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Unconditionality of orthogonal bases in B_0-algebras

Abstract. Unconditionality of an orthogonal basis in a B_0-algebra has been characterized in terms of certain inequalities satisfied by a defining sequence of its seminorms. An extension of these inequalities to non-metrizable locally convex algebras has also been considered. An example is given to show that an orthogonal basis in a B_0-algebra, even in a Banach algebra need not be unconditional.

Introduction

Let A be a complex topological algebra (i.e., an algebra which is a topological vector space with jointly continuous multiplication). If the underlying topological vector space is locally convex, A is called a *locally convex algebra*. The topology of a locally convex algebra can be described by a family $P = \{p\}$ of seminorms. If, in addition, each seminorm $p \in P$ satisfies the inequality: $p(xy) \leqslant p(x)p(y)$ for all $x, y \in A$, then A is called *locally m-convex*. A complete metrizable locally convex algebra is called a *B_0-algebra* and a complete metrizable locally m-convex algebra is called a *Fréchet algebra* (cf. [4], [6]).

A sequence $\{e_n\}$ in a topological vector space E is called a *basis* if for each $x \in E$ there exists a unique sequence $\{e_n^*(x)\}$ of scalars such that $x = \sum_{n=1}^{\infty} e_n^*(x)e_n$. A basis $\{e_n\}$ in a topological algebra A is called *orthogonal* if $e_m e_n = \delta_{mn} e_m$ for all $m, n \in N$, where δ_{mn} is the Kronecker's delta. This concept of orthogonal bases in a topological algebra was first introduced by the second author and studied in collaboration with his coworkers, [5], [6]. It is worthwhile to note that a topological algebra with an orthogonal basis is always commutative [6].

If each series $x = \sum_{n=1}^{\infty} e_n^*(x)e_n$ in a topological vector space E with a

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basis $\{e_n\}$ is unconditionally convergent [9], [13], the basis $\{e_n\}$ is called an *unconditional basis.* A number of characterizations of unconditionality of a basis in a Banach space are well-known [9], [13].

Here we give some necessary and sufficient conditions for unconditionality of a basis in a Fréchet space (Proposition 1.1). A seminorm p on a locally convex algebra A is called *squarely submultiplicative* if $p^2(xy) \leqslant p(x^2)p(y^2)$ for all $x, y \in A$. We show that an orthogonal basis in a B_0-algebra A is unconditional iff each seminorm of a defining sequence of seminorms of A satisfies this inequality (Theorem 2.5).

A basis in a Banach space need not be unconditional [13]. In 2.1 we show by examples that even an orthogonal basis in a Banach algebra need not be unconditional.

We also study briefly some further consequences of the above inequality in Section 3.

For details regarding bases in Banach spaces and topological vector spaces, consult [9], [13] and for orthogonal bases in topological algebras, see [5] and [6].

1. Conditions that convert Fréchet spaces into Fréchet algebras

A complete metrizable locally convex space is called a *Fréchet space.* Indeed, each Banach space is a Fréchet space. First we prove some necessary and sufficient conditions for unconditionality of a basis in a Fréchet space, generalizing those known ([13], p. 461) for Banach spaces. We omit the proofs of most implications since they are similar to the Banach space case (cf. [9] or [13]).

1.1. PROPOSITION. *Let E be a Fréchet space whose topology is generated by a family $\{\|\cdot\|_\alpha : \alpha \in \mathscr{P}\}$ of seminorms, and let $\{e_n\}$ be a basis in E. Then:*

(a) *The following statements are equivalent:*

(i) *The basis $\{e_n\}$ is unconditional.*

(ii) *For every $x \in E$, $\lim_{J\in\Gamma} \sum_{n\in J} e_n^*(x)e_n$ exists, where Γ is the collection of all finite subsets $J \subset N$ directed by inclusion and $\{\sum_{n\in J} e_n^*(x)e_n\}_{J\in\Gamma}$ is the resulting net of all unordered finite partial sums.*

(iii) *For every $f \in E'$* (the topological dual of *E*) *and $x \in E$, $\sum_{n=1}^{\infty} |e_n^*(x)|\,|f(e_n)| < \infty$.*

(iv) *If $A_\alpha = \{f \in E' : \|f\|_\alpha \equiv \sup_{\|x\|_\alpha \leqslant 1} |f(x)| \leqslant 1\}$, $\alpha \in N$, then the convergence in* (iii) *is uniform on each A_α, i.e., for $x \in E$, $\alpha \in N$ and $\varepsilon > 0$ there exists $N \in N$ such that $\sum_{k=m+1}^{n} |e_k(x)|\,|f(e_k)| < \varepsilon$ for every $f \in A_\alpha$ and $n > m \geqslant N$.*

(v) *Let* $T_\alpha = \{n \in N \colon \|e_n\|_\alpha \neq 0\}$, $\alpha \in N$; *then for every* $x \in E$ *and every sequence* $\{\beta_n\}$ *of scalars such that each of the sets* $\{\beta_n \colon n \in T_\alpha\}$, $\alpha \in N$, *is bounded, the series* $\sum_{n=1}^{\infty} \beta_n e_n^*(x) e_n$ *converges.*

(vi) *For every* $x \in E$ *and every bounded sequence* $\{\beta_n\}$ *of scalars,* $\sum_{n=1}^{\infty} \beta_n e_n^*(x) e_n$ *converges.*

(vii) *For every* $x \in E$ *and every sequence* $\{\varepsilon_n\}$ *of scalars with* $\varepsilon_n = \pm 1$ *for all* $n \in N$, $\sum_{n=1}^{\infty} \varepsilon_n e_n^*(x) e_n$ *converges.*

(b) *Let the basis* $\{e_n\}$ *be unconditional. For* $x \in E$, $\alpha \in N$ *put*

$$\|x\|'_\alpha = \sup_{f \in \Lambda_\alpha} \sum_{n=1}^{\infty} |e_n^*(x)|\, |f(e_n)|, \qquad \|x\|''_\alpha = \sup_{J \in \Gamma} \Big\| \sum_{n \in J} e_n^*(x) e_n \Big\|_\alpha .$$

Then each of $\{\|\cdot\|'_\alpha \colon \alpha \in N\}$ *and* $\{\|\cdot\|''_\alpha \colon \alpha \in N\}$ *is a family of seminorms on* E *generating the original topology and* $\|x\|_\alpha \leqslant \|x\|''_\alpha \leqslant \|x\|'_\alpha$ *for all* $x \in E$ *and* $\alpha \in N$.

