ANNALES SOCIETATIS MATHEMATICAE POLONAE
Series I: COMMENTATIONES MATHEMATICAE XXVII (1987)
ROCZNIKI POLSKIEGO TOWARZYSTWA MATEMATYCZNEGO
Seria I: PRACE MATEMATYCZNE XXVII (1987)

Zbigniew Olszak (Wrocław)

Curvature properties of four-dimensional Hermitian manifolds

Abstract. In the paper we study Hermitian manifolds of dimension 4. We obtain basic identities for the Riemann curvature tensors, the Ricci curvature tensors and the scalar curvatures of such manifolds. Then certain sufficient conditions for a 4-dimensional Hermitian manifold to be Kählerian are derived. Also two examples of Hermitian structures are given, one of them is not locally conformal Kählerian and the other is flat, globally conformal Kählerian and not Kählerian.

1. Preliminaries. Let M be an almost Hermitian manifold with almost complex structure J and Hermitian metric g. Let Ω indicate the fundamental form of M, given by $\Omega(X, Y) = g(X, JY)$ ($X, Y, \ldots$ are always vector fields on M, if it is not otherwise stated). Moreover, denote by N the torsion of J, i.e., the Nijenhuis tensor field defined by

$$N(X, Y) = [JX, JY]-[X, Y]-J[JX, Y]-J[X, JY]. \tag{1.1}$$

M is said to be *Hermitian* if the structure J arises from a complex structure on M. A well-known theorem states that M is Hermitian if and only if it is without torsion, i.e., $N = 0$. It is also shown ([4]) that M is Hermitian if and only if $\nabla_{JX}(J)JY = \nabla_X(J)Y$, where ∇ is the Riemannian connection of M.

When $\dim M = 4$, another characterisation for M to be Hermitian is found (see [3], Theorem 3.1). Namely, we have the following proposition.

Proposition 1.1. *A 4-dimensional almost Hermitian manifold M is Hermitian if and only if the covariant derivative of Ω is of the form*

$$\nabla_X(\Omega)(Y, Z) = -\tfrac{1}{2}\{g(X, Y)\delta\Omega(Z)-g(X, Z)\delta\Omega(Y) -g(X, JY)\delta\Omega(JZ)+g(X, JZ)\delta\Omega(JY)\}, \tag{1.2}$$

where $\delta\Omega$ denotes the codifferential of the form Ω.

Throughout the rest of the paper we shall assume that M is a 4-dimensional Hermitian manifold if it is not otherwise stated.

Let ω be the Lee form of M defined by $\omega = \delta\Omega\circ J$ and B the contravariant field of ω called the *Lee field*. Then the identity (1.2) can be expressed equivalently in the following form

$$(1.3)\qquad 2\nabla_X(J)Y = g(X, Y)JB - g(X, JY)B - \omega(Y)JX + \omega(JY)X.$$

Define Φ to be the tensor field on M given by

$$(1.4)\qquad \Phi(X, Y) = -2\omega(\nabla_X(J)Y).$$

As a consequence of (1.3) and (1.4), one obtains that Φ is a 2-form on M and

$$(1.5)\qquad \Phi = |B|^2\Omega - \omega \wedge \theta,$$

where $\theta = \omega \circ J$. Let $\operatorname{Ann}\Phi$ be the annihilator of the form Φ, i.e., it is the distribution $M \ni m \mapsto \operatorname{Ann}\Phi(m)$, where

$$\operatorname{Ann}\Phi(m) = \{X \in T_m M \mid \Phi(X, Y) = 0 \text{ for all } Y \in T_m M\}.$$

Consider also another distribution on M. Namely, let $K(M)$ be the Kähler-nullity distribution $M \ni m \mapsto K(m)$ (see [2]), where

$$K(m) = \{X \in T_m M \mid \nabla_X(J)Y = 0 \text{ for all } Y \in T_m M\}.$$

With the help of equalities (1.3)–(1.5) the following proposition can be proved. The proof goes just as the proof of Proposition 1.1 in [7].

