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The Faedo–Galerkin method in thermal stresses theory

Abstract. Five boundary-initial value problems for thermal stresses equations of classical and generalized thermomechanics describing inhomogeneous, anisotropic medium occupied bounded domain $G \subset \mathbf{R}^r$ ($r = 1, 2, 3$) have been formulated according to the classification of the V. D. Kupradze. The solvability and properties of the weak solutions of these boundary-initial value problems in the Sobolev spaces have been investigated using the Faedo–Galerkin method.

1. Introduction. The initial-boundary value problems in the thermal stresses theory have been investigated by V. D. Kupradze (cf. [24]), W. Nowacki (cf. [34], [35], [37], [38], [39]), J. C. Podstrigač (cf. [45]) in the class of smooth functions using the method of integral transformation and the method of integral equations. The initial-boundary value problems in classical linear thermoelasticity have been studied by C. M. Dafermos (cf. [5]) using the method of Hilbert space and by G. Duvaut and J. L. Lions (cf. [9]) using the method of the variational inequalities.

In this paper, using the Faedo–Galerkin method, the solvability and properties of the weak solutions of the five boundary-initial value problems for thermal stresses equations of classical and generalized linear thermomechanics have been investigated in anisotropic Sobolev spaces. These boundary-initial value problems have been formulated according to the classification of V. D. Kupradze (cf. [24]). We restrict our attention to classical and generalized linear thermal stresses equations for inhomogeneous anisotropic materials.

After an introductory section in which the initial-boundary value problems are formulated we proceed to investigate existence and uniqueness of weak solutions. We prove the existence and uniqueness theorems of the weak solutions and the continuous dependence of these solutions on given data for the five boundary-initial value problems for the thermal stresses equations of classical and generalized linear thermomechanics (Sections 4 and 5), respectively.

In the final sections, we study the regularity of the weak solutions with respect to the space and time variables in the case of classical and generalized thermal stresses theory (Sections 6 and 7).

Basing on the proved theorems, we have obtained the sufficient conditions in order to make the weak solution of the first boundary-initial value problem in the case of isotropic and homogeneous medium the classical solution (Example 6.1 in Section 6).

The proofs of all existential theorems have been carried out using the Faedo–Galerkin method.

2. Sobolev spaces. By r we denote the dimension of the Euclidean space E^r in which the configuration of the thermoelastic medium is embedded. The analysis will be carried out for general r though the model is physically meaningful only for $r = 1, 2, 3$. By x we denote the typical point of E^r and by $x_1, \ldots, x_r$ the coordinates of x with respect to a fixed Cartesian coordinate system. By $\alpha = (\alpha_1, \ldots, \alpha_r)$ we denote multiindex and by $|\alpha| = \alpha_1 + \ldots + \alpha_r$ its length. We introduce the following notation for derivatives with respect to the space variables.

$$\partial^\alpha = \partial_1^{\alpha_1} \ldots \partial_r^{\alpha_r}, \quad \text{where } \partial_j = \partial/\partial x_j \quad \text{for } j = 1, \ldots, r.$$

Time derivatives are denoted by $\partial_t^s = \partial^s/\partial t^s$, where $s = 1, 2$.

Let G be an open bounded set in E^r (cf. [12], p. 13) with regular boundary ∂G.

$L^p(G)$ is the space of [1] (equivalence classes of) measurable functions u such that (p being given with $1 \leqslant p \leqslant \infty$)

$$\|u\|_{L^p(G)} = \Big(\int_G |u(x)|^p\,dx\Big)^{1/p} < \infty, \qquad 1 \leqslant p < \infty, \tag{2.1}$$

$$\|u\|_{L^\infty(G)} = \operatorname*{ess\,sup}_{x \in G} |u(x)|, \qquad p = \infty, \tag{2.2}$$

Taken with the norm (2.1) or (2.2), $L^p(G)$ is a Banach space; if $p = 2$, $L^2(G)$ is a Hilbert space, where the scalar product corresponding to the norm (2.1) (where $p = 2$) is given by

$$(u, v)_{L^2(G)} = \int_G u(x)\,v(x)\,dx. \tag{2.3}$$

The Sobolev space $W_p^m(G)$ (cf. [4], p. 29–38, [48], p. 53–64), $1 \leqslant p < \infty$, consists of functions u belonging to $L^p(G)$ with weak derivatives $\partial^\alpha u$, $|\alpha| \leqslant m$, belonging to $L^p(G)$

$$W_p^m(G) = \{u\colon\ u \in L^p(G)\colon\ \partial^\alpha u \in L^p(G);\ |\alpha| \leqslant m\}, \tag{2.4}$$

With the norm

$$\|u\|_{W_p^m(G)} = \Big(\sum_{|\alpha| \leqslant m} \|\partial^\alpha u\|_{L^p(G)}^p\Big)^{1/p} \tag{2.5}$$

it is a Banach space.

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[1] All functions considered here are real-valued.

The case $p = 2$ is fundamental. To simplify the writing, we shall put

$$W_2^m(G) = H^m(G);$$

with the scalar product

$$(2.6) \qquad (u, v)_{H^m(G)} = \sum_{|\alpha| \leqslant m} (\partial^\alpha u, \partial^\alpha v)_{L^2(G)}$$

it is a Hilbert space. The norm in this space is given by

$$(2.7) \qquad \|u\|_{H^m(G)} = \Big(\sum_{|\alpha| \leqslant m} \|\partial^\alpha u\|^2_{L^2(G)} \Big)^{1/2}.$$

Let $C_0^\infty(G)$ denote the space of infinitely differentiable real-valued functions defined on G consisting of those elements with compact support contained in G. By $H_0^m(G)$ we denote the Hilbert space obtained as the completion of $C_0^\infty(G)$ by means of the norm $\|\cdot\|_{H^m(G)}$ given by (2.7). $H_0^m(