Generalization of representation theorem of Erdős and Surányi

In the paper it is stated and proved the following

**Theorem.** Let \( f(x) \) be a polynomial with rational coefficients such that for any \( n \in \mathbb{Z} \), \( f(n) \) is an integer. If the greatest common factor of the terms of the sequence \( (f(n))_{n \geq 1} \) is equal to 1, then any integer \( k \) can be represented in infinitely many ways in the form

\[
k = \pm f(1) \pm f(2) \pm \ldots \pm f(m)
\]

for certain natural numbers \( m \) and certain choices of the signs \( + \) and \( - \).

This result generalizes the theorem of Erdős und Surányi, which is obtained by taking \( f(x) = x^2 \) (see [2], Problem 250). In [1] Jerzy Mitek gave a first generalization of this theorem, corresponding to \( f(x) = x^2 \) (\( s = 1 \)).

Let \( s = \text{degree of } f(x) \). For \( s = 0 \), the only polynomials verifying theorem's hypotheses are \( f(x) = 1 \) and \( f(x) = -1 \), for which theorem's conclusion obviously holds.

Let now \( s \geq 1 \). By use of Newton's interpolation on the nodes 1, 2, ..., \( s \), \( f(x) \) can be written in the form

\[
f(x) = a_0 + \sum_{i=1}^{s} a_i (x - 1)(x - 2) \ldots (x - i),
\]

where \( a_0, a_1, \ldots, a_s \in \mathbb{Q} \). A couple of lemmas are necessary to prove our result.

**Lemma 1.** The greatest common divisor of the terms of the sequence \( (f(n))_{n \geq 1} \) is equal to \( d = (0, a_0, 1, a_1, \ldots, s, a_s) \), where \((q_1, q_2, \ldots, q_k)\) denotes the greatest common divisor of the numbers \( q_1, q_2, \ldots, q_k \).

**Proof.** We are going to show by induction on \( s \) that \( (f(n+1), f(n+2), \ldots, f(n+s+1)) = d \) for any \( n \in \mathbb{Z} \), what will be sufficient to derive the result stated. For \( s = 1 \), \( (f(n+1), f(n+2)) = (f(n+1), f(n+2) - f(n+1)) = (a_0 + na_1, a_1) = (a_0, a_1) \). Let us now consider \( s > 1 \). \( d' = (f(n+1), f(n+2), \ldots, f(n+s+1)) = (f(n+1), g(n+1), g(n+2), \ldots, g(n+s)) \), where
\( g(x) = f(x+1) - f(x) = a_1 + \sum_{i=1}^{s-1} (i+1) a_{i+1} (x-1)(x-2) \ldots (x-i) \). Since degree of \( g(x) = s-1 \), according to the inductive hypothesis \((g(n+1), g(n+2), \ldots, g(n+s)) = (0!a_1, 1!a_2, \ldots, (s-1)!a_s)\). Then \( d' = (f(n+1), 1!a_1, 2!a_2, \ldots, s!a_s) = d \) because \( f(n+1) - \sum_{i=1}^{s} \binom{n}{i} i! a_i = a_0 \).

**Lemma 2.** \( f(jms!+n) \equiv f(n)(\text{mod } m) \) \((\forall) j, n \in \mathbb{Z} \).

**Proof.** It is well known that \( a_k k! \in \mathbb{Z} \) and then \( a_k s! \in \mathbb{Z} \) for any \( k \in \{0, 1, \ldots, s\} \). Since

\[
(jms!+n-1) \ldots (jms!+n-i) \equiv (n-1) \ldots (n-i)(\text{mod } ms!),
\]

we deduce that

\[
a_i (jms!+n-1) \ldots (jms!+n-i) \equiv a_i (n-1) \ldots (n-i)(\text{mod } m)
\]

and therefore

\[
f(jms!+n) \equiv f(n)(\text{mod } m).
\]