Proof. We only show that (iv)$\Rightarrow$(v)$\Rightarrow$(vi). Assume that (iv) holds and let $\{\beta_n\}$ be as in (v). For each $\alpha \in N$, put $M_\alpha = \sup_{n \in T_\alpha} |\beta_n|$. For $x \in E$, $\alpha \in N$ and $\varepsilon > 0$ let $N \in N$ be such that $\sum_{k=m+1}^{n} |e_k^*(x)|\, |f(e_k)| < \varepsilon / M_\alpha$ whenever $n > m \geqslant N$ and $f \in \Lambda_\alpha$. For every such m, n, it follows from a well-known corollary of the Hahn–Banach theorem [12] that there exists $g_{m,n} \in \Lambda_\alpha$ such that

$$g_{m,n}\Big(\sum_{k=m+1}^{n} \beta_k e_k^*(x) e_k \Big) = \Big\| \sum_{k=m+1}^{n} \beta_k e_k^*(x) e_k \Big\|_\alpha$$

and hence

$$\Big\| \sum_{k=m+1}^{n} \beta_k e_k^*(x) e_k \Big\|_\alpha \leqslant \sum_{k=m+1}^{n} |\beta_k|\, |e_k^*(x)|\, |g_{m,n}(e_k)| = \sum_{\substack{k=m+1 \\ k \in T_\alpha}}^{n} |\beta_k|\, |e_k^*(x)|\, |g_{m,n}(e_k)|,$$

where the last equality holds since for $k \in N \setminus T_\alpha$ we have $|g_{m,n}(e_k)| \leqslant \|e_k\|_\alpha = 0$. It follows that

$$\begin{aligned} \Big\| \sum_{k=m+1}^{n} \beta_k e_k^*(x) e_k \Big\|_\alpha &\leqslant \sum_{k=m+1}^{n} |\beta_k|\, |e_k^*(x)|\, |g_{m,n}(e_k)| \\ &\leqslant M_\alpha \sum_{\substack{k=m+1 \\ k \in T_\alpha}}^{n} |e_k^*(x)|\, |g_{m,n}(e_k)| \\ &\leqslant M_\alpha \sum_{k=m+1}^{n} |e_k^*(x)|\, |g_{m,n}(e_k)| < \varepsilon \end{aligned}$$

whenever $n > m \geqslant N$. Since E is complete, $\sum_{n=1}^{\infty} \beta_n e_n^*(x) e_n$ converges. Thus (iv)$\Rightarrow$(v). The desired conclusion follows since the implication (v)$\Rightarrow$(vi) is obvious.

Now we extend a result due to Husain and Watson [6] (Proposition 4.1) from Banach spaces to Fréchet spaces. It should be pointed out that the unconditional basis in the hypothesis of Proposition 4.1 [6] should be assumed to be normalized which is not possible in general as it will show in the example given below.

1.2. PROPOSITION. *Let* $\{e_n\}$ *be an unconditional basis in Fréchet space* E. *The following statements are equivalent*:

(i) E *has a generating family* $\{\|\cdot\|_\alpha\colon \alpha\in N\}$ *of seminorms such that for every* $\alpha\in N$, 0 *is not a limit point of the set* $\{\|e_n\|_\alpha\colon n\in N\}$.

(ii) E *can be endowed with a multiplication that makes* E *into a Fréchet algebra with* $\{e_n\}$ *as an orthogonal* (*unconditional*) *basis.*

Proof. (ii)$\Rightarrow$(i) If $\{\|\cdot\|_\alpha\colon \alpha\in N\}$ is a defining family of submultiplicative seminorms, then for every $\alpha\in N$, $n\in N$ we have $\|e_n\|_\alpha=\|e_n^2\|_\alpha\leqslant\|e_n\|_\alpha^2$ and so $\|e_n\|_\alpha=0$ or $\|e_n\|_\alpha\geqslant 1$.

(i)$\Rightarrow$(ii) From Proposition 1.1 we have

$$|e_n^*(x)|\,\|e_n\|_\alpha=\|e_n^*(x)e_n\|_\alpha\leqslant\sup_{J\in\Gamma}\Big\|\sum_{n\in J}e_n^*(x)e_n\Big\|_\alpha=\|x\|_\alpha''\leqslant\|x\|_\alpha'$$

for every $x\in E$ and $\alpha, n\in N$. By hypothesis, $\delta_\alpha=\inf_{n\in N}\{\|e_n\|_\alpha\colon \|e_n\|_\alpha\neq 0\}>0$, and hence, for each fixed $\alpha\in N$,

$$|e_n^*(x)|\leqslant\frac{\|x\|_\alpha'}{\|e_n\|_\alpha}\leqslant\frac{\|x\|_\alpha'}{\delta_\alpha}<\infty$$

for every $n\in N$ with $\|e_n\|_\alpha\neq 0$. Thus the sequence $\{e_n^*(x)\}$ of scalars satisfies the condition described in statement (a) (v) of Proposition 1.1. It follows that for $x=\sum_{n=1}^{\infty}e_n^*(x)e_n$, $y=\sum_{n=1}^{\infty}e_n^*(y)e_n\in E$ the series $\sum_{n=1}^{\infty}e_n^*(x)e_n^*(y)e_n$ converges in E. We define a multiplication on E by

$$xy=\sum_{n=1}^{\infty}e_n^*(x)e_n^*(y)e_n.$$

Under this multiplication, E is a Fréchet algebra. Indeed, for $\alpha\in N$, x, $y\in E$ we have

$$\|xy\|_\alpha'=\sup_{f\in A_\alpha}\sum_{n=1}^{\infty}|e_n^*(x)|\,|e_n^*(y)|\,|f(e_n)|=\sup_{f\in A_\alpha}\sum_{\substack{n=1\\ n\in T_\alpha}}^{\infty}|e_n^*(x)|\,|e_n^*(y)|\,|f(e_n)|$$

$$\leqslant\frac{\|x\|_\alpha'}{\delta_\alpha}\sup_{f\in A_\alpha}\sum_{\substack{n=1\\ n\in T_\alpha}}^{\infty}|e_n^*(y)|\,|f(e_n)|=\frac{\|x\|_\alpha'}{\delta_\alpha}\sup_{f\in A_\alpha}\sum_{n=1}^{\infty}|e_n^*(y)|\,|f(e_n)|=\frac{\|x\|_\alpha'}{\delta_\alpha}\|y\|_\alpha',$$

