) $\Rightarrow$ (ii) is obvious.

Remark 1.3. Let A: $B \to B$ be a linear continuous operator. It is easy to see that

(a) If $A(B) = B$ and $B = B_A \oplus B^A$, then $A(B_A) = B_A$ and $B^{A^{-1}} = B^A$, $B_{A^{-1}} = B_A$, where A^{-1} is the inverse operator to A.

(b) If $\dim B_A < \infty$, then $B = B_A \oplus B^A \Leftrightarrow B_A \cap B^A = \{\theta\}$.

EXAMPLE 1.2. Let M_i $(i = 1, 2)$ be two Orlicz functions: $[0, +\infty) \to [0, +\infty)$, i.e., continuous convex non-decreasing functions with $M_i(0) = 0$ and $M_i \not\equiv 0$. Let $L_{M_2}(0, 1)$ be an Orlicz space of equivalence classes of such measurable functions h: $[0, 1] \to (-\infty, +\infty)$ for which

$$\|h\| = \inf\{t > 0: \int_0^1 M_2(|h(x)|/t)\,dx \leqslant 1\} < +\infty.$$

Let $B = L_{M_1}([0, 1]; L_{M_2}(0, 1))$ be the Orlicz space of the equivalence classes of strongly measurable functions f: $[0, 1] \to L_{M_2}(0, 1)$ for which

$$|||f||| = \inf\{t > 0: \int_0^1 M_1(\|f(x)\|/t)\,dx \leqslant 1\} < +\infty.$$

Now, we shall define for every $n \in N$ a map τ_n: $[0, 1] \to [0, 1]$ as follows: if $n = 1$, then $\tau_1(x) = x$ for every $x \in [0, 1]$; if $n \geqslant 2$, then

$$\tau_n(x) = \begin{cases} x + \dfrac{1}{n} & \text{if } x \in \left(0, \dfrac{n-1}{n}\right); \\[2mm] x - \dfrac{n-1}{n} & \text{if } x \in \left[\dfrac{n-1}{n}, 1\right]; \\[2mm] 1 & \text{if } x = 0. \end{cases}$$

An operator $Q(x)$: $L_{M_2}(0, 1) \to L_{M_2}(0, 1)$ defined for every $x \in (0, 1]$ by $(Q(x)h)y = h(\tau_{[1/x]}(y))$ (where $h \in L_{M_2}(0, 1)$, $[1/x]$ is the greatest integer of the number $1/x$, $y \in [0, 1]$), and $Q(0)h = h$, for $x = 0$, is a linear isometry.

Then the operator $A\colon B \to B$, such that $Af(x) = Q(x)\big(f(\tau_2(x))\big)$ for each $f \in B$, $x \in [0, 1]$ is a linear isometry of B onto itself.

B^A will be a subspace of all functions $f \in B$, satisfying for every $n \in N$ the property:

$$\big(f(x)\big)(\tau_n(y)) = \big(f(\tau_2(x))\big)(y) \quad \text{for } \mu\text{-almost every } x \in \left(\frac{1}{n+1}, \frac{1}{n}\right]$$

and $y \in [0, 1]$ relative to Lebesgue measure μ.

B_A will be a subspace of all functions $f \in B$ satisfying for every $n \in N \setminus \{1\}$ the property

$$\sum_{k=0}^{n-1} \big(f(x) + f(x+\tfrac{1}{2})\big)(y + k/n) = 0 \tag{5}$$

for μ-almost every $x \in \big(1/(n+1), 1/n\big]$ and $y \in [0, 1/n]$ relative to Lebesgue measure μ.

It is easy to verify that the subspaces B^A and B_A are infinite-dimensional and that the operator A satisfies condition (iii) from Proposition 1.2.

Now, let P be a projection from B onto B_A which can be defined in the following manner: $Pf(x) = f(x)$ if $x \in [0, \frac{1}{2})$; if $n \geq 2$ and $x - \frac{1}{2} \in \big(1/(n+1), 1/n\big]$, then

$$\big(Pf(x)\big)y = \begin{cases} \big(f(x)\big)y & \text{if } y \in \left[0, \dfrac{n-1}{n}\right); \\ -\displaystyle\sum_{k=0}^{n-1} \big(f(x-\tfrac{1}{2}) + f(x)\big)\left(y - \frac{k}{n}\right) & \text{if } y \in \left[\dfrac{n-1}{n}, 1\right]. \end{cases}$$

By Theorem 1.1 and Proposition 1.2 the projection $\tilde{P}\colon B \to B_A$ defined by (1) is a minimal one. By virtue of results of Section 2, $\|\tilde{P}\| \leq 2$.