PROPOSITION 1.2. *Let M be a 4-dimensional Hermitian manifold. Then* $K(M) = \operatorname{Ann}\Phi$ *and, moreover,*

(a) *the singular points of $K(M)$ are the vanishing points of ω and at such a point $K(m) = T_m M$,*

(b) *at every non-singular point m, $K(m)$ is 2-dimensional and is generated by B and $A = -JB$.*

2. Basic curvature identities. A general curvature identity for a Hermitian manifold is of the form

$$(2.1)\qquad R_{XYZW} + R_{JXJYJZJW} - R_{JXJYZW} - R_{JXYJZW} - R_{JXYZJW} - R_{XJYJZW} - R_{XJYZJW} - R_{XYJZJW} = 0,$$

where $R_{XYZW} = g(R_{XY}Z, W)$ and R_{XY} is the curvature transformation $[\nabla_X, \nabla_Y] - \nabla_{[X,Y]}$. Identity (2.1) was obtained by Nagao and Koto [5], Theorem 2.1, and Gray [2], Corollary 3.2. In our Corollary 2.2 we shall prove that a stronger equality holds good, if the manifold is of dimension four.

But firstly we introduce some notations.

And so, let L be the linear operator (i.e., the tensor field of type (1, 1)) on M defined by

$$(2.2)\qquad LX = \nabla_X B + \tfrac{1}{2}\omega(X)B.$$

Moreover, if X is a vector field on M, then denote by σ_X its covariant field (i.e., σ_X is the 1-form on M such that $\sigma_X(Y) = g(X, Y)$ for all Y). Note easily that $\nabla_X(\sigma_Y) = \sigma_{\nabla_X Y}$ and $\sigma_{fX} = f\sigma_X$, if f is a function on M.

Assume also the following convention: $[\nabla_X, T] = \nabla_X \circ T - T \circ \nabla_X = \nabla_X(T)$, if T is a linear operator on M.

THEOREM 2.1. *For a 4-dimensional Hermitian manifold M we have*

$$(2.3)\qquad 2[R_{XY}, J] = \tfrac{1}{2}|B|^2\{Y\otimes\sigma_{JX} - X\otimes\sigma_{JY} + (JY)\otimes\sigma_X - (JX)\otimes\sigma_Y\}$$
$$+(JX)\otimes\sigma_{L'Y} - (JY)\otimes\sigma_{L'X} - (JL'Y)\otimes\sigma_X + (JL'X)\otimes\sigma_Y$$
$$+X\otimes\sigma_{JL'Y} - Y\otimes\sigma_{JL'X} - (L'Y)\otimes\sigma_{JX} + (L'X)\otimes\sigma_{JY}.$$

Proof. At first, using the Jacobi identity, we find

$$(2.4)\qquad [R_{XY}, J] = [\nabla_X, [\nabla_Y, J]] - [\nabla_Y, [\nabla_X, J]] - [\nabla_{[X,Y]}, J].$$

From (1.3) it follows that

$$2[\nabla_X, J] = 2\nabla_X(J) = (JB)\otimes\sigma_X + B\otimes\sigma_{JX} - (JX)\otimes\sigma_B - X\otimes\sigma_{JB},$$

which used in (2.4) gives

$$(2.5)\qquad 2[R_{XY}, J] = [\nabla_X, (JB)\otimes\sigma_Y + B\otimes\sigma_{JY} - (JY)\otimes\sigma_B - Y\otimes\sigma_{JB}]$$
$$-[\nabla_Y, (JB)\otimes\sigma_X + B\otimes\sigma_{JX} - (JX)\otimes\sigma_B - X\otimes\sigma_{JB}]$$
$$-(JB)\otimes\sigma_{[X,Y]} - B\otimes\sigma_{J[X,Y]} + (J[X, Y])\otimes\sigma_B + [X, Y]\otimes\sigma_{JB}.$$

On the other hand, one has

$$(2.6)\qquad [\nabla_X, (JB)\otimes\sigma_Y] - [\nabla_Y, (JB)\otimes\sigma_X] - (JB)\otimes\sigma_{[X,Y]}$$
$$= \nabla_X((JB)\otimes\sigma_Y) - \nabla_Y((JB)\otimes\sigma_X) - (JB)\otimes\sigma_{[X,Y]}$$
$$= (\nabla_X(J)B + J\nabla_X B)\otimes\sigma_Y - (\nabla_Y(J)B + J\nabla_Y B)\otimes\sigma_X$$
$$= (JL'X + \tfrac{1}{2}\omega(JX)B - \tfrac{1}{2}|B|^2 JX)\otimes\sigma_Y$$
$$-(JL'Y + \tfrac{1}{2}\omega(JY)B - \tfrac{1}{2}|B|^2 JY)\otimes\sigma_X,$$