**Proof of the theorem.** proceeds by induction on \( s \).

We have already mentioned that the theorem for \( s = 0 \); hence let us take \( s \geq 1 \). We write \( g(x) = f(2x-1) - f(2x) \) and let \( d \) be the greatest common divisor of the terms of the sequence \((g(n))_{n \geq 1}\). Since degree \( g(x) = s-1 \) if \( d = 1 \), it follows that according to the inductive hypothesis — for any \( k \in \mathbb{Z} \) there exist infinitely many representations of the form

\[
k = \pm g(1) \pm g(2) \pm \ldots \pm g(m)
\]

\[
= \pm (f(1) - f(2)) \pm (f(3) - f(4)) \pm \ldots \pm (f(2m-1) - f(2m)).
\]

Let us consider \( d > 1 \) and \( k \) an arbitrary even integer. Since \((f(1), f(2), \ldots, f(s+1)) = 1\), there exist \( u_1, u_2, \ldots, u_{s+1} \in \mathbb{Z} \) such that

\[
\sum_{i=1}^{s+1} u_i f(i) = k/2.
\]

For \( u_i \neq 0 \), we write \( v_i = |u_i| \) and \( w_i = |u_i|/u_i \). It follows from Lemma 2 that, for \( u_i \neq 0 \),

\[
\sum_{j=1}^{v_i} f(jds!+i) = v_i f(i)(\text{mod } d)
\]

and then

\[
N = \sum_{i=1}^{s+1} w_i \sum_{j=1}^{v_i} f(jds!+i) \equiv \sum_{i=1}^{s+1} u_i f(i) = \frac{1}{2} k (\text{mod } d).
\]

Let \( p \in \mathbb{N}, 2p = 0 \pmod{d}, 2pds! \geq v_i ds! + i \) for any \( i \in \{1, 2, \ldots, s+1\} \), and \( w: \mathbb{Z} \rightarrow \{-1, 1\}, w(i) = w_i \) for \( i \in \{1, 2, \ldots, s+1\}, u_i \neq 0 \) and the function \( w \) is periodical of period \( ds! \). The sum \( M = \sum_{i=1}^{2pds!} w(i) f(i) \) contains all the
terms of $N$ and
\[
M = \sum_{j=0}^{2p-1} \sum_{i=1}^{d!} w(jd! + i) f(jd! + i) \equiv \sum_{j=0}^{2p-1} \sum_{i=1}^{d!} w(i) f(i)
\]
\[
= 2p \sum_{i=1}^{d!} w(i) f(i) = 0(\text{mod } d).
\]
Therefore $2N - M \equiv k (\text{mod } d)$ and $2N - M$ is of the form $2N - M = \sum_{i=1}^{2r} \pm f(i)$, where $r = pds!$

The degree of the polynomial $g(r + x)/d$ is equal to $s - 1$ and the greatest common divisor of the terms of the associated sequence equals to 1. According to the inductive hypothesis, $(k + M - 2N)/d$ can be expressed in infinitely many ways in the form
\[
(k + M - 2N)/d = \pm g(r + 1)/d \pm g(r + 2)/d \pm \ldots \pm g(r + m)/d
\]
or
\[
k = 2N - M \pm g(r + 1) \pm g(r + 2) \pm \ldots \pm g(r + m)
\]
\[
= \sum_{i=1}^{2r} \left[ \pm f(i) \right] + \sum_{i=1}^{m} \left[ \pm \left( f(2r + 2i - 1) - f(2r + 2i) \right) \right]
\]
\[
= \sum_{i=1}^{2r + 2m} \left[ \pm f(i) \right].
\]

Let now $k$ be an odd integer and let $f(t)$ be the first odd term of the sequence $(f(n))_{n \geq 1}$. Then $k - f(1) - f(2) - \ldots - f(t)$ is even and, since the polynomial $f(t + x)$ verifies the conditions of the theorem, it follows from what we have already proved that there exist infinitely many representations of the form
\[
k - f(1) - f(2) - \ldots - f(t) = \pm f(t + 1) \pm f(t + 2) \pm \ldots \pm f(t + m)
\]
which implies (1), and hence the proof is complete.

References


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