where the second equality from the left and the second equality from the right hold since $|f(e_n)| \leqslant \|e_n\|_\alpha = 0$ for $f \in \Lambda_\alpha$ and $n \notin T_\alpha = \{n \in N: \|e_n\|_\alpha \neq 0\}$. Set $p_\alpha(z) = \|z\|'_\alpha/\delta_\alpha$ for $\alpha \in N$ and $z \in E$; then $\{p_\alpha: \alpha \in N\}$ is a family of seminorms generating the original topology on E, since $\{\|\cdot\|'_\alpha: \alpha \in N\}$ is such a family by Proposition 1.1. Moreover, each p_α is submultiplicative since from the last inequality we have

$$p_\alpha(xy) = \frac{1}{\delta_\alpha}\|xy\|'_\alpha \leqslant \frac{1}{\delta_\alpha}\frac{\|x\|'_\alpha\|y\|'_\alpha}{\delta_\alpha} = p_\alpha(x)\,p_\alpha(y).$$

The proof is completed by the simple observation that the basis $\{e_n\}$ is indeed orthogonal under the multiplication introduced.

A weaker condition of orthogonality of a basis $\{e_n\}$ in a topological algebra A is that $e_m e_n = c_m \delta_{mn} e_m$, where c_m, $m \in N$ are non-zero scalars. This type of orthogonality was also discussed in [4], [5] and [6]. Here we call a basis with this property *quasiorthogonal*. This definition leads to the following:

1.3. PROPOSITION. *Let $\{e_n\}$ be an unconditional basis in a Fréchet space E. Then E can be endowed with a multiplication that makes E into a Fréchet algebra with $\{e_n\}$ as a quasiorthogonal (unconditional) basis.*

Proof. Let $\{\|\cdot\|_\alpha: \alpha \in N\}$ be an increasing sequence of submultiplicative seminorms generating the topology of E.

For each fixed n, put $\gamma_n = \|e_n\|_{\alpha_n}$, where α_n is the smallest positive integer such that $\|e_n\|_{\alpha_n} \neq 0$ and set $e'_n = \gamma_n^{-1} e_n$. Then for any fixed $\alpha \in N$ we have $\|e'_n\|_\alpha = \|e_n\|_\alpha/\gamma_n \geqslant 1$ for all $n \in N$ with $\|e_n\|_\alpha \neq 0$. Thus, condition (i) of Proposition 1.2 is satisfied for the unconditional basis $\{e'_n\}$ and the generating family $\{\|\cdot\|_\alpha: \alpha \in N\}$ of seminorms. Hence E can be endowed with a multiplication under which E is a Fréchet algebra with $\{e'_n\}$ as an orthogonal basis. Now $e_n^2 = (\gamma_n e'_n)^2 = \gamma_n^2 e'^2_n = \gamma_n^2 e'_n = \gamma_n(\gamma_n e'_n) = \gamma_n e_n$ and $e_m e_n = \gamma_m e'_n \cdot \gamma_n e'_n = 0$ for $m \neq n$.

If E in Propositions 1.2 and 1.3 is a Banach space with a norm $\|\cdot\|$, then $\|e_n\| \neq 0$ for all $n \in N$ and so we have:

1.4. COROLLARY. *Let $\{e_n\}$ be an unconditional basis in a Banach space $(E, \|\cdot\|)$. Then E can be endowed with a multiplication that makes it into a Banach algebra with $\{e_n\}$ as an orthogonal (unconditional) basis iff* $\inf\limits_{n \in N} \|e_n\| > 0$.

1.5. EXAMPLE. (i) For an example of a Banach space with an unconditional basis $\{e_n\}$ which cannot be made into a Banach algebra with $\{e_n\}$ as an orthogonal basis, we consider E to be the Banach space of all sequences $x = \{x(n)\}$ of complex numbers such that $\|x\| = \sum\limits_{n=1}^{\infty} |x(n)|/n < \infty$. Clearly, the canonical basis $e_n = \{\delta_{nm}\}$, $n \in N$ is unconditional. If E is a Banach algebra with $\{e_n\}$ as an orthogonal basis, then $xy = \sum x(n)y(n)e_n \in E$ for $x, y \in E$. But this multiplication is not possible if $x(n) = n^{1/3}$, $n = k^3$ for some $k \in N$ and $= 0$, otherwise since

$$\sum_{k=1}^{\infty} \frac{|x(k)|}{k} = \sum \frac{k}{k^3} < \infty$$

but x^2 does not exist because $\sum k^2/k^3 = \infty$. However, following Proposition 1.3, the product

$$x \cdot y = \sum \frac{x(n)y(n)}{n} e_n$$

converts E into a Banach algebra with $\{e_n\}$ as quasiorthogonal basis.

(ii) The Fréchet space $H(D)$ of all functions holomorphic on the open unit disc with the compact-open topology has the unconditional basis $e_n(z) = z^n$ $(n \in N)$. With the multiplication $xy = \sum e_n^*(x)e_n^*(y)e_n$, $H(D)$ is a B_0-algebra (but not a Fréchet algebra) in which $\{e_n\}$ is an orthogonal basis. The increasing sequence $\{q_n\}$ of seminorms, where $q_n(x) = \sup\{|x(z)|: |z| \leqslant n/(n+1)\}$, defines the topology of $H(D)$. In view of Proposition 1.4, put $\gamma_n = q_1(e_n) = (\frac{1}{2})^n$, then the multiplication $xy = \sum 2^{-n} e_n^*(x)e_n^*(y)e_n$ converts $H(D)$ into a Fréchet algebra with $\{e_n\}$ as quasiorthogonal basis.

2. Unconditionality of orthogonal bases in B_0-algebras

In this section we have necessary and sufficient conditions for an orthogonal basis in a B_0-algebra (in particular Banach algebra) to be unconditional. Each of the classical Banach algebras l_p, $1 \leqslant p < \infty$ of complex sequences $x = \{x(n)\}$ with the norm $\|x\|_p = (\sum_{n=1}^{\infty} |x(n)|^p)^{1/p} < \infty$ and the Banach algebra c_0 of complex sequences $x = \{x(n)\}$ such that $\lim_n x(n) = 0$, with $\|x\|_\infty = \sup_n |x(n)|$, under the coordinate algebraic operations, have an orthogonal basis $\{e_n\}$, where $e_n = \{\delta_{nm}\}$, $n, m \in N$, which is unique [6] and unconditional. However, as we shall shortly show by examples, an orthogonal basis in a B_0-algebra (and even in the special case of a Banach algebra) need not be unconditional. First we have:

2.1. EXAMPLES. 1. For $1 \leqslant p < \infty$, the convolution algebra $L_p(T)$ over the torus group T is a Banach algebra. For $1 < p < \infty$, we have

$$x = \lim_N \sum_{n=-N}^{N} \hat{x}(n) e^{in(\cdot)} \tag{2.1.1}$$

in the L_p-norm, where $\hat{x}$: $Z \to C$ is the Fourier transform of $x \in L_p(T)$ [7].