where we applied (1.3) and used the notation (2.2). Similarly, one can obtain

$$(2.7)\qquad [\nabla_X, B\otimes\sigma_{JY}] - [\nabla_Y, B\otimes\sigma_{JX}] - B\otimes\sigma_{J[X,Y]} = (L'X)\otimes\sigma_{JY}$$
$$-(L'Y)\otimes\sigma_{JX} - g(X, JY)B\otimes\sigma_B + \tfrac{1}{2}\omega(JY)B\otimes\sigma_X - \tfrac{1}{2}\omega(JX)B\otimes\sigma_Y,$$

$$(2.8)\qquad -[\nabla_X, (JY)\otimes\sigma_B] + [\nabla_Y, (JX)\otimes\sigma_B] + (J[X, Y])\otimes\sigma_B$$
$$= (JX)\otimes\sigma_{L'Y} - (JY)\otimes\sigma_{L'X} + g(X, JY)B\otimes\sigma_B - \tfrac{1}{2}\omega(JY)X\otimes\sigma_B$$
$$+\tfrac{1}{2}\omega(JX)Y\otimes\sigma_B,$$

$$(2.9)\qquad -[\nabla_X, Y\otimes\sigma_{JB}] + [\nabla_Y, X\otimes\sigma_{JB}] + [X, Y]\otimes\sigma_{JB}$$
$$= X\otimes\sigma_{JL'Y} - Y\otimes\sigma_{JL'X} - \tfrac{1}{2}\omega(JX)Y\otimes\sigma_B + \tfrac{1}{2}\omega(JY)X\otimes\sigma_B$$
$$+\tfrac{1}{2}|B|^2(Y\otimes\sigma_{JX} - X\otimes\sigma_{JY}).$$

Finally, (2.5) implies (2.3) in virtue of (2.6)–(2.9). The proof is complete.

COROLLARY 2.2. *The curvature tensor of a 4-dimensional Hermitian manifold* *satisfies the equality*

(2.10) $$R_{XYJZJW} - R_{XYZW} = \tfrac{1}{4}|B|^2\{g(X, JW)g(Y, JZ) - g(X, JZ)g(Y, JW) - g(X, W)g(Y, Z) + g(X, Z)g(Y, W)\} + \tfrac{1}{2}\{g(X, W)L(Y, Z) - g(X, Z)L(Y, W) + g(Y, Z)L(X, W) - g(Y, W)L(X, Z) - g(X, JW)L(Y, JZ) + g(X, JZ)L(Y, JW) - g(Y, JZ)L(X, JW) + g(Y, JW)L(X, JZ)\},$$

where we assumed (cf. (2.2))

(2.11) $$L(X, Y) = g(L'X, Y) = \nabla_X(\omega)Y + \tfrac{1}{2}\omega(X)\omega(Y).$$

Proof. We have in general

$$R_{XYJZJW} - R_{XYZW} = g([R_{XY}, J]Z, JW).$$

Thus, using Theorem 2.1, one can easily derive (2.10), completing the proof.

Remark 2.3. Tensor field L defined by (2.11) is not symmetric in general (cf. Example A in Section 5). As $L(X, Y) - L(Y, X) = d\omega(X, Y)$, we see that L is symmetric if and only if M is locally conformal Kählerian (cf. [6]).

Define the Ricci curvature tensor, the Ricci $*$-curvature tensor, the scalar curvature and the scalar $*$-curvature of M by

$$\varrho(X, Y) = \sum_s R_{E_s XYE_s}, \qquad \varrho^*(X, Y) = -\tfrac{1}{2}\sum_s R_{E_s JE_s XJY},$$

$$\tau = \sum_s \varrho(E_s, E_s), \qquad \tau^* = \sum_s \varrho^*(E_s, E_s),$$

respectively, where $\{E_1, \ldots, E_4\}$ is an orthonormal frame. Note that using the first Bianchi identity one can easily get

(2.12) $$\varrho^*(X, Y) = \sum_s R_{E_s XJYJE_s}.$$

COROLLARY 2.4. *The Ricci tensors of a* 4-*dimensional Hermitian manifold* M *fulfil the relation*

(2.13)
$$\varrho^*(Y, Z) - \varrho(Y, Z) = \tfrac{1}{2}\{L(Y, Z) - L(JY, JZ)\} + \tfrac{1}{2}(\operatorname{div} B - \tfrac{1}{2}|B|^2)g(Y, Z),$$

where $\operatorname{div} B = \sum_s g(\nabla_{E_s} B, E_s)$.

