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On the Dieudonné property *

Abstract. We investigate the relationships between the so-called Dieudonné property and other isomorphic properties of Banach spaces. Consequences of this property concerning operators are pointed out.

Introduction. In his famous paper [6] Grothendieck considered an isomorphic property of B- (= Banach) spaces called by him *Dieudonné property*. The present brief note is mostly devoted to the study of the relationships between the class of B-spaces with the above cited property and other classes of B-spaces. Moreover, we also point out some consequences of the Dieudonné property concerning operators defined on B-spaces with this property.

Preliminaries. For the sake of brevity in the sequel we shall use the following abbreviations concerning operators T defined on a B-space E with values into a B-space F: $T\ wC = T$ is weakly compact; $T\ wcc = T$ is weakly completely continuous; $T\ uc = T$ is unconditionally converging; $T\ nfc_0 = T$ does not fix a copy of c_0. Hence, we shall say that a B-space E has the Dieudonné property ([6]), the strict Dieudonné property ([9]), property (V) of Pełczyński ([11]) (perhaps new), property (∗) iff, respectively, for every B-space F and every operator $T\colon E \to F$ the following implications are true: $T\ wcc \Rightarrow T\ wC$, $T\ nfc_0 \Rightarrow T\ wC$, $T\ uc \Rightarrow T\ wC$, $T\ uc \Rightarrow T\ wcc$. Let (D), (sD), (V), $(*)$ denote the respective classes of B-spaces.

Relationships between the class (D) and the other classes. Our first result of this section is the following

Theorem 1. $(sD) = (V) \subset (D)$.

Proof. It is easy to prove that $T\ wcc \Rightarrow T\ uc$; moreover, in [4], p. 54, it is proved that $T\ uc \Leftrightarrow T\ nfc_0$. This completes the proof.

Another class of B-spaces belonging to the class (D) is given in the following

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THEOREM 2. *If a B-space E does not contain l^1, then $E \in (D)$.*

Proof. A well-known result of Rosenthal ([12]) says that any bounded sequence in E has a weak Cauchy subsequence. Hence, the implication $T\ wcc \Rightarrow T\ wC$ is easily true.

The above results cannot be reversed; indeed, the B-space J ([7]) is in the class (D) since it does not contain l^1 and does not belong to the class (V) because its dual space J^* is not weakly sequentially complete, whereas in [11] it is shown that a space in (V) must have a weakly sequentially complete dual; moreover, the space $C([0, 1])$ is in (D), *via* Theorem 1, but it contains l^1.

Now, using property $(*)$ we characterize the Dieudonné property.

THEOREM 3. *The classes of B-spaces $(*) \cap (D)$ and (V) coincide.*

Proof. If $E \in (*)$, then for every operator $T: E \to F$ we have $T\ uc \Rightarrow T\ wcc$; if $E \in (D)$ we have $T\ wcc \Rightarrow T\ wC$; hence, one has $(*) \cap (D) \subset (V)$. But $(V) \subset (D)$ by Theorem 1 and $(V) \subset (*)$. The proof is over.

Property $(*)$ is new as far as we know; so we think that a brief digression on it is in order. We shall observe that the class of B-spaces having the so-called property (u) ([10]) is contained in $(*)$; we recall that a B-space E is said to have *property* (u) iff for any weak Cauchy sequence (x_n) in E there is a sequence (y_n) in E such that the series $\sum y_n$ is weakly unconditionally convergent and the sequence $(x_n - \sum_{k=1}^{n} y_k)$ converges weakly to θ in E.

THEOREM 4. *If a B-space E has property (u), then $E \in (*)$.*

Proof. Let $T: E \to F$, F an arbitrary B-space, be an *uc* operator. If (x_n) is a weak Cauchy sequence in E, we consider a sequence (y_n) as in the definition of the property (u); hence, the sequence $(T(x_n) - \sum_{k=1}^{n} T(y_k))$ converges weakly to θ in F. But $\sum T(y_n)$ is unconditionally convergent in F to a $z \in F$: so $T(x_n) \to z$ weakly in F, i.e., T is *wcc*.