Let the sets $\{e_n: n = 0, 1, \ldots\} \subset L_p(T)$ and $\{e_n^*: n = 0, 1, \ldots\} \subset L'_p(T)$ (the topological dual of $L_p(T)$) be given by: $e_0(t) \equiv 1$, $e_0^*(x) = \hat{x}(0)$; $e_{2k-1}(t) = e^{-ikt}$, $e_{2k-1}^*(x) = \hat{x}(-k)$; $e_{2k}(t) = e^{ikt}$, $e_{2k}^*(x) = \hat{x}(k)$; $t \in T$, $x \in L_p(T)$, $k = 1, 2, \ldots$ It is easy to see that the series $\sum_{n=0}^{\infty} e_n^*(x) e_n$ which is a one-sided rearrangement of (2.1.1) also converges to x in the L_p-norm and hence $\{e_n\}$ is a basis for $L_p(T)$, $1 < p < \infty$. This basis is orthogonal since for $m, n \in N \cup \{0\}$ there exist $r, s \in Z$ with $e_m = e^{ir(\cdot)}$, $e_n = e^{is(\cdot)}$ and $r = s$ iff $m = n$, consequently

$$(e_m e_n)(t) = \frac{1}{2\pi} \int_0^{2\pi} e^{ir(t-u)} e^{isu}\, du = e^{irt} \frac{1}{2\pi} \int_0^{2\pi} e^{-i(r-s)u}\, du$$

$$= \delta_{rs} e^{irt} = \delta_{mn} e_m(t).$$

For $1 < p < \infty$, $p \neq 2$ this basis is not unconditional. Indeed, there exist $x \in L_p(T)$ and $\varepsilon: Z \to \{-1, 1\}$ such that $\varepsilon\hat{x}: Z \to C$ is not the Fourier transform of any $y \in L_p(T)$ [3], contrary to statement (vii) of part (a), Proposition 1.1.

2. Let w_0 (also denoted by bv_0) be the set of all sequences $x = \{x(n)\}$ of complex numbers such that $\lim_n x(n) = 0$ and $\sum_{n=1}^{\infty} |x(n) - x(n+1)| < \infty$. The function $\|\cdot\|: w_0 \to R$ given by $\|x\| = \sup_n |x(n)| + \sum_{n=1}^{\infty} |x(n) - x(n+1)|$ is a norm on w_0 such that $(w_0, \|\cdot\|)$ is a Banach algebra under the coordinatewise operations, which is (algebraically) a subalgebra of c_0 [14]. It is easy to see that the coordinate unit vectors $\{e_n\}$ form an orthogonal basis in w_0. This basis is not unconditional since

$$\sum_{n=1}^{\infty} \frac{1}{n} e_n$$

converges in w_0 while its subseries

$$\sum_{n=1}^{\infty} \frac{1}{2n-1} e_{2n-1}$$

does not. (Notice that

$$\sum_{n=1}^{\infty} \left| \frac{1}{n} - \frac{1}{n+1} \right| = \sum_{n=1}^{\infty} \frac{1}{n(n+1)} < \infty \quad \text{while} \quad \sum_{n=1}^{\infty} \left| \frac{\varepsilon_n}{n} - \frac{\varepsilon_{n+1}}{n+1} \right| = \infty,$$

where $\varepsilon_n = 1$, n odd and $\varepsilon_n = 0$, n even.)

2.2. Definitions. Let A be an algebra (no topological structure assumed).

(i) A subset $S \subset A$ is said to be a *squarely idempotent set* (in short, s.i. set) if $xy \in S$ whenever $x^2, y^2 \in S$.

(ii) A seminorm $\|\cdot\|$ on A is said to be *squarely submultiplicative* (s.s. for short) if $\|xy\|^2 \leqslant \|x^2\|\,\|y^2\|$ for all $x, y \in A$.

2.3. PROPOSITION. *Let A be as in Definition 2.2 and let S be a circled, convex and absorbing subset of A with gauge $\|\cdot\|$. Then S is squarely idempotent iff the seminorm $\|\cdot\|$ is squarely submultiplicative.*

Proof. Assume that $\|\cdot\|$ is squarely submultiplicative and let $x, y \in A$ be such that $x^2, y^2 \in S$, then $\|xy\| \leqslant \|x^2\|^{1/2}\|y\|^{1/2} \leqslant 1$ and so $xy \in S$.

Conversely, if S is an s.i. set and $x, y \in A$ such that $\|x^2\| \neq 0$ and $\|y^2\| \neq 0$, then $x^2/\|x^2\|$, $y^2/\|y^2\| \in S$ and so

$$\frac{x}{\|x^2\|^{1/2}}\frac{y}{\|y^2\|^{1/2}} \in S.$$

Hence $\left\|xy/\|x^2\|^{1/2}\|y^2\|^{1/2}\right\| \leqslant 1$ and consequently $\|xy\| \leqslant \|x^2\|^{1/2}\|y^2\|^{1/2}$. If one of $\|x^2\|$ and $\|y^2\|$ (say $\|x^2\|$) is zero, then for any $\lambda > 0$ we have $\dfrac{x^2}{\lambda^2/\mu_0} \in S$, where $\mu_0 > 0$ is such that $y^2/\mu_0 \in S$. Hence,

$$\frac{xy}{\lambda} = \frac{x}{\lambda/\sqrt{\mu_0}} \cdot \frac{y}{\sqrt{\mu_0}} \in S$$

and so $xy \in \lambda S$. Since $\lambda > 0$ is arbitrary, we have $\|xy\| = 0 \leqslant \|x^2\|^{1/2}\|y^2\|^{1/2}$.

2.4. DEFINITIONS. (i) A topological (locally convex) algebra A is said to be a *topological* (locally convex) *s-algebra* if A has a 0-neighborhood base $\mathscr{U} = \{U\}$ such that each U is an s.i. set, in addition to being circled (convex) and closed.