Proof. Put $X = W = E_s$ into (2.10) and sum. Then applying (2.12), we have

(2.14) $$\varrho^*(Y, Z)-\varrho(Y, Z) = \tfrac{1}{2}\{L(Y, Z)-L(JY, JZ)\} + \tfrac{1}{2}(\operatorname{trace} L-|B|^2)g(Y, Z)-\tfrac{1}{2}\sum_s L(E_s, JE_s)g(Y, JZ).$$

Again, putting $Y = E_s$, $Z = JE_s$ into (2.14) and summing, we obtain

(2.15) $$\sum_s L(E_s, JE_s) = 0.$$

Here we used the relations $\sum_s \varrho^*(E_s, JE_s) = \sum_s \varrho(E_s, JE_s) = 0$. Moreover, from (2.11) it follows that

(2.16) $$\operatorname{trace} L = \operatorname{div} B+\tfrac{1}{2}|B|^2.$$

By virtue of equations (2.15) and (2.16), identity (2.14) reduces to (2.13), completing the proof.

Our next statement is a consequence of Corollary 2.4.

COROLLARY 2.5. *For the scalar curvature and *-curvature of a 4-dimensional Hermitian manifold M we have*

(2.17) $$\tau^*-\tau = 2(\operatorname{div} B-\tfrac{1}{2}|B|^2).$$

3. Certain curvature properties.

PROPOSITION 3.1. *Let M be a 4-dimensional Hermitian manifold. If the trajectories of the vector fields B and $A = -JB$ are holomorphically planar curves* (especially: *geodesics*), *then on the open set on which $\omega \neq 0$, the Kähler-nullity distribution $K(M)$ is integrable.*

Proof. Since both B and A belong to $K(M)$, we have

$$[B, A] = \nabla_B A-\nabla_A B = -\nabla_B JB-\nabla_A JA = -J(\nabla_B B+\nabla_A A).$$

Assume now that the trajectories of B and A are holomorphically planar curves (for the notion of such a curve see [9], p. 258). Then

$$\nabla_A A = f_1 A+f_2 JA, \qquad \nabla_B B = h_1 B+h_2 JB$$

for certain functions f_1, f_2, h_1, h_2. These equalities used in the above give

$$[B, A] = (h_1+f_2)A+(h_2-f_1)B.$$

Now, because B and A generate the distribution $K(M)$ on the open set on which $\omega \neq 0$ (cf. Proposition 1.2), by the Frobenius theorem, the distribution is integrable. The proof is complete.

One of the most interesting curvature equalities occurring in certain classes of almost Hermitian manifolds is (cf. [2])

(3.1) $$R_{XYJZJW} = R_{XYZW}.$$

Any Kählerian manifold fulfils (3.1), but not every almost Hermitian manifold (even if it is Hermitian) satisfying (3.1) is Kählerian (cf. ibidem). Concerning this equality we prove here the following result.

Proposition 3.2. *Let M be a 4-dimensional Hermitian manifold. Then, M satisfies* (3.1) *if and only if*

$$(3.2)\qquad \text{(a)}\quad L(JX, JY) = L(X, Y), \qquad \text{(b)}\quad \operatorname{trace} L = |B|^2.$$

Proof. Let (3.1) be satisfied for M. Consequently, we have $\varrho^* - \varrho = 0$ and $\tau^* - \tau = 0$. Therefore, one can deduce (3.2) (a), (b) from (2.13), (2.16) and (2.17).

