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Some estimations for sequences of Fourier coefficients belonging to generalized Orlicz sequence spaces

1. There are well-known theorems on convergence of a series $\sum_{n=1}^{\infty} n^{\beta}|c_n(f)|^{\gamma}$, $\beta, \gamma > 0$, where $c(f) = (c_n(f))_{n=1}^{\infty}$ is the sequence of Fourier coefficients of function f with respect to an orthogonal system $\psi = (\psi_n)_{n=1}^{\infty}$, supposing f to be of bounded variation with modulus of continuity satisfying some conditions (see e.g. [10], Chapter 6, and [8], Chapter 1). Following these ideas, we shall investigate here the problem of convergence of the series $\sum_{n=1}^{\infty} \varphi_n(\lambda|c_n(f)|)$ for some $\lambda > 0$, where $(\varphi_n)_{n=1}^{\infty}$ is a sequence of φ-functions. This convergence is equivalent to the statement that the sequence $c(f)$ belongs to the generalized sequence space l^{φ} with $\varphi = (\varphi_n)_{n=1}^{\infty}$ (see e.g. [6], Definition 7.2). The inequalities obtained in course of the investigations give an estimation of the modular

$$\varrho(x) = \sum_{n=1}^{\infty} \varphi_n(|c_n|), \qquad x = (c_n)_{n=1}^{\infty}$$

by means of some expressions including modulus of continuity and Φ-variation of the function f (see [6], Definition 10.4). This enables us to get theorems on continuity of the Fourier coefficients operator from some two-modular spaces or two-norm spaces (see [7] and [1], [9]) of functions f into l^{φ}. We present here results for the trigonometric, Haar and Rademacher systems.

2. In this section we shall deal with the trigonometric system starting with $\cos 2t$, i.e., $\cos 2t$, $\sin 2t$, $\cos 3t$, $\sin 3t, \ldots$, denoting the Fourier coefficients by

$$a_n(f) = \frac{1}{\pi}\int_0^{2\pi} f(t)\cos nt\,dt, \qquad b_n(f) = \frac{1}{\pi}\int_0^{2\pi} f(t)\sin nt\,dt, \qquad n = 2, 3, \ldots,$$

$$a(f) = (a_n(f))_{n=2}^{\infty}, \qquad b(f) = (b_n(f))_{n=2}^{\infty};$$

moreover, let

$$(1) \qquad \varrho(x) = \sum_{n=2}^{\infty} \varphi_n(|c_n|) \quad \text{for } x = (c_n)_{n=2}^{\infty},$$

where $(\varphi_n)_{n=2}^{\infty}$ is a sequence of φ-functions, $\varphi = (\varphi_n)_{n=2}^{\infty}$, and l^{φ} is the generalized Orlicz sequence space generated by the modular ϱ.

2.1. Let Φ be a φ-function. In the following we shall assume that f is 2π-periodic and complex-valued and that the Φ-variation $\bigvee_0^{2\pi} \Phi(\lambda f)$ of λf in $\langle 0, 2\pi\rangle$ is finite for some $\lambda > 0$; this implies f to be bounded (see [6], 10.7 (a)) and $\bigvee_{\alpha}^{\beta} \Phi(\lambda f) < \infty$ for any real $\alpha < \beta$. Moreover, we shall denote

$$\Psi(u) = |u|^p/\Phi(u) \quad \text{for } u > 0, \qquad \Psi(0) = \lim_{u \to 0+} \Psi(u)$$

assuming Ψ to be a non-decreasing function of $u \geqslant 0$ for some p, $1 < p \leqslant 2$, and we take $q = p/(p-1)$.

2.2. Lemma. *Let f satisfy Assumptions 2.1. Then*

$$\sum_{j=2^{k-1}+1}^{2^k} \left(|a_j(\lambda f)|^q + |b_j(\lambda f)|^q\right) \leqslant 2^{-q/2} \left\{\bigvee_{-2\pi}^{4\pi} \Phi(\lambda f)\right\}^{q/p} 2^{-qk/p} \Psi^{q/p}\left(\omega(\lambda f, \pi/2^k)\right),$$

for $k = 1, 2, \ldots$, where $\omega(f, \delta)$ is the modulus of continuity of f.