From Proposition 2.3 we easily see that a locally convex s-algebra can be equivalently defined as a locally convex algebra whose topology can be generated by a family of squarely submultiplicative seminorms.

(ii) A locally m-convex algebra is said to be a *locally m-convex s-algebra* if A has a 0-neighborhood base $\mathscr{U} = \{U\}$ in which each U is an s.i. set, in addition to being circled, convex, idempotent ($U^2 \subset U$) and closed.

Clearly, the topology of a locally m-convex s-algebra can be generated by a family of submultiplicative, squarely submultiplicative seminorms.

Consequently, a B_0 s-algebra is a complete metrizable locally convex s-algebra, a Fréchet s-algebra is a complete metrizable locally m-convex s-algebra, a normed s-algebra is an algebra which is topologized by a submultiplicative, squarely submultiplicative norm and a Banach s-algebra is a complete normed s-algebra.

Now we prove our main result:

2.5. THEOREM. *An orthogonal basis $\{e_n\}$ in a B_0-algebra A is unconditional iff A is a B_0 s-algebra.*

Proof. Let $\{\|\cdot\|_\alpha\colon \alpha\in \boldsymbol{N}\}$ be a generating family of squarely submultiplicative seminorms on A. If $I\subset J\subset \boldsymbol{N}$, J is finite and λ_k, $k\in J$ are scalars, then because of the orthogonality of the basis $\{e_n\}$ we have $\sum\limits_{k\in I}\lambda_k e_k = \sum\limits_{k\in I}\lambda_k^{1/2} e_k\cdot\sum\limits_{k\in J}\lambda_k^{1/2} e_k$ and so for every $\alpha\in\boldsymbol{N}$,

$$\|\sum_{k\in I}\lambda_k e_k\|_\alpha^2 \leqslant \|\sum_{k\in I}\lambda_k e_k\|_\alpha\cdot\|\sum_{k\in J}\lambda_k e_k\|_\alpha. \tag{2.5.1}$$

If $\|\sum\limits_{k\in I}\lambda_k e_k\|_\alpha = 0$, then we trivially have

$$\|\sum_{k\in I}\lambda_k e_k\|_\alpha \leqslant \|\sum_{k\in J}\lambda_k e_k\|_\alpha. \tag{2.5.2}$$

If $\|\sum\limits_{k\in I}\lambda_k e_k\|_\alpha \neq 0$, then (2.5.2) follows by dividing both sides of (2.5.1) by $\|\sum\limits_{k\in I}\lambda_k e_k\|_\alpha$. Let $\sum\limits_{j=1}^{\infty}\lambda_{k_j}e_{k_j}$ be a subseries of a convergent series $\sum\limits_{k=1}^{\infty}\lambda_k e_k$. Given $\alpha\in\boldsymbol{N}$ and $\varepsilon>0$, there exists $N\in\boldsymbol{N}$ such that $\|\sum\limits_{k=m+1}^{n}\lambda_k e_k\|_\alpha<\varepsilon$ whenever $n>m\geqslant N$. Let $N'\in\boldsymbol{N}$ be so large that $k_j\geqslant N$ for $j\geqslant N'$, then for $j>i\geqslant N'$ we have $k_j>k_i\geqslant N$ and consequently

$$\|\sum_{r=i+1}^{j}\lambda_{k_r}e_{k_r}\|_\alpha \leqslant \|\sum_{k=k_i+1}^{k_j}\lambda_k e_k\|_\alpha<\varepsilon,$$

in which the first inequality follows from (2.5.2). Hence the sequence of partial sums of the subseries $\sum\limits_{j=1}^{\infty}\lambda_{k_j}e_{k_j}$ is Cauchy and, from the completeness of A, it is convergent. From Proposition 1.1 (a) it follows that the basis $\{e_n\}$ is unconditional.

Conversely, if the basis $\{e_n\}$ is unconditional then by Proposition 1.1 (b) we have an equivalent family $\{\|\cdot\|'_\alpha\colon \alpha\in\boldsymbol{N}\}$ of seminorms on A given by

$$\|x\|'_\alpha = \sup_{f\in A_\alpha}\sum_{n=1}^{\infty}|e_n^*(x)|\,|f(e_n)|,\qquad x\in A,\ \alpha\in\boldsymbol{N}$$

(see Proposition 1.1 for A_α). We show that each of the seminorms $\|\cdot\|'_\alpha$ is squarely submultiplicative. For each fixed $\alpha\in\boldsymbol{N}$ and $f\in A_\alpha$ we can define a measure μ_f on the power set of $\boldsymbol{N}$ by $\mu_f(T)=\sum\limits_{n\in T}|f(e_n)|$, and each $x\in A$ can be thought of as a function $x\colon \boldsymbol{N}\to\boldsymbol{C}$ given by $x(n)=e_n^*(x)$, $n\in\boldsymbol{N}$. Since

$\sum_{n=1}^{\infty} |e_n^*(x)|^2 |f(e_n)| = \sum_{n=1}^{\infty} |e_n^*(x^2)| |f(e_n)| \leqslant \|x^2\|'_\alpha < \infty$ for all $x \in A$, it follows from the Hölder's inequality that for $x, y \in A$,

$$\begin{aligned}\sum_{n=1}^{\infty} |e_n^*(xy)| |f(e_n)| &= \sum_{n=1}^{\infty} |e_n^*(x)| |e_n^*(y)| |f(e_n)| \\ &\leqslant \Big(\sum_{n=1}^{\infty} |e_n^*(x)|^2 |f(e_n)|\Big)^{1/2} \Big(\sum_{n=1}^{\infty} |e_n^*(y)|^2 |f(e_n)|\Big)^{1/2} \\ &= \Big(\sum_{n=1}^{\infty} |e_n^*(x^2)| |f(e_n)|\Big)^{1/2} \Big(\sum_{n=1}^{\infty} |e_n^*(y^2)| |f(e_n)|\Big)^{1/2}.\end{aligned}$$

Whence, we have

$$\begin{aligned}\|xy\|'_\alpha &= \sup_{f \in A_\alpha} \sum_{n=1}^{\infty} |e_n^*(xy)| |f(e_n)| \\ &\leqslant \Big(\sup_{f \in A_\alpha} \sum_{n=1}^{\infty} |e_n^*(x^2)| |f(e_n)|\Big)^{1/2} \Big(\sup_{f \in A_\alpha} \sum_{n=1}^{\infty} |e_n^*(y^2)| |f(e_n)|\Big)^{1/2} \\ &= \|x^2\|'^{1/2}_\alpha \|y^2\|'^{1/2}_\alpha,\end{aligned}$$

thus proving that A is a $B_0 s$-algebra.