Conversely, suppose we have (3.2) (a), (b). We want to prove (3.1). From (2.10) we see that it is sufficient to prove the following

$$(3.3)\qquad -g(X, W)L(Y, Z) + g(X, Z)L(Y, W) - g(Y, Z)L(X, W) + g(Y, W)L(X, Z)$$
$$+g(X, JW)L(Y, JZ) - g(X, JZ)L(Y, JW) + g(Y, JZ)L(X, JW) - g(Y, JW)L(X, JZ)$$
$$= \tfrac{1}{2}|B|^2\{g(X, JW)g(Y, JZ) - g(X, JZ)g(Y, JW) - g(X, W)g(Y, Z) + g(X, Z)g(Y, W)\}.$$

It will be convenient to use the linear operator L' defined by $g(X, L'Y) = L(X, Y)$. Note that (3.2) (a) is equivalent to the commutativity of operators J and L', i.e., $JL' = L'J$. We prove now that

$$(3.4)\qquad (JZ)\wedge(JL'W) + (JL'Z)\wedge(JW) - Z\wedge(L'W) - (L'Z)\wedge W = \tfrac{1}{2}|B|^2\{(JZ)\wedge(JW) - Z\wedge W\},$$

where $\wedge$ is the exterior product of the vector fields (i.e., $X\wedge Y = X\otimes Y - Y\otimes X$). Our first observation is as follows: in virtue of $JL' = L'J$, (3.4) is satisfied if $W = JZ$, and, moreover, if Z and W satisfy (3.4), then JZ and JW do the same. Take a local orthonormal frame $\{E_1, \ldots, E_4\}$ such that $E_3 = JE_1$ and $E_4 = JE_2$. It will be sufficient to prove (3.4) for two pairs E_1, E_2 and E_1, E_4. But this is a straighforward verification if we write down explicitly the operator L' with respect to the local orthonormal frame. By (3.2) (a), (b) and (2.15), the operator L' is given by

$$L'E_1 = aE_1 + bE_2 + cE_3 + dE_4, \quad L'E_2 = eE_1 + (\tfrac{1}{2}|B|^2 - a)E_2 + fE_3 - cE_4,$$

$$L'E_3 = -cE_1 - dE_2 + aE_3 + bE_4, \quad L'E_4 = -fE_1 + cE_2 + eE_3 + (\tfrac{1}{2}|B|^2 - a)E_4.$$

Define now the scalar product

$$\langle Y\wedge X, Z\wedge W\rangle = g(Y, Z)g(X, W) - g(Y, W)g(X, Z).$$

Finally, projecting (3.4) onto $Y \wedge X$ with respect to product $\langle \cdot, \cdot \rangle$ one obtains (3.3). This completes the proof.

Another interesting curvature identity considered in almost Hermitian manifolds is (cf. [2], [8])

$$R_{JXJYJZJW} = R_{XYZW}. \tag{3.5}$$

It is clear that (3.1) implies (3.5), and as it is known the converse statement fails in general.

PROPOSITION 3.3. *Let M be a 4-dimensional Hermitian manifold whose curvature tensor satisfies* (3.5). *Then*

$$L(JX, JY) + L(JY, JX) - L(X, Y) - L(Y, X) = 0. \tag{3.6}$$

Proof. For the Ricci tensor ϱ^* we always have $\varrho^*(JX, JY) = \varrho^*(Y, X)$. And for the Ricci tensor ϱ from (3.5) follows $\varrho(JX, JY) = \varrho(X, Y)$. Consequently, puting JZ, JY instead of Y, Z into (2.13), we find

$$\varrho^*(Y, Z) - \varrho(Z, Y) = \tfrac{1}{2}\{L(JZ, JY) - L(Z, Y)\} + \tfrac{1}{2}(\operatorname{div} B - \tfrac{1}{2}|B|^2)g(Z, Y),$$

which together with (2.13) gives (3.6). The proof is complete.

4. Certain sufficient conditions for M to be Kählerian.

THEOREM 4.1. *Let M be a conformally flat compact 4-dimensional Hermitian manifold. If the scalar curvature τ of M is non-positive, then τ vanishes identically and M is Kählerian and, moreover, M is locally a product of two surfaces with mutually opposite constant Gauss curvatures.*

Proof. By the vanishing of the Weyl conformal curvature tensor we have

$$\begin{aligned} R_{XYZW} = \tfrac{1}{2}\{&\varrho(X, W)g(Y, Z) - \varrho(X, Z)g(Y, W) + \varrho(Y, Z)g(X, W) \\ &- \varrho(Y, W)g(X, Z)\} - \tfrac{1}{6}\tau\{g(X, W)g(Y, Z) - g(X, Z)g(Y, W)\}. \end{aligned} \tag{4.1}$$

Put E_s, JZ, JE_s instead of X, Z, W into (4.1) and sum. Then, using also (2.12), one gets

$$\varrho^*(Y, Z) = \tfrac{1}{2}\{\varrho(Y, Z) + \varrho(JY, JZ)\} - \tfrac{1}{6}\tau g(Y, Z).$$