2. Evaluation of norm of minimal projection, generated by isometry. In this section, S_X (resp. U_X) will denote the unit sphere (resp. the unit ball) of a real Banach space X.

Let D, K be subspaces of a Banach space B such that $B = D \oplus K$. Let $x \in S_K$ and $D_x = D \oplus \operatorname{span}\{x\}$. Let $f \in S_{D_x^*}$ be such that $f^{-1}(0) = D$.

For every $a \in [0, 1]$, let

$$W_a^x = f^{-1}(a), \qquad C_a^x = U_{D_x} \cap W_a^x, \qquad \varrho_D^x(a) = \inf_{z \in W_a^x} \sup_{y \in C_a^x} \|z - y\|,$$

C_a^x and $\varrho_D^x(a)$ will be called, respectively, the *hypercircle* in D and the *Chebyshev radius* of C_a^x (cf. [6]).

Next write C_D^x for $\sup_{a \in [0,1)} \varrho_D^x(a)$ and C_D^K for $\sup_{x \in S_k} C_D^x$. Consider now the function g: $[1, 2] \to [1, 2]$ (cf. [6]) defined as

$$(6) \qquad g(t) = \begin{cases} 1+\frac{1}{2}\left((t-1)+\sqrt{(t-1)^2+8(t-1)}\right) & \text{if } 1 \leqslant t \leqslant \sqrt{17}-3, \\ 1+\dfrac{8(t-1)}{t^2+4(t-1)} & \text{if } \sqrt{17}-3 < t \leqslant 2. \end{cases}$$

Note that g is strictly increasing and concave. Moreover, $g(1) = 1$, $g(2) = 2$, $g(t) \geqslant t$ for each $t \in [1, 2]$. In terms of the function g and the number C_D^K we can evaluate the norm of the minimal projection $\tilde{P}$ defined by (1).

THEOREM 2.1. *Let A be a linear isometry of Banach space B onto itself such that $B = \operatorname{Im}(I-A) \oplus \operatorname{Ker}(I-A)$. Let $D = \operatorname{Im}(I-A)$, $K = \operatorname{Ker}(I-A)$. Let $\tilde{P}$ be a projection from B onto D along K. Then*

$$(7) \qquad 1 \leqslant C_D^K \leqslant \lambda(B, D) = \|\tilde{P}\| \leqslant g(C_D^K) \leqslant 2.$$

Proof. Let $x \in S_K$ and $D_x = D \oplus \operatorname{span}\{x\}$. By Remark 1.3 the operator $A_x = A|_{D_x}$ is an isometry of D_x onto itself with $\operatorname{Im}(I-A) = D$, $\operatorname{Ker}(I-A) = \operatorname{span}\{x\}$. By Theorem 1.1, the projection P_D^x: $D_x \to D$ along $\operatorname{span}\{x\}$ is a minimal projection, i.e., $\|P_D^x\| = \lambda(D_x, D)$. In view of the fact that $\operatorname{codim}_{D_x} D = 1$ and by a result of C. Franchetty ([6], Theorem 3) we have

$$1 \leqslant C_D^x \leqslant \lambda(D_x, D) \leqslant g(C_D^x) \leqslant 2.$$

Observe also that in view of the inequality $\lambda(D_x, D) \leqslant \lambda(B, D)$, we obtain $1 \leqslant C_D^K \leqslant \lambda(B, D)$. Next we use the fact that the projection $\tilde{P}$ is minimal and $\|\tilde{P}\| = \lambda(B, D)$ (cf. Theorem 1.1).

If $\|\tilde{P}\| = 1$, then the theorem follows from the identity $g(1) = 1$.

