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## Generalization of representation theorem of Erdős and Surányi

In the paper it is stated and proved the following

**THEOREM.** *Let  $f(x)$  be a polynomial with rational coefficients such that for any  $n \in \mathbf{Z}$ ,  $f(n)$  is an integer. If the greatest common factor of the terms of the sequence  $(f(n))_{n \geq 1}$  is equal to 1, then any integer  $k$  can be represented in infinitely many ways in the form*

$$k = \pm f(1) \pm f(2) \pm \dots \pm f(m)$$

for certain natural numbers  $m$  and certain choices of the signs  $+$  and  $-$ .

This result generalizes the theorem of Erdős und Surányi, which is obtained by taking  $f(x) = x^2$  (see [2], Problem 250). In [1] Jerzy Mitek gave a first generalization of this theorem, corresponding to  $f(x) = x^2$  ( $s \geq 1$ ).

Let  $s = \text{degree of } f(x)$ . For  $s = 0$ , the only polynomials verifying theorem's hypotheses are  $f(x) = 1$  and  $f(x) = -1$ , for which theorem's conclusion obviously holds.

Let now  $s \geq 1$ . By use of Newton's interpolation on the nodes  $1, 2, \dots, s$ ,  $f(x)$  can be written in the form

$$f(x) = a_0 + \sum_{i=1}^s a_i (x-1)(x-2) \dots (x-i),$$

where  $a_0, a_1, \dots, a_s \in \mathbf{Q}$ . A couple of lemmas are necessary to prove our result.

**LEMMA 1.** *The greatest common divisor of the terms of the sequence  $(f(n))_{n \geq 1}$  is equal to  $d = (0!a_0, 1!a_1, \dots, s!a_s)$ , where  $(q_1, q_2, \dots, q_k)$  denotes the greatest common divisor of the numbers  $q_1, q_2, \dots, q_k$ .*

**Proof.** We are going to show by induction on  $s$  that  $(f(n+1), f(n+2), \dots, f(n+s+1)) = d$  for any  $n \in \mathbf{Z}$ , what will be sufficient to derive the result stated. For  $s = 1$ ,  $(f(n+1), f(n+2)) = (f(n+1), f(n+2) - f(n+1)) = (a_0 + na_1, a_1) = (a_0, a_1)$ . Let us now consider  $s > 1$ .  $d' = (f(n+1), f(n+2), \dots, f(n+s+1)) = (f(n+1), g(n+1), g(n+2), \dots, g(n+s))$ , where

$g(x) = f(x+1) - f(x) = a_1 + \sum_{i=1}^{s-1} (i+1)a_{i+1}(x-1)(x-2)\dots(x-i)$ . Since degree of  $g(x) = s-1$ , according to the inductive hypothesis  $(g(n+1), g(n+2), \dots, g(n+s)) = (0!a_1, 1!2a_2, \dots, (s-1)!sa_s)$ . Then  $d' = (f(n+1), 1!a_1, 2!a_2, \dots, s!a_s) = d$  because  $f(n+1) - \sum_{i=1}^s \binom{n}{i} i!a_i = a_0$ .

LEMMA 2.  $f(jms! + n) \equiv f(n) \pmod{m} \ (\forall) j, n \in \mathbf{Z}$ .

Proof. It is well known that  $a_k k! \in \mathbf{Z}$  and then  $a_k s! \in \mathbf{Z}$  for any  $k \in \{0, 1, \dots, s\}$ . Since

$$(jms! + n - 1) \dots (jms! + n - i) \equiv (n-1) \dots (n-i) \pmod{ms!},$$

we deduce that

$$a_i (jms! + n - 1) \dots (jms! + n - i) \equiv a_i (n-1) \dots (n-i) \pmod{m}$$

and therefore

$$f(jms! + n) \equiv f(n) \pmod{m}.$$

Proof of the theorem proceeds by induction on  $s$ .