2.6. EXAMPLES. 1. Since the canonical orthogonal basis in each of the Banach algebras l_p, $1 \leqslant p < \infty$ is unconditional, these are B_0 s-algebras by Theorem 2.5. To verify this fact directly, we see that for $x = \{x(n)\}$, $y = \{y(n)\} \in l_p$ we have:

$$\begin{aligned}\|xy\|_p &= \Big(\sum_{n=1}^{\infty} |x(n)y(n)|^p\Big)^{1/p} \leqslant \Big[\Big(\sum_{n=1}^{\infty} |x(n)|^{2p}\Big)^{1/2} \Big(\sum_{n=1}^{\infty} |y(n)|^{2p}\Big)^{1/2}\Big]^{1/p} \\ &= \Big[\Big(\sum_{n=1}^{\infty} |x^2(n)|^p\Big)^{1/p} \Big(\sum_{n=1}^{\infty} |y^2(n)|^p\Big)^{1/p}\Big]^{1/2} = \|x^2\|_p^{1/2} \|y^2\|_p^{1/2},\end{aligned}$$

and hence l_p is a Banach s-algebra. The same is true for the Banach algebra c_0 because

$$\begin{aligned}\|xy\|_\infty &= \sup_n |x(n)y(n)| \leqslant \sup_n |x(n)| \cdot \sup_n |y(n)| \\ &= \big(\sup_n |x(n)|^2\big)^{1/2} \big(\sup_n |y(n)|^2\big)^{1/2} = \|x^2\|_\infty^{1/2} \|y^2\|_\infty^{1/2}.\end{aligned}$$

2. The Banach algebras $L_p(T)$, $1 \leqslant p \leqslant \infty$, $p \neq 2$ (Example 2.1.1) are not B_0 s-algebras. For the case $p \neq 1$, this follows from Theorem 2.5 since $L_p(T)$, $1 < p < \infty$, $p \neq 2$ has an orthogonal basis which is not unconditional. For the case $p = 1$, we notice that the Banach algebra $L_1(T)$ has a bounded approximate identity [8]. Let $\{x_\alpha\}$ be an approximate identity with

$\|x_\alpha\| \leqslant K$ for all α and some $K > 0$. If $\|\cdot\|'$ were an equivalent squarely submultiplicative seminorm on $L_1(T)$ with $M_1\|x\| \leqslant \|x\|' \leqslant M_2\|x\|$ for some $M_1, M_2 > 0$ and for all $x \in L_1(T)$, then

$$\begin{aligned} \|x\| &\leqslant \frac{1}{M_1}\|x\|' = \frac{1}{M_1}\|\lim_\alpha xx_\alpha\|' = \frac{1}{M_1}\lim_\alpha\|xx_\alpha\|' \\ &\leqslant \frac{1}{M_1}\sup_\alpha\|xx_\alpha\|' \leqslant \frac{1}{M_1}\sup_\alpha\|x^2\|'^{1/2}\|x_\alpha^2\|'^{1/2} \leqslant \frac{(M_2\|x^2\|)^{1/2}}{M_1}\sup_\alpha(M_2\|x_\alpha^2\|)^{1/2} \\ &= \frac{M_2}{M_1}\|x^2\|^{1/2}\sup_\alpha\|x_\alpha^2\|^{1/2} \leqslant \frac{M_2}{M_1}\|x^2\|^{1/2}\sup_\alpha\|x_\alpha\| \leqslant \frac{KM_2}{M_1}\|x^2\|^{1/2}, \end{aligned}$$

where the second inequality from the right follows from the submultiplicativity of $\|\cdot\|$. It would follow that $\|x\|^2 \leqslant (KM_2/M_1)^2\|x^2\|$, thus leading to the false conclusion that the Fourier transform $L_1(T) \to \hat{L}_1(T) \subset C_0(\boldsymbol{Z})$ is a topological isomorphism [8].

3. Let B be the locally convex algebra of all bounded complex sequences $x = \{x(n)\}$ with coordinatewise operations, whose topology is generated by the family of seminorms $p_\varphi(x) = \sup_n|\varphi(n)x(n)|$, where φ ranges over the set φ of all sequences $\{\varphi(n)\}$ with $\varphi(n) \geqslant 0$ for all n and $\lim_n\varphi(n) = 0$. This topology is known as the "strict topology" and was introduced by R. C. Buck [2] in a more general setting. B is complete [2] and, clearly, it has an orthogonal basis (the coordinate unit vector basis) and an identity. Hence, if B were locally m-convex, it would follow that B, as a set, coincides with the set of all complex sequences [6], a contradiction. Hence B is not locally m-convex. However, B is a complete locally convex s-algebra. Indeed, for $x, y \in A$ and $\varphi \in \Phi$ we have

$$\begin{aligned} p_\varphi(xy) &= \sup_n|\varphi(n)x(n)y(n)| = \big(\sup_n|\varphi^2(n)x^2(n)y^2(n)|\big)^{1/2} \\ &= \big(\sup_n|\varphi(n)x^2(n)\cdot\varphi(n)y^2(n)|\big)^{1/2} \leqslant \big(\sup_n|\varphi(n)x^2(n)|\big)^{1/2}\big(\sup_n|\varphi(n)y^2(n)|\big)^{1/2} \\ &= p_\varphi^{1/2}(x^2)\,p_\varphi^{1/2}(y^2). \end{aligned}$$

4. The set $C(N)$ endowed with the pointwise algebraic operations and the product topology is a Fréchet algebra which is usually denoted by s. It is a Fréchet s-algebra since its topology is generated by the family of submultiplicative, squarely submultiplicative seminorms $\{p_J\colon J \subset N,\ J \text{ is finite}\}$, where $p_J(x) = \max_{n\in J}|x(n)|$, $x \in s$.