Next assume that $\|\tilde{P}\| > 1$. Then for any $\varepsilon > 0$ with $\varepsilon < \|\tilde{P}\| - 1$, there exists $x_0 \in S_B$ such that $\|\tilde{P}(x_0)\| + \varepsilon > \|\tilde{P}\|$. Let y_D and y_K be such that $x_0 = y_D + y_K$ with $y_D \in D$ and $y_K \in K$. Evidently, $y_K \neq \theta$.

Let $z = y_K/\|y_K\|$. Then $\|\tilde{P}(x_0)\| = \|P_D^z(x_0)\| \leqslant \|P_D^z\| = \lambda(D_z, D)$, where P_D^z is the projection from $D_z = D \oplus \operatorname{span}\{z\}$ onto D along $\operatorname{span}\{z\}$. Clearly, $\lambda(B, D) < \lambda(D_z, D) + \varepsilon$. Since the function g is strictly increasing, we get

$$\lambda(B, D) \leqslant \sup_{x \in S_K} \lambda(D_x, D) \leqslant \sup_{x \in S_K} g(C_D^x) \leqslant g(\sup_{x \in S_K} C_D^x) = g(C_D^K) \leqslant 2.$$

By Theorem 2.1 and a result of Franchetty in [6] (Theorem 4), taking into account the form of function g, we get directly:

COROLLARY 2.1. *Let A be a linear isometry of a Banach space B onto itself and $B = \operatorname{Im}(I-A) \oplus \operatorname{Ker}(I-A)$. Let $D = \operatorname{Im}(I-A)$, $K = \operatorname{Ker}(I-A)$. Then*

$$\text{(I)} \qquad \lambda(B, D) = 1 \Leftrightarrow C_D^K = 1 \Leftrightarrow \forall x \in S_K\colon C_D^x = 1 \Leftrightarrow \forall a \in (0, 1) \text{ and } \forall x \in S_K\colon \varrho_D^x(a) \leqslant 1,$$

$$\text{(II)} \qquad \lambda(B, D) < 2 \Leftrightarrow C_D^K < 2 \Leftrightarrow \forall x \in S_K \; \exists a_x \in (0, 1)\colon \varrho_D^x(a_x) < 1 + a_x,$$

$$\text{(III)} \qquad \lambda(B, D) = 2 \Leftrightarrow C_D^K = 2.$$

In the next example we give a realization of cases (II) and (III) of Corollary 2.1. For case (I), see [11] (Proposition 3.a.4).

EXAMPLE 2.1 Let B be a Banach space with the symmetric norm (2) (relative to a normal basis $(e_i)_1^\infty$ ([17], [18])). Let $(j(i))_{i=1}^\infty$ be a strictly increasing sequence of natural numbers so that $j(1) = 1$ and $k(i) = j(i+1) - j(i) \geqslant 3$. Now, let $A\colon B \to B$ be a linear isometry such that $A(e_s) = e_{s+1}$ for every $s \in N$ and $j(i) \leqslant s < j(i+1)-1$ and $A_{j(i+1)-1} = e_{j(i)}$ $(i = 1, 2, \ldots)$.

Let

$$D_{k(i)} = \Big\{x = (\alpha_1, \alpha_2, \ldots) \in B\colon \sum_{\nu=j(i)}^{j(i+1)-1} \alpha_\nu = 0,\ \alpha_\nu = 0 \quad \text{if } (\nu < j(i)) \text{ or } (\nu \geqslant j(i+1))\Big\},$$

$$B_{k(i)} = \operatorname{span}\{e_{j(i)}, \ldots, e_{j(i+1)-1}\}, \qquad i = 1, 2, \ldots$$

It is easy to see that $\dim(B_{k(i)}/D_{k(i)}) = 1$. From Theorem 1.1 it follows immediately that the isometry A generates in the subspace $B_{k(i)}$ the minimal projection P_i from $B_{k(i)}$ onto $D_{k(i)}$ and $\|P_i\| = \lambda(B_{k(i)}, D_{k(i)})$.

Now, in view of Proposition 1.2 it is easy to check that its condition (ii) holds. Hence, by Theorem 1.1 the projection $\tilde{P}$ from B onto $D = \operatorname{Im}(I-A) = \bigoplus_{i=1}^{\infty} D_{k(i)}$ along $\operatorname{Ker}(I-A)$ is a minimal projection.