Hence it follows that $\tau^* = \tfrac{1}{3}\tau$, which used in (2.17) yields

$$\tau = 3(\tfrac{1}{2}|B|^2 - \operatorname{div} B). \tag{4.2}$$

Now, integrating (4.2) along the manifold M with respect to the natural volume element and using famous theorem of Green, we obtain

$$\int \tau = \tfrac{3}{2}\int |B|^2 \geqslant 0.$$

So, if $\tau \leqslant 0$, then $\tau = 0$ and $B = 0$. In this case, by (1.3), it must be $\nabla J = 0$.

Thus, M is Kählerian. The remaining part of our assertion follows from [1], Remark 2. The proof is complete.

THEOREM 4.2. *Any 4-dimensional compact Hermitian manifold fulfilling the condition $R_{XYJZJW} = R_{XYZW}$ is Kählerian.*

Proof. Under our condition, from Proposition 3.2, we have trace $L = |B|^2$. So, by (2.16), it holds that $\operatorname{div} B = \frac{1}{2}|B|^2$. Now, integrating this equality along the manifold M and using the Greenn theorem, we see that $B = 0$. This completes the proof.

From Theorems 4.1 or 4.2 the following corollary obviously follows.

COROLLARY 4.3. *Any locally flat 4-dimensional compact Hermitian manifold is Kählerian.*

Remark 4.4. Note that in Theorems 4.1 and 4.2 and Corollary 4.3 the compactness of the manifold M can be replaced by the assumption $\operatorname{div} B \leqslant 0$. Our Example B in Section 5 shows that the compactness (or $\operatorname{div} B \leqslant 0$) cannot be omitted in the above.

THEOREM 4.5. *Let M be a 4-dimensional Hermitian manifold fulfilling the condition $R_{JXJYJZJW} = R_{XYZW}$. If B is an infinitesimal automorphism of the structure J, then M is Kählerian.*

Proof. From Proposition 3.3, by (2.11), we obtain

$$\nabla_{JX}(\omega)(JY) + \nabla_{JY}(\omega)(JX) - \nabla_X(\omega)(Y) - \nabla_Y(\omega)(X) + \omega(JX)\,\omega(JY) - \omega(X)\,\omega(Y) = 0.$$

Hence it follows that

$$g(\nabla_{JX}B - J\nabla_X B,\ JY) + g(\nabla_{JY}B - J\nabla_Y B,\ JX) = \omega(X)\,\omega(Y) - \omega(JX)\,\omega(JY). \tag{4.3}$$

As $\nabla_B(J) = 0$ (cf. Proposition 1.2) we find

$$[B, JX) - J[B, X] = \nabla_B JX - \nabla_{JX}B - J\nabla_B X + J\nabla_X B = -\nabla_{JX}B + J\nabla_X B,$$

which used in (4.3) becomes

$$g([B, JX] - J[B, X],\ JY) + g([B, JY] - J[B, Y],\ JX) = -\omega(X)\,\omega(Y) + \omega(JX)\,\omega(JY). \tag{4.4}$$

On the other hand, one has $\mathcal{L}_B(J)X = [B, JX] - J[B, X]$, where $\mathcal{L}$ indicates the Lie derivative. Let us now assume B is an infinitesimal automorphism of J, i.e., $\mathcal{L}_B(J) = 0$. Then, (4.4) gives $\omega(X)\omega(Y) = \omega(JX)\omega(JY)$, which immediately implies $\omega = 0$. This completes the proof.

THEOREM 4.6. *Let M be a 4-dimensional Hermitian manifold fulfilling the condition $R_{JXJYJZJW} = R_{XYZW}$. If the trajectories of the vector fields B and A are geodesics, then M is Kählerian.*

Proof. We shall use equality (4.4). Taking there $X = Y = B$, we find

$$2g([B, A], A) = -|B|^4. \tag{4.5}$$

Let now the trajectories of B and A be geodesics, that is, $\nabla_B B = 0 = \nabla_A A$. As we have seen in the proof of Proposition 3.1, it holds good that $[B, A] = -J(\nabla_B B + \nabla_A A)$. So, in our case $[B, A] = 0$. Using this in (4.5) we get $B = 0$, completing the proof.

