Dănuț Marcu (Bucharest, Romania)

Note on the square of a graph

Abstract. It is shown in this paper that the square of every graph with minimum degree at least 2 contains a semi-factor.

The terminology and notation of this paper follow those of the recent book by Behzad, Chartrand and Lesniak-Foster [1]. Let $G$ be a graph with $V(G)$ the set of vertices and $E(G)$ the set of edges.

By $\delta(G)$ we shall mean the minimum degree among the vertices of $G$. The vertex set of $G$ having degree $k$ is denoted by $V(G, k)$.

If $U$ is a non-empty subset of $V(G)$, then the subgraph $\langle U \rangle$ of $G$ induced by $U$ is the graph having vertex set $U$ and whose edge set consists of those edges of $G$ incident with two elements of $U$.

A semi-factor of $G$ is a spanning subgraph $H$ of $G$ such that every vertex in $H$ has degree 2.

Given a path $P: v_0, v_1, \ldots, v_n - 1, v_n$, we let $P^* = \{v_1, v_2, \ldots, v_{n-1}\}$, i.e., the set of internal vertices of $P$, and $V(P) = \{v_0, v_1, \ldots, v_n\}$.

To distinguish between a path $P$ and the graph whose vertices and edges are exactly those of $P$, we denote the graph by $G[P]$.

The $m$-th power $G^m$ of $G$, where $m \geq 1$, is that graph with $V(G^m) = V(G)$ for which $uv \in E(G^m)$ if and only if $1 \leq d(u,v) \leq m$, where $d(u,v)$ is the distance between $u$ and $v$. The graph $G^2$ will be called the square of $G$.

A caterpillar is a tree $T$ such that $\langle V(T) - V(T, 1) \rangle$ is a path or the empty graph.

According to [2], we have the following

Lemma. If $T$ is a tree, then $T^2$ is Hamiltonian if and only if $T$ is a caterpillar with $|V(T)| \geq 3$.

The main result of this paper is the following

Theorem. If $G$ is a graph with $\delta(G) \geq 2$, then $G^2$ contains a semi-factor.

Proof. Obviously, since $\delta(G) \geq 2$, the longest path in $G$ must have length $\geq 2$.

By the above lemma, to prove the theorem it is sufficient to find a spanning forest of $G$ in which each tree is a caterpillar with at least 3 vertices. We choose the longest path $P_1$ of $G$. Having chosen the paths
$P_1, P_2, \ldots, P_n$ in $G$, then, if $V(G) - \bigcup_{k=1}^{n} V(P_k)$ is not empty, we choose the longest path $P_{n+1}$ in $\langle V(G) - \bigcup_{k=1}^{n} V(P_k) \rangle$. Obviously, the end vertices of $P_{n+1}$ are adjacent only to vertices in $(\bigcup_{k=1}^{n} P_k^* ) \cup V(P_{n+1})$.

Since $G$ is finite, we find that we have chosen the paths $P_1, P_2, \ldots, P_m$ such that $V(G) - \bigcup_{k=1}^{m} V(P_k)$ is empty.

For each end vertex $v$ of each path of length 0 or 1, we choose an internal vertex of a longer path to which $v$ is adjacent through an edge $e_v$. Obviously, such an internal vertex must exist because all the vertices of $G$ are of degree $\geq 2$. Now, we form a spanning forest whose trees are caterpillars by deleting the edges from the graphs in $\{G[P_1], G[P_2], \ldots, G[P_m]\}$, which have length 1 and adding the edges $e_v$. Since every one of the obtained caterpillars includes the vertices of a path with an internal vertex, every caterpillar has at least 3 vertices, and the theorem is proved.

References


FACULTY OF MATHEMATICS, UNIVERSITY OF BUCHAREST, BUCHAREST, ROMANIA