It is obvious that

$$\|\tilde{P}\| = \sup_i \|P_i\| \tag{8}$$

(because for each $x = \sum_{i=1}^{\infty} \alpha_i e_i$ we have $\|x\| \geqslant \Big\|\sum_{\nu=j(i)}^{j(i+1)-1} \alpha_\nu e_\nu\Big\|$, $i = 1, 2, \ldots$, [18]).

Now, let $B = l_1$ or $B = c_0$. We prove that $\|P_i\| = 2 - 2/k(i)$ $(i = 1, 2, \ldots)$. Indeed, if $B = l_1$, then there exists a linear isometry $F_1\colon B_{k(i)} \xrightarrow{\text{onto}} l_1^{k(i)}$, so $F_1(D_{k(i)}) = f_{1,i}^{-1}(0)$, where $f_{1,i} = (1, \ldots, 1) \in (l_1^{k(i)})^*$. If $B = c_0$, then there exists a linear isometry $F_2\colon B_{k(i)} \xrightarrow{\text{onto}} l_\infty^{k(i)}$, so $F_2(D_{k(i)}) = f_{2,i}^{-1}(0)$, where $f_{2,i} = (1/k(i), \ldots, 1/k(i)) \in (l_\infty^{k(i)})^*$.

By the result of [3] we get in both cases: $\lambda(B_{k(i)}, D_{k(i)}) = 2 - 2/k(i)$. By (8), $\|\tilde{P}\| = 2 - \inf_i(2/k(i))$.

Therefore, $\|\tilde{P}\| = 2$ and $C_D^K = 2$, where $K = \operatorname{Ker}(I-A)$, if $\sup_i k(i) = +\infty$. If $\sup_i(k(i)) < +\infty$, then for each $x \in S_K$ there exists $a_x \in (0, 1)$ such that $\varrho_D^x(a_x) < 1 + a_x$.

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(2) Let E be the set all $\varepsilon = (\varepsilon_1, \varepsilon_2, \ldots)$ with $\varepsilon_i \in \{-1, 1\}$ $(i = 1, 2, \ldots)$. Let Π be the set of all permutations $\sigma\colon N \to N$. A Banach space B with a normal basis $(e_i)_1^\infty$ is said to be *symmetric* iff $\|\varepsilon\sigma x\| = \|x\|$ for every $x \in B$, $\sigma \in \Pi$, $\varepsilon \in E$ (cf. [18]).

Remark 2.1. Note, if $B = c_0$, then the projection $\tilde{P}$ from Example 2.1 is the unique minimal projection onto D.

If $B = l_1$, then the uniqueness of the minimal projection $\tilde{P}$ onto D (from Example 2.1) fails, although the projections P_i are unique minimal projections from $B_{k(i)}$ onto $D_{k(i)}$, $i = 1, 2, \ldots$ (Cf. [14], [15].)

From Theorem 2.1 we get immediately:

COROLLARY 2.2. *Suppose D be a complemented subspace of Banach space B such that $\lambda(B, D) > 2$, Then there exists no linear isometry A of B onto itself such that $D = \mathrm{Im}(I-A)$ and $B = D \oplus \mathrm{Ker}(I-A)$.*

EXAMPLE 2.2. Let $B = \tilde{C}([0, 2\pi])$ be the space of all continuous 2π-periodic real-valued functions defined on $[0, 2\pi]$, and D_n $(n \geqslant 1)$, be the subspace in B consisting of all trigonometric polinomials of degree $\leqslant n$.

We shall prove that for $n \geqslant 8$ there exists no linear isometry A of B onto itself such that $\mathrm{Im}(I-A) = D_n$. Suppose that for some $n \geqslant 8$ there exists such a linear isometry A.

By Proposition 1.1 and Example 1.1 it follows that $B = D_n \oplus \mathrm{Ker}(I-A)$.

Hence, by Theorem 2.1, $\lambda(B, D_n) \leqslant 2$. On the other hand, $\lambda(B, D_n)$ is equal to the Lebesgue Constant ϱ_n such that $\varrho_n = (4/\pi^2)\log n + 1.27033 + \varepsilon_n$, where $0.166 > \varepsilon_n \downarrow 0$ (cf., for example, [4]). Taking into account that $n \geqslant 8$, we have $\varrho_n > 2$, a contradiction.

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