The presence in E of property $(*)$ has the following consequence:

THEOREM 5. *Let E belong to $(*)$. Any complemented subspace F of E not containing c_0 is weakly sequentially complete. Hence E is weakly sequentially complete iff it is in the class $(*)$ and does not contain c_0.*

Proof. Let P be the projection from E onto F; since $F \not\supset c_0$, P is *uc*; but $E \in (*)$ and so P is *wcc*; $P_{|F}$ = identity on F and so the first part of the theorem is proved. The second equivalence is easily true, since any weakly sequentially complete space E has property (u) and hence property $(*)$ and it does not contain c_0. Furthermore, the converse follows from the first part of the theorem. The proof is over.

Theorem 5 improves a well-known result concerning weak sequential completeness of a B-lattice with order continuous norm; indeed, it is known that such a lattice has property (u) and hence property $(*)$ ([8]). Moreover, an easy consequence of the same result of [8] is that in order continuous B-lattices the following equivalence is true "$E \in (V) \Leftrightarrow E \in (D)$", *via* Theorem 3. We still observe that the space J, which belongs to (D), does not contain c_0 and it is not weakly sequentially complete; hence J is not in $(*)$.

At the end of the paper we shall also give an example of a B-space which is in $(*)$, but not in (D). In such a way, we shall prove that the classes (D) and $(*)$ are different.

Using the same techniques as the one employed in [2] we can prove

THEOREM 6. *Let K be a scattered compact Hausdorff space. Then $C(K, E)$ is in $(*)$ iff E is.*

Consequences of the Dieudonné property. The second part of the paper is devoted to individuating some consequences of the Dieudonné property concerning operators defined on B-spaces belonging to (D).

THEOREM 7. *Let E belong to (D). Each operator T: $E \to F$, F a weakly sequentially complete B-space, is wC.*

Proof. The proof is very easy and we omit it.

The next result due to Gamlen ([5]) is a corollary of Theorem 7:

COROLLARY 1. *Let K be a compact Hausdorff space and E be a B-space such that E^* has the Radon–Nikodym property. Hence, any operator T from $C(K, E)$ to a weakly sequentially complete B-space F is wC.*

Proof. Let T: $C(K, E) \to F$ be an operator and (f_n) be a bounded sequence in $C(K, E)$. We put $A = \overline{\text{span}}\{f_n(t)\colon n \in N,\ t \in K\}$ and observe that A is a closed separable subspace of E; hence, A^* is separable ([3]). A result of Bombal and Cembranos ([1]) implies that $C(K, A) \in (D)$; moreover, $(f_n) \subset C(K, A)$. Since the restriction of T to $C(K, A)$ is wC, *via* Theorem 7, we can conclude the proof by arbitrarity of (f_n).

Another consequence of Theorem 7 is the following

COROLLARY 2. *Any complemented closed subspace F of a B-space E, $E \in (D)$, is itself an element of (D). Hence F is reflexive provided that it is weakly sequentially complete. Moreover, any E belonging to (D) cannot contain complemented copies of l^1 and $L^1([0, 1])$.*

We omit the simple proof.

At the end, we furnish the announced example of a B-space not in (D), but in $(*)$; we can take the spaces l^1 and $L^1([0, 1])$.

Addendum. Let X, Y be two Banach spaces. By $K(X, Y)$ we shall denote the Banach space of all compact operators from X into Y equipped with the operator norm, whereas $K_{w^*}(X^*, Y)$ will denote the Banach space of all compact, weak*-weak continuous operators from X^* into Y, also equipped with the operator norm. Here we give a result concerning the Dieudonné property in the space $K(X, Y)$

THEOREM. *Let X be an $\mathscr{L}_\infty$-space and Y be a B-space such that Y^* has the Radon–Nikodym property. Then, $K(X^*, Y) \in (D)$.*

Proof. It is known that $K(X^*, Y)$ is isometrically isomorphic to $K_{w^*}(X^{***}, Y)$ (for this we refer the reader [13]). But X^{**} is a $C(K)$ space, for a suitable compact Hausdorff space K; hence, $K_{w^*}(X^{***}, Y)$ is isometrically isomorphic to $C(K, Y)$ (see [13]) and this last space is in the class (D) by a result of Bombal and Cembranos (see [1]). The proof is over.

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