We have already mentioned that the theorem for  $s = 0$ ; hence let us take  $s \geq 1$ . We write  $g(x) = f(2x-1) - f(2x)$  and let  $d$  be the greatest common divisor of the terms of the sequence  $(g(n))_{n \geq 1}$ . Since degree  $g(x) = s-1$  if  $d = 1$ , it follows that according to the inductive hypothesis – for any  $k \in \mathbf{Z}$  there exist infinitely many representations of the form

$$\begin{aligned} k &= \pm g(1) \pm g(2) \pm \dots \pm g(m) \\ &= \pm (f(1) - f(2)) \pm (f(3) - f(4)) \pm \dots \pm (f(2m-1) - f(2m)). \end{aligned}$$

Let us consider  $d > 1$  and  $k$  an arbitrary even integer. Since  $(f(1), f(2), \dots, f(s+1)) = 1$ , there exist  $u_1, u_2, \dots, u_{s+1} \in \mathbf{Z}$  such that

$\sum_{i=1}^{s+1} u_i f(i) = k/2$ . For  $u_i \neq 0$ , we write  $v_i = |u_i|$  and  $w_i = |u_i|/u_i$ . It follows from

Lemma 2 that, for  $u_i \neq 0$ ,  $\sum_{j=1}^{v_i} f(jds! + i) = v_i f(i) \pmod{d}$  and then

$$N = \sum_{\substack{i=1 \\ u_i \neq 0}}^{s+1} w_i \sum_{j=1}^{v_i} f(jds! + i) \equiv \sum_{i=1}^{s+1} u_i f(i) = \frac{1}{2} k \pmod{d}.$$

Let  $p \in \mathbf{N}$ ,  $2p \equiv 0 \pmod{d}$ ,  $2pds! \geq v_i ds! + i$  for any  $i \in \{1, 2, \dots, s+1\}$  and  $w: \mathbf{Z} \rightarrow \{-1, 1\}$ ,  $w(i) = w_i$  for  $i \in \{1, 2, \dots, s+1\}$ ,  $u_i \neq 0$  and the function  $w$  is periodical of period  $ds!$ . The sum  $M = \sum_{i=1}^{2pds!} w(i) f(i)$  contains all the

terms of  $N$  and

$$\begin{aligned} M &= \sum_{j=0}^{2p-1} \sum_{i=1}^{ds!} w(jds! + i) f(jds! + i) \equiv \sum_{j=0}^{2p-1} \sum_{i=1}^{ds!} w(i) f(i) \\ &= 2p \sum_{i=1}^{ds!} w(i) f(i) = 0 \pmod{d}. \end{aligned}$$

Therefore  $2N - M \equiv k \pmod{d}$  and  $2N - M$  is of the form  $2N - M = \sum_{i=1}^{2r} \pm f(i)$ , where  $r = pds!$

The degree of the polynomial  $g(r+x)/d$  is equal to  $s-1$  and the greatest common divisor of the terms of the associated sequence equals to 1. According to the inductive hypothesis,  $(k+M-2N)/d$  can be expressed in infinitely many ways in the form

$$(k+M-2N)/d = \pm g(r+1)/d \pm g(r+2)/d \pm \dots \pm g(r+m)/d$$

or

$$\begin{aligned} k &= 2N - M \pm g(r+1) \pm g(r+2) \pm \dots \pm g(r+m) \\ &= \sum_{i=1}^{2r} [\pm f(i)] + \sum_{i=1}^m [\pm (f(2r+2i-1) - f(2r+2i))] \\ &= \sum_{i=1}^{2r+2m} [\pm f(i)]. \end{aligned}$$

Let now  $k$  be an odd integer and let  $f(t)$  be the first odd term of the sequence  $(f(n))_{n \geq 1}$ . Then  $k - f(1) - f(2) - \dots - f(t)$  is even and, since the polynomial  $f(t+x)$  verifies the conditions of the theorem, it follows from what we have already proved that there exist infinitely many representations of the form

$$k - f(1) - f(2) - \dots - f(t) = \pm f(t+1) \pm f(t+2) \pm \dots \pm f(t+m)$$

which implies (1), and hence the proof is complete.

#### References

- [1] J. Mitek, *Generalization of a theorem of Erdős and Surányi*, Ann. Soc. Math. Polon. Ser. I: Comment. Math. 21 (1979), 173-175.
- [2] W. Sierpiński, *250 Problems in Elementary Number Theory*, American Elsevier Publ. Co., New York 1970.