Remark 4.7. Theorems 4.2, 4.5 and 4.6 for Hermitian manifolds being locally conformal Kählerian are proved by Vaisman in [7].

5. Examples.

A. Let $\mathscr{A}$ be the 4-dimensional Lie algebra whose skew-symmetric multiplication is given by

$$[e_1, e_2] = e_3, \quad [e_2, e_3] = e_1, \quad [e_2, e_4] = 2e_3, \quad \text{and} \quad [e_i, e_j] = 0 \quad \text{in other cases,} \tag{5.1}$$

where $\{e_1, \ldots, e_4\}$ is certain fixed basis of $\mathscr{A}$.

Consider a connected Lie subgroup G of general linear group $GL(k, R)$, for certain natural k, such that the Lie algebra $\mathfrak{g}$ of G is isomorphic to $\mathscr{A}$. Let $s\colon \mathscr{A} \to \mathfrak{g}$ be the isomorphism. Let $\{E_1, \ldots, E_4\}$ be the basis of $\mathfrak{g}$ formed by left invariant vector fields on G such that $s(e_i) = E_i$, $1 \leqslant i \leqslant 4$. Then we have

$$[E_1, E_2] = E_3, \quad [E_2, E_3] = E_1, \quad [E_2, E_4] = 2E_3, \quad \text{and} \quad [E_i, E_j] = 0 \quad \text{in other cases.} \tag{5.2}$$

Let (J, g) be the left invariant almost Hermitian structure on G defined by $JE_1 = E_3$, $JE_2 = E_4$, $JE_3 = -E_1$, $JE_4 = -E_2$, and $g(E_i, E_j) = \delta_{ij}$, $1 \leqslant i, j \leqslant 4$. Using (1.1) and (5.2), by strightforward calculation, one verifies that $N(E_i, E_j) = 0$ for $1 \leqslant i, j \leqslant 4$. So, (J, g) is a Hermitian structure on G.

We need here the Riemannian connection with respect to the metric g. This is given by

$$\begin{gathered} \nabla_{E_2} E_1 = -E_3, \quad \nabla_{E_2} E_3 = E_1 - E_4, \quad \nabla_{E_2} E_4 = E_3, \quad \nabla_{E_3} E_2 = -E_4, \\ \nabla_{E_3} E_4 = E_2, \quad \nabla_{E_4} E_3 = E_2, \quad \nabla_{E_4} E_2 = -E_3, \\ \text{and} \quad \nabla_{E_i} E_j = 0 \quad \text{in other cases.} \end{gathered} \tag{5.3}$$

From (1.3) we find $\sum_s \nabla_{E_s}(J) E_s = JB$. Hence $B = -\sum_s (J \nabla_{E_s} JE_s + \nabla_{E_s} E_s)$. Consequently, using (5.3), we obtain $B = 2E_1$. Therefore, with the help of (2.11), (2.2) and (5.3), one gets for the tensor field L

$$L(E_2, E_3) - L(E_3, E_2) = g(\nabla_{E_2} B, E_3) - g(\nabla_{E_3} B, E_2) = -2.$$

We may conclude that L is not symmetric and our Hermitian structure (J, g) is not locally conformal Kählerian.

Finally, we describe an example of a Lie group whose Lie algebra is isomorphic to the Lie algebra $\mathscr{A}$ of the form (5.1). Namely, note that $\mathscr{A}$ may be represented as a subalgebra of $\mathrm{gl}(4, \boldsymbol{R})$. It is sufficient to put

$$e_1 = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}, \quad e_2 = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix},$$

$$e_3 = \begin{bmatrix} 0 & 0 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}, \quad e_4 = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & -2 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}.$$

Hence, $\mathscr{A}$ is formed by the following matrices

$$\begin{bmatrix} 0 & 0 & x^2 & 0 \\ x^3 & 0 & x^1 & 0 \\ -x^2 & 0 & 0 & 0 \\ 0 & 0 & 0 & x^4 \end{bmatrix}, \qquad x^1, \ldots, x^4 \in \boldsymbol{R}.$$

The corresponding Lie group G is formed by the matrices

$$\begin{bmatrix} b & 0 & c & 0 \\ d & 1 & a & 0 \\ -c & 0 & b & 0 \\ 0 & 0 & 0 & e \end{bmatrix},$$

where $a, b, c, d, e \in \boldsymbol{R}$, $b^2+c^2 = 1$ and $e > 0$. We see that G is a connected Lie subgroup of $\mathrm{GL}(4, \boldsymbol{R})$.