Proof. We may limit ourselves to $\lambda = 1$. Applying the usual procedure with Hausdorff–Young inequality, we obtain easily

$$\sum_{j=2^{k-1}+1}^{2^k} \left(|a_j(f)|^q + |b_j(f)|^q\right)$$

$$\leqslant 2^{-q/2} \pi^{-q/p} \left\{\int_0^{2\pi} |f(t+\pi/2^{k+1}) - f(t-\pi/2^{k+1})|^p \, dt\right\}^{q/p}$$

$$\leqslant 2^{-q/2} \pi^{-q/p} \Psi^{q/p}\left(\omega(f, \pi/2^k)\right) \left\{\int_0^{2\pi} \Phi\left(|f(t+\pi/2^{k+1}) - f(t-\pi/2^{k+1})|\right) dt\right\}^{q/p}$$

$$= 2^{-q/2} \pi^{-q/p} \Psi^{q/p}\left(\omega(f, \pi/2^k)\right) \left\{\int_0^{2^{-k}} \sum_{j=1}^{2^{k+1}} \Phi\left|f\left(t + \frac{2j-1}{2^{k+1}}\pi\right) - f\left(t + \frac{2j-3}{2^{k+1}}\pi\right)\right| dt\right\}^{q/p}$$

$$\leqslant 2^{-q/2} \Psi^{q/p}\left(\omega(f, \pi/2^k)\right) 2^{-qk/p} \left\{\bigvee_{-2\pi}^{4\pi} \Phi(f)\right\}^{q/p}.$$

2.3. Theorem. *Let Assumptions 2.1 be satisfied and let $\bar{\varphi}_n(u) = \varphi_n(u^{1/q})$ be concave φ-functions, $n = 2, 3, \ldots$ Moreover, let one of the following two cases hold:*

1° $\varphi_n(u) \leqslant \varphi_{n+1}(u)$ *for* $u \geqslant 0$, $n = 2, 3 \ldots$

or

2° $\varphi_n(u) \geqslant \varphi_{n+1}(u)$ *for* $u \geqslant 0$, $n = 2, 3, \ldots$
Then for $\lambda > 0$

(2) $$\varrho(\lambda a(f)) \leqslant \tfrac{1}{2} \sum_{k=1}^{\infty} \varrho_k(\lambda f) \quad \textit{and} \quad \varrho(\lambda b(f)) \leqslant \tfrac{1}{2} \sum_{k=1}^{\infty} \varrho_k(\lambda f),$$

where

$$\varrho_k(f) = 2^k \varphi_{m_k} \{2^{-k} \big(\bigvee_{-2\pi}^{2\pi} \Phi(f)\big)^{1/p} \Psi^{1/p}(\omega(f, \pi/2^k))\},$$

with $m_k = 2^k$ *in case* 1° *and* $m_k = 2^{k-1}+1$ *in case* 2°. *In particular, if* $\sum_{k=1}^{\infty} \varrho_k(\lambda f) < \infty$ *for some* $\lambda > 0$, *then* $a(f) \in l^{\varphi}$ *and* $b(f) = l^{\varphi}$.

Proof. We limit ourselves to the case $a(f)$ and 1°. By Jensen's inequality and Lemma 2.2, we have

$$\begin{aligned}\varrho(\lambda a(f)) &\leqslant \sum_{k=1}^{\infty} \sum_{j=2^{k-1}+1}^{2^k} \varphi_{2^k}(|a_j(\lambda f)|) \\ &\leqslant \sum_{k=1}^{\infty} 2^{k-1} \bar{\varphi}_{2^k}\big(2^{-k+1} \sum_{j=2^{k-1}+1}^{2^k} |a_j(\lambda f)|^q\big) \\ &\leqslant \sum_{k=1}^{\infty} 2^{k-1} \varphi_{2^k} \{2^{1/q-1/2} \big(\bigvee_{-2\pi}^{4\pi} \Phi(\lambda f)\big)^{1/p} 2^{-k} \Psi^{1/p}(\omega(\lambda f, \pi 2^{-k}))\} \\ &\leqslant \tfrac{1}{2} \sum_{k=1}^{\infty} \varrho_k(\lambda f).\end{aligned}$$

Let us remark that the assumptions of 2.3 imply φ_n to satisfy the condition (Δ_2): $\varphi_n(2u) \leqslant 2^q \varphi_n(u)$, $u \geqslant 0$.