5. The Arens Algebra L^w is defined as $L^w = \bigcap_{1\leqslant p<\infty} L_p[0, 1]$ with pointwise algebraic operations, topologized by the family $\{\|\cdot\|_p\colon 1 \leqslant p < \infty\}$ of

seminorms where $\|x\|_p = (\int_0^1 |x(t)|^p dt)^{1/p}$, $x \in L^w$, $1 \leqslant p < \infty$ [1]. L^w is a B_0-algebra. If $x, y \in L^w$ and $1 \leqslant p < \infty$, then $x, y \in L_{2p}[0, 1]$ and so x^{2p}, $y^{2p} \in L_1[0, 1]$ but then $x^p, y^p \in L_2[0, 1]$. It follows from the Hölder's inequality that

$$\begin{aligned}\|xy\|_p^p &= \int_0^1 |x(t)|^p |y(t)|^p dt \\ &\leqslant \Big(\int_0^1 |x(t)|^{2p} dt\Big)^{1/2} \Big(\int_0^1 |y(t)|^{2p} dt\Big)^{1/2} = [\|x^2\|_p^p \, \|y^2\|_p^p]^{1/2}\end{aligned}$$

and so $\|xy\|_p \leqslant \|x^2\|_p^{1/2} \|y^2\|_p^{1/p}$. Hence L^w is a B_0 s-algebra. However, L^w is not locally m-convex and hence is not a Fréchet algebra [1].

6. The Banach algebra w_0 (Example 2.1 (2)) is not a B_0 s-algebra by Theorem 2.5 since it has an orthogonal basis which is not unconditional.

3. Locally convex s-algebras

In this section, we briefly discuss some consequences of the definition of a locally convex s-algebra. Further details of such algebras will be studied elsewhere. We assume in this section that all algebras are commutative.

3.1. PROPOSITION. *Let* $\|\cdot\|$ *be a squarely submultiplicative seminorm on an algebra* A. *For every* $n \in N$ *and* $x \in A$, *set* $\|x\|_n = \|(x)^{2^n}\|^{1/2^n}$. *Then:*

(i) *Each* $\|\cdot\|_n$ *is a squarely submultiplicative seminorm on* A.

(ii) *If, in addition,* $\|\cdot\|$ *is submultiplicative, so is every* $\|\cdot\|_n$ *and* $\|x\| \geqslant \|x\|_1 \geqslant \|x\|_2 \geqslant \ldots$ *for all* $x \in A$.

Proof. (i) Clearly $\|0\|_n = 0$ and $\|\alpha x\|_n = |\alpha| \, \|x\|_n$ for all $n \in N$, $\alpha \in N$ and $x \in A$. It remains to show that each $\|\cdot\|_n$ is squarely submultiplicative and satisfies the triangular inequality. For this, we use an induction argument. Assume that for some $n \in N$, $\|\cdot\|_n$ satisfies the triangular inequality and is squarely submultiplicative, then

$$\begin{aligned}\|x+y\|_{n+1} &= \|(x+y)^{2^{n+1}}\|^{1/2^{n+1}} = (\|[(x+y)^2]^{2^n}\|^{1/2^n})^{1/2} = \|(x+y)^2\|_n^{1/2} \\ &= \|x^2 + 2xy + y^2\|_n^{1/2} \leqslant (\|x^2\|_n + 2\,\|xy\|_n + \|y^2\|_n)^{1/2} \\ &\leqslant (\|x^2\|_n + 2\,\|x^2\|_n^{1/2} \, \|y^2\|_n^{1/2} + \|y^2\|_n)^{1/2} = \|x^2\|_n^{1/2} + \|y^2\|_n^{1/2}.\end{aligned}$$

Since

$$\|x\|_{n+1} = \|(x)^{2^{n+1}}\|^{1/2^{n+1}} = (\|(x^2)^{2^n}\|^{1/2^n})^{1/2} = \|x^2\|_n^{1/2}, \tag{3.1.1}$$

it follows from the last inequality that $\|x+y\|_{n+1} \leqslant \|x\|_{n+1} + \|y\|_{n+1}$, which is the triangular inequality for $\|\cdot\|_{n+1}$. From (3.1.1) and the square submultiplicativity of $\|\cdot\|_n$ we have $\|xy\|_{n+1} = \|x^2 y^2\|_n^{1/2} \leqslant (\|(x^2)^2\|_n^{1/2} \, \|(y^2)^2\|_n^{1/2})^{1/2}$

$=(\|x^2\|_{n+1}\|y^2\|_{n+1})^{1/2}$ and so $\|\cdot\|_{n+1}$ is squarely submultiplicative. This completes the proof of part (i) since, by hypothesis, the original norm $\|\cdot\|$ $(=\|\cdot\|_0)$ is squarely submultiplicative.

(ii) If, in addition, $\|\cdot\|$ is submultiplicative, then for $n \in N$, $x, y \in A$ we have

$$\|xy\|_n = \|(xy)^{2^n}\|^{1/2^n} = \|(x)^{2^n}(y)^{2^n}\|^{1/2^n} \leqslant (\|(x)^{2^n}\| \cdot \|(y)^{2^n}\|)^{1/2^n} = \|x\|_n \|y\|_n$$

and so each $\|\cdot\|_n$ is submultiplicative. It then follows from (3.1.1) that $\|x\|_{n+1} = \|x^2\|_n^{1/2} \leqslant (\|x\|_n\|x\|_n)^{1/2} = \|x\|_n$ for all $x \in A$, $n \in N$.

3.2. Corollary. *Let U be a squarely idempotent, convex, circled and absorbing subset of an algebra A. For each $n \in N$ define the function f_n on A by $f_n\colon x \to (x)^{2^n}$, then $f_n^{-1}(U)$ is also squarely idempotent, convex, circled and absorbing.*

Proof. Let $\|\cdot\|$ be the gauge of U. From Proposition 2.3 we see that $\|\cdot\|$ is a squarely submultiplicative seminorm on A and hence so is every $\|\cdot\|_n$, by Proposition 3.1. Thus, for every $n \in N$ we have

$$\begin{aligned} f_n^{-1}(U) &= \{x \in A\colon (x)^{2^n} \in U\} = \{x \in A\colon \|(x)^{2^n}\| \leqslant 1\} \\ &= \{x \in A\colon \|(x)^{2^n}\|^{1/2^n} \leqslant 1\} = \{x \in A\colon \|x\|_n \leqslant 1\} \end{aligned}$$

is a squarely idempotent, convex, circled and absorbing subset of A.

3.3. Proposition. *Let $\mathscr{P}$ be a family of squarely submultiplicative seminorms on an algebra A. Let $\mathscr{P}_1$ be the family of squarely submultiplicative seminorms given by $\mathscr{P}_1 = \{p_1\colon p \in \mathscr{P}\}$, $p_1(x) = p^{1/2}(x^2)$. Then A is a topological algebra under $\mathscr{P}$ iff $\mathscr{P}$ generates a stronger topology on A than $\mathscr{P}_1$.*

Proof. If multiplication is jointly continuous under $\mathscr{P}$, then for every $p \in \mathscr{P}$ there exists $q \in \mathscr{P}$ such that $p(xy) \leqslant q(x)q(y)$ for all $x, y \in A$. In particular, $p(x^2) \leqslant (q(x))^2$ and so $p_1(x) = p^{1/2}(x^2) \leqslant q(x)$ for all $x \in A$.