B. Let $M = \boldsymbol{R}^4 - \{0\}$ and let $(x^1, \ldots, x^4)$ be the Cartesian coordinates system of $\boldsymbol{R}^4$ restricted to M. There is the standard Kählerian structure (J, g') on M defined by

$$J\frac{\partial}{\partial x^1} = \frac{\partial}{\partial x^3}, \quad J\frac{\partial}{\partial x^2} = \frac{\partial}{\partial x^4},$$

$$J\frac{\partial}{\partial x^3} = -\frac{\partial}{\partial x^1}, \quad J\frac{\partial}{\partial x^4} = -\frac{\partial}{\partial x^2} \quad \text{and} \quad g'\left(\frac{\partial}{\partial x^i}, \frac{\partial}{\partial x^j}\right) = \delta_{ij}, \quad 1 \leqslant i, j \leqslant 4.$$

Below we deform the metric g' conformally to a new metric g being flat too. Before doing this recall the formulas from conformal Riemannian geometry.

Let $g = \alpha^2 g'$, $\alpha > 0$ being a function on M. Assume that $\beta = d(\log\alpha)$ and C is the vector field defined by the condition $\beta(X) = g'(C, X)$. Then for the Riemannian connections ∇ and ∇' we have

$$\nabla_X Y = \nabla'_X Y + \beta(X)Y + \beta(Y)X - g'(X, Y)C.$$

Put

$$s(X, Y) = \nabla'_X(\beta)(Y) - \beta(X)\beta(Y) + \tfrac{1}{2}\beta(C)g'(X, Y), \qquad g'(SX, Y) = s(X, Y).$$

Then for the curvature operators R and R' we have

(5.5) $R_{XY}Z = R'_{XY}Z - s(Y, Z)X + s(X, Z)Y - g'(Y, Z)SX + g'(X, Z)SY.$

Coming back to our considerations suppose additionally $\alpha = (\sum_r (x^r)^2)^{-1}$. Then one has

$$\beta = -2\alpha\sum_r x^r dx^r, \qquad C = -2\alpha\sum_r x^r \frac{\partial}{\partial x^r}, \qquad \beta(C) = 4\alpha,$$

$$\nabla'_{\frac{\partial}{\partial x^i}}(\beta)\left(\frac{\partial}{\partial x^j}\right) = 4\alpha^2 x^i x^j - 2\alpha\delta_{ij}.$$

Consequently, $s(\partial/\partial x^i, \partial/\partial x^j) = 0$, i.e., $s = 0$ and $S = 0$. Therefore and by $R' = 0$, from (5.5) we have $R = 0$. Thus, (J, g) is a flat Hermitian and non-Kählerian manifold.

References

[1] A. Derdziński, *Self-dual Kähler manifolds and Einstein manifolds of dimension four*, Composito Math. 49 (1983), 405–433.

[2] A. Gray, *Curvature identities for Hermitian and almost Hermitian manifolds*, Tôhoku Math. J. 28 (1976), 601–612.

[3] — and L. M. Hervella, *The sixteen classes of almost Hermitian manifolds and their linear invariants*, Ann. Mat. Pura Appl. (IV) 123 (1980), 35–58.

[4] S. Kotô, *Some theorems on almost Kähler spaces*, J. Math. Soc. Japan 12 (1960), 422–433.

[5] M. Nagao and S. Kotô, *Curvatures in almost Hermitian manifolds*, Memoirs of the Faculty of Education, Niigata University, 15 (1973), 1–6.

[6] I. Vaisman, *On locally conformal almost Kähler manifolds*, Israel J. Math. 24 (1976), 338–351.

[7] —, *Some curvature properties of locally conformal Kähler manifolds*, Trans. Amer. Math. Soc. 259 (1980), 439–447.

[8] L. Vanhecke, *Almost Hermitian manifolds with J-invariant Riemannian curvature tensor*, Rend. Sem. Mat. Univers. Politecn. Torino 34 (1975–1976), 487–498.

[9] K. Yano, *Differential geometry on complex and almost complex spaces*, International Series of Monographs in Pure and Applied Mathematics, Vol. 49 (1965).

INSTYTUT MATEMATYKI POLITECHNIKI WROCŁAWSKIEJ