Taking in 2.3, $\varphi_k(u) = k^{\beta} |u|^{\gamma}$ with $0 < \gamma \leqslant q$ and arbitrary real β, we obtain the following

2.4. Corollary. *If Assumptions* 2.1 *are satisfied and*

$$\sum_{k=1}^{\infty} 2^{(1+\beta-\gamma)k} \Psi^{\gamma/p}(\omega(\lambda f, \pi 2^{-k})) < \infty$$

for a $\lambda > 0$, *then*

(3) $$\sum_{n=2}^{\infty} n^{\beta} (|a_n(f)|^{\gamma} + |b_n(f)|^{\gamma}) < \infty.$$

2.5. Remark. Taking $\beta = 0$, $p = 2$, we obtain Theorem 2 from [4] with $\lambda_n = n$. Supposing $\Phi(u) = |u|^r$ with $1 \leqslant r \leqslant p$ in 2.4, and $f \in \operatorname{Lip} \alpha$ with some $\alpha > 0$, we obtain (3) if $\gamma > p(1+\beta)(p+\alpha p-\alpha r)^{-1}$, which gives the result 4.41

of [5] with $\lambda_n = n$, $m = 1$. In particular, for $\beta = 0$, $\gamma = 1$, $p = 2$, $r = 1$ we obtain the well-known Zygmund's theorem (see e.g. [10], Chapter VI, Theorem 3.6, or [8], Theorem 1.3). Taking in 2.4, $\beta = 0$, $\gamma = 1$, $p = 2$, $\Phi(u) = |u|$, we obtain Salem's theorem (see [8], Theorem 1.5).

2.6. We shall apply now inequalities (2) in order to obtain an embedding result. Let $a\colon f \to a(f)$ and $b\colon f \to b(f)$ be the operators associating with any function $f \in L^1_{2\pi}$ the sequences of its cosinus and sinus Fourier coefficients, respectively. Let $V^{\Phi}_{2\pi}$ be the space of 2π-periodic functions f such that $\bigvee_0^{2\pi} \Phi(\lambda f) < \infty$ for some $\lambda > 0$. Then $\varrho^{(1)}(f) = \big(\bigvee_{-2\pi}^{4\pi} \Phi(f)\big)^{1/p}$ is a pseudomodular in $V^{\Phi}_{2\pi}$. We may define in $V^{\Phi}_{2\pi}$ another pseudomodular $\varrho^{(2)}(f)$ as:

$$\varrho^{(2)}(f) = \sum_{k=1}^{\infty} 2^k \varphi_{m_k} \{2^{-k} \Psi^{1/p}(\omega(f, \pi 2^{-k}))\},$$

where $m_k = 2^k$ in case 2.3.1° and $m_k = 2^{k-1}+1$ in case 2.3.2°. Thus, there is defined in $V^{\Phi}_{2\pi}$ the notion of two-modular convergence, or γ-convergence $\langle V^{\Phi}_{2\pi}, \varrho^{(1)}, \varrho^{(2)}\rangle$ (see [7], 1.2, or [6], p. 169): if $f_n \in V^{\Phi}_{2\pi}$, then $f_n \xrightarrow{\gamma} 0$ if the sequence (f_n) is $\varrho^{(1)}$-bounded and $\varrho^{(2)}$-convergent to 0. Now, $\varrho^{(1)}$-boundedness of (f_n) implies existence of constants k_1, $M > 0$ such that $\varrho^{(1)}(k_1 f_n) \leqslant M$ for $n = 1, 2, 3, \ldots$ (see [3], Proposition 1.3, or [6], Theorem 5.5). Let

$$X_{\varrho^{(2)}} = \{f \in V^{\Phi}_{2\pi}\colon \varrho^{(2)}(\lambda f) \to 0 \quad \text{as} \quad \lambda \to 0^+\}.$$

2.7. THEOREM. *Let all Assumptions of* 2.3 *be satisfied.*

Then $a\colon f \to a(f)$ and $b\colon f \to b(f)$ are linear, continuous operators from the space $X_{\varrho^{(2)}}$ provided with the two-modular convergence $\langle V^{\Phi}_{2\pi}, \varrho^{(1)}, \varrho^{(2)}\rangle$ into l^{φ} provided with the modular convergence (or equivalently, with the norm convergence) generated by the modular (1).