Conversely, if $\mathscr{P}$ generates a stronger topology on A than $\mathscr{P}_1$, the joint continuity of multiplication under $\mathscr{P}$ follows from $p(xy) \leqslant p^{1/2}(x^2)p^{1/2}(y^2) = p_1(x)p_1(y)$, $x, y \in A$, $p \in \mathscr{P}$.

3.4. Corollary. *Let A be a locally convex s-algebra with $\mathscr{P}$ as a generating family of squarely submultiplicative seminorms. Then each of the families $\mathscr{P}_n = \{p_n\colon p \in \mathscr{P}\}$, $p_n(x) = (p[(x)^{2^n}])^{1/2^n}$, is equivalent to $\mathscr{P}$, provided that A has a bounded approximate identity.*

Proof. From (3.1.1) we have $p_{n+1}(x) = p_n^{1/2}(x^2)$ for all $p \in \mathscr{P}$ and $x \in A$ and so $\mathscr{P}_{n+1}$ is derived from $\mathscr{P}_n$ the same way as $\mathscr{P}_1$ is derived from $\mathscr{P}$. Hence, the proof for all $n \in N$ follows inductively once we have a proof for $n = 1$. From Proposition 3.3, each p_1, $p \in \mathscr{P}$ is a continuous squarely submultiplicative seminorm and hence if $\{x_\alpha\}$ is a bounded approximate identity in A, then for each $p_1 \in \mathscr{P}_1$ there exists $M_{p_1} > 0$ such that $p_1(x_\alpha) \leqslant M_{p_1}$ for all α. Thus for each $p \in \mathscr{P}$ and $x \in A$ we have

$$\begin{aligned} p(x) &= p(\lim_\alpha xx_\alpha) = \lim_\alpha p(xx_\alpha) \leqslant \sup_\alpha p(xx_\alpha) \\ &\leqslant \sup_\alpha p^{1/2}(x^2)\, p^{1/2}(x_\alpha^2) = p_1(x) \sup_\alpha p_1(x_\alpha) \\ &\leqslant M_{p_1} p_1(x). \end{aligned}$$

Hence $\mathscr{P}_1$ generates a stronger topology on A than $\mathscr{P}$ and so the two topologies are equivalent by Proposition 3.3

3.5. PROPOSITION. *Let p be a squarely submultiplicative seminorm on an algebra A. Then p is submultiplicative iff $p_1(x) = p^{1/2}(x^2) \leqslant p(x)$ for all $x \in A$.*

Proof. If $p_1(x) = p^{1/2}(x^2) \leqslant p(x)$ for all $x \in A$, then $p(xy) \leqslant p^{1/2}(x^2)\, p^{1/2}(y^2) = p_1(x)\, p_1(y) \leqslant p(x)\, p(y)$ for all $x, y \in A$. Conversely, if $p(xy) \leqslant p(x)\, p(y)$ for all $x, y \in A$ then, in particular, $p(x^2) \leqslant p^2(x)$ and so $p_1(x) = p^{1/2}(x^2) \leqslant p(x)$ for all $x \in A$.

3.6. COROLLARY. *Let A be a semisimple Banach s-algebra. Consider the following:*

(i) *A has a bounded approximate identity.*
(ii) *$(A, \|\cdot\|_n)$ is complete for some $n \in N$.*
(iii) *$\|\cdot\|$ and $\|\cdot\|_1$ are equivalent.*
(iv) *All the norms $\|\cdot\|$ and $\|\cdot\|_n$, $n \in N$ are equivalent.*
(v) *The Gelfand map φ: $A \to \hat{A}$ is a topological isomorphism.*
Then (i) $\Rightarrow$ (ii) $\Leftrightarrow$ (iii) $\Leftrightarrow$ (iv) $\Leftrightarrow$ (v).

Proof. By Proposition 3.1, $(A, \|\cdot\|_n)$ is a normed s-algebra for each $n \in N$. (ii) $\Rightarrow$ (iii) follows from the inequality $\|x\| \geqslant \|x\|_1 \geqslant \|x\|_n$ for all $x \in A$ (Proposition 3.1 (ii)) and the Banach Open Mapping theorem. (iii) $\Rightarrow$ (iv) follows inductively from (3.1.1) as in the proof of Corollary 3.4, and (iv) $\Rightarrow$ (ii) is trivial. The equivalence of (iii) and (v) follows from the fact that φ: $A \to \hat{A}$ is a topological isomorphism iff there exists $M > 0$ such that $\|x\|^2 \leqslant M\|x^2\|$ for all $x \in A$ [8]. Finally, the implication (i) $\Rightarrow$ (iv) follows from Corollary 3.4.

3.7. COROLLARY. (a) *If in addition to the hypothesis of Corollary* 3.6, *A is self-adjoint, then from the Gelfand–Naimark theorem* [8], (v) *can be replaced by:*

(vi) *A is topologically isomorphic with $c_0(\Delta)$, where Δ is the maximal ideal space of A.*

(b) *If in addition to the hypothesis of Corollary* 3.6, *A has an orthogonal basis $\{e_n\}$, then* (v) *can be replaced by:*

(vii) *A is topologically isomorphic with c_0.*

Indeed, in this case Δ is homeomorphic with N in the discrete topology [6] and in addition to being closed in $c_0(\Delta) = c_0$ [8], A is dense in c_0 since it contains all elements of c_0 with finitely many non-zero coordinates. Moreover, in this case, all the statements in Corollary 3.6 (with (v) replaced by (vii)) are equivalent since c_0 has a bounded approximate identity $\{\sum_{k\in J} e_k\}_{J\in\Gamma}$, where Γ is the set of all finite subsets of N directed by inclusion. This characterizes c_0 (up to a topological isomorphism) as the semisimple Banach s-algebra with an orthogonal basis and an approximate identity. (Notice that w_0 has all these properties except the square submultiplicativity for the norm.)

It would be interesting to know when the topology of a metrizable locally convex or B_0-algebra can be described by a sequence of squarely submultiplicative seminorms, as is the case for the Arens algebra L^w.

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