Proof. Linearity of a and b being obvious, let us suppose that $f_n \in X_{\varrho^{(2)}}$ for $n = 1, 2, \ldots$ and $f_n \xrightarrow{\gamma} 0$. Let $k_1, k_2 > 0$ and $M \geqslant 1$ be chosen so that $\varrho^{(1)}(k_1 f_n) \leqslant M$ and $\varrho^{(2)}(k_2 f_n) \to 0$ as $n \to \infty$. Taking a positive integer N so that $M \leqslant 2^N$, we have $\varphi_{m_k}(Mu) \leqslant 2^{Nq} \varphi_{m_k}(u)$ for $u \geqslant 0$. Hence, taking $0 < \lambda \leqslant \min(k_1, k_2)$ and applying inequality (2), we obtain

$$\varrho(\lambda a(f_n)) \leqslant 2^{Nq-1} \varrho^{(2)}(\lambda f_n) \to 0,$$

which shows the continuity of the operator a.

2.8. The above result may be presented also in the form of two-norm convergence (see [1], [9]) in place ot two-modular convergence, assuming additionally that the function Φ satisfies the condition (Δ_2) for all $u \geqslant 0$, since in this case boundedness of (f_n) in the sense of the pseudonorm generated by $\varrho^{(1)}$ implies $\varrho^{(1)}(\lambda f_n) \leqslant \bar{M}$ for $n = 1, 2, \ldots$ and for any $\lambda > 0$, with $\bar{M}$ depending on λ.

3. Let $\chi_1 = \chi_0^{(0)}$, $\chi_n = \chi_k^{(j)}$ $(n = 2^k+j,\ k = 0, 1, \ldots;\ j = 1, 2, 3, \ldots, 2^k)$ be the Haar orthonormal system, and let $a_k^{(j)}(f)$ be the Fourier coefficients with respect to this system of a 1-periodic function integrable in $\langle 0, 1\rangle$, $a_1(f) = a_0^{(0)}(f)$, $a_n(f) = a_k^{(j)}(f)$ with $n = 2^k+j$, as above. Denoting by $i_k^{(j)}$ the interval $((j-1)2^{-k-1}, j2^{-k-1})$, we have then

$$a_k^{(j)}(f) = -2^{k/2} \int_{i_k^{(2j-1)}} [f(t+2^{-k-1})-f(t)]\,dt$$

(see [2], p. 63). Let ϱ be defined by (1), $a(f) = (a_n(f))_{n=2}^{\infty}$.

3.1. There will be assumed in the following that f is a 1-periodic function with Φ-variation $\bigvee_0^1 \Phi(\lambda f) < \infty$ for some $\lambda > 0$, Φ being a φ-function. Moreover, we define Ψ as in 2.1, where $p \geqslant 1$ is arbitrary, and we shall assume Ψ to be a non-decreasing function of $u \geqslant 0$.

3.2. Theorem. *Let Assumptions* 3.1 *be satisfied. Let $\bar{\varphi}_n(u) = \varphi_n(u^{1/p})$ be concave φ-functions, $n = 2, 3, \ldots$ Moreover, let one of the following two cases hold*:

1° $\varphi_n(u) \leqslant \varphi_{n+1}(u)$ *for* $u \geqslant 0$, $n = 2, 3, \ldots$

or

2° $\varphi_n(u) \geqslant \varphi_{n+1}(u)$ *for* $u \geqslant 0$, $n = 2, 3, \ldots$

Finally, let

$$\varrho_k(f) = 2^k\, \varphi_{m_k}\Big\{\tfrac{1}{2}\Big(\bigvee_0^1 \Phi(f)\Big)^{1/p} 2^{-k(1/p+1/2)}\, \Psi^{1/p}\big(\omega(f, 1/2^{k+1})\big)\Big\}$$

with $m_k = 2^{k+1}$ *in case* 1° *and* $m_k = 2^k+1$ *in case* 2°. *Then*

$$\varrho(\lambda a(f)) \leqslant \sum_{k=0}^{\infty} \varrho_k(\lambda f) \quad \text{for } \lambda > 0. \tag{4}$$

In particular, if $\sum_{k=0}^{\infty} \varrho_k(\lambda f) < \infty$ *for some* $\lambda > 0$, *then* $a(f) \in l^{\varphi}$.

Proof. We have

$$\sum_{j=1}^{2^k} |a_k^{(j)}(f)|^p \leqslant 2^{1-p} \sum_{j=1}^{2^k} 2^{k(1-p/2)} \int_{i_k^{(2j-1)}} |f(t+2^{-k-1})-f(t)|^p\,dt$$

$$\leqslant 2^{1-p}\, 2^{k(1-p/2)}\, \Psi\big(\omega(f, 1/2^{k+1})\big) \sum_{j=1}^{2^k} \int_{i_k^{(2j-1)}} \Phi\big(|f(t+2^{-k-1})-f(t)|\big)\,dt$$

$$\leqslant 2^{-p}2^{-kp/2}\Psi\left(\omega\left(f,\frac{1}{2^{k+1}}\right)\right)\int_0^1\sum_{j=1}^{2^k}\Phi\left(\left|f\left(\frac{s}{2^{k+1}}+\frac{2j-1}{2^{k+1}}\right)\right.\right.$$

$$\left.\left.-f\left(\frac{s}{2^{k+1}}+\frac{2j-2}{2^{k+1}}\right)\right|\right)ds$$

$$\leqslant 2^{-p}2^{-kp/2}\Psi\big(\omega(f,1/2^{k+1})\big)\bigvee_0^1\Phi(f).$$

Limiting ourselves to case 1° and arguing as in the proof of 2.3 with p in place of q, we thus obtain

$$\varrho(\lambda a(f))\leqslant\sum_{k=0}^{\infty}\sum_{j=1}^{2^k}\varphi_{2^{k+1}}\big(\lambda|a_k^{(j)}(f)|\big)\leqslant\sum_{k=0}^{\infty}\varrho_k(\lambda f).$$

Now, let us take $\varphi_k(u)=k^\beta|u|^\gamma$ with $0<\gamma\leqslant p$ and arbitrary real β. Then 3.2 gives

3.3. Corollary. *If Assumptions* 3.1 *are satisfied and*

$$\sum_{k=0}^{\infty}2^{k(1+\beta-\gamma/p-\gamma/2)}\Psi^{\gamma/p}\big(\omega(\lambda f,1/2^{k+1})\big)<\infty$$

for a $\lambda>0$, *then*

$$\sum_{n=1}^{\infty}n^\beta|a_n(f)|^\gamma<\infty.$$

3.4. Remark. Taking here $\beta=0$, $\gamma=1$, $p=2$ and $\Phi(u)=|u|^r$ with $1\leqslant r<2$, we obtain that if $\sum_{k=0}^{\infty}\omega^{1-r/2}(f,2^{-k-1})<\infty$, then $\sum_{n=1}^{\infty}|a_n(f)|<\infty$, which is equivalent to [2], Theorem 3.

Applying inequality (4) and arguing analogously as in 2.6 and 2.7, we obtain

3.5. Theorem. *Let the assumptions of* 3.2 *be satisfied. Let*

$$\varrho^{(1)}(f)=\Big(\bigvee_0^1\Phi(f)\Big)^{1/p},$$

$$\varrho^{(2)}(f)=\sum_{k=0}^{\infty}2^k\varphi_{m_k}\{2^{-k(1/p+1/2)}\Psi^{1/p}(\omega(f,2^{-k-1}))\},$$

where $m_k=2^{k+1}$ *in case* 3.2.1° *and* $m_k=2^k+1$ *in case* 3.2.2°. *Let* V_1^Φ *be the space of* 1-*periodic functions* f *such that* $\bigvee_0^1\Phi(\lambda f)<\infty$ *for some* $\lambda>0$ *and let*

$$X_{\varrho^{(2)}}=\{f\in V_1^\Phi:\ \varrho^{(2)}(\lambda f)\to 0\ \text{as}\ \lambda\to 0^+\}.$$

Then $a\colon f \to a(f)$ *is a linear, continuous operator from* $X_{\varrho^{(2)}}$ *provided with the two-modular convergence* $\langle V_1^{\Phi}, \varrho^{(1)}, \varrho^{(2)}\rangle$ *into* l^{φ} *provided with the modular convergence generated by the modular* (1).

Let us note that Remark 2.8 remains valid also in our case.

4. We investigate now the case of Rademacher system $r_n(t) = \operatorname{sgn}\sin 2^n \pi t$, $n = 1, 2, 3, \ldots$, $0 \leqslant t \leqslant 1$. Here, the Fourier coefficients of a 1-periodic function f integrable in $\langle 0, 1\rangle$ are

(5) $$a_k(f) = -\sum_{j=1}^{2^k} \int_{i_k^{(2j-1)}} [f(t+2^{-k-1})-f(t)]\,dt, \qquad a(f) = \big(a_k(f)\big)_{k=1}^{\infty},$$

where $i_k^{(j)}$ are defined in 3. Let

$$\varrho(x) = \sum_{n=1}^{\infty} \varphi_n(|c_n|) \quad \text{for } x = (c_n)_{n=1}^{\infty}.$$

4.1. Theorem. *Let Assumptions* 3.1 *be satisfied. Then, writing*

$$\varrho_k(f) = \varphi_k\Big\{\tfrac{1}{2}2^{-k/p}\big(\bigvee_0^1 \Phi(f)\big)^{1/p}\Psi^{1/p}\big(\omega(f, 1/2^{k+1})\big)\Big\}$$

for $k = 1, 2, \ldots$, *we have*

(6) $$\varrho\big(\lambda a(f)\big) \leqslant \sum_{k=1}^{\infty} \varrho_k(\lambda f) \quad \textit{for } \lambda > 0.$$

In particular, if $\sum_{k=1}^{\infty} \varrho_k(\lambda f) < \infty$ *for some* $\lambda > 0$, *then* $a(f) \in l^{\varphi}$.

Proof. We have

$$|a_k(f)|^p \leqslant 2^{1-p}\sum_{j=1}^{2^k}\int_{i_k^{(2j-1)}} |f(t+2^{-k-1})-f(t)|^p\,dt$$
$$\leqslant 2^{-p}2^{-k}\Psi\big(\omega(f, 1/2^{k+1})\big)\bigvee_0^1 \Phi(f),$$

whence

$$\varphi_k\big(\lambda|a_k(f)|\big) \leqslant \varphi_k\Big\{\tfrac{1}{2}2^{-k/p}\big(\bigvee_0^1 \Phi(f)\big)^{1/p}\Psi^{1/p}\big(\omega(\lambda f, 1/2^{k+1})\big)\Big\}$$

for $k = 1, 2, \ldots$, and the result follows.

Taking $\varphi_k(u) = k^{\beta}|u|^{\gamma}$ with $0 < \gamma \leqslant p$ and an arbitrary real β in 4.1, and taking into account the fact that the sequence $\big(\Psi^{\gamma/p}(\omega(\lambda f, 2^{-k-1}))\big)_{k=1}^{\infty}$ is then bounded, we get

4.2. COROLLARY. *If Assumptions* 3.1 *are satisfied, then*

$$\sum_{n=1}^{\infty} n^{\beta} |a_n(f)|^{\gamma} < \infty$$

for any real β *and* $0 < \gamma \leqslant p$.

4.3. Remark. Specifying $\Phi(u) = |u|^r$ with any $r \geqslant 1$ and taking for a given $\gamma > 0$, $p = \max(r, \gamma)$, we see that if $\bigvee_0^1 \Phi(f) < \infty$, then 3.1 is satisfied. Hence 4.2 implies that $\sum_{n=1}^{\infty} n^{\beta} |a_n(f)|^{\gamma} < \infty$ for every real β and $\gamma > 0$. In case $\beta = 0$ this gives [2], Theorem 5.

4.4. THEOREM. *Let the assumptions of* 4.1 *be satisfied and let* Ψ *be s-convex with some* $s \in (0, 1\rangle$ (see [6], 1.9.I). *Let* $\varrho^{(1)}(f) = \left(\bigvee_0^1 \Phi(f)\right)^{1/p}$,

$$\varrho^{(2)}(f) = \sum_{k=1}^{\infty} \varphi_k \left\{2^{-k/p} \Psi^{1/p}\left(\omega(f, 2^{-k-1})\right)\right\}.$$

Let V_1^{Φ} *and* $X_{\varrho^{(2)}}$ *be defined as in* 3.5. *Then the thesis of 3.5 holds, where a is given by* (5).

Proof. Supposing $f_n \in X_{\varrho^{(2)}}$ for $n = 1, 2, \ldots$ and $f_n \xrightarrow{\gamma} 0$, there are $k_1, k_2 > 0$ and $M \geqslant 1$ such that $\varrho^{(1)}(k_1 f_n) \leqslant M$ and $\varrho^{(2)}(k_2 f_n) \to 0$ as $n \to \infty$. Since Ψ is s-convex, so $M\Psi(u) \leqslant \Psi(M_1 u)$ for $u \geqslant 0$, where $M_1 = M^{1/s}$. Hence, taking $0 < \lambda \leqslant \min(k_1, k_2 M_1^{-1})$ and applying inequality (6), we get

$$\varrho\left(\lambda a(f_n)\right) \leqslant \varrho^{(2)}(\lambda M_1 f_n) \leqslant \varrho^{(2)}(k_2 f_n) \to 0,$$

whence a is continuous.

Also, 4.4 can be given in the two-norm case, as in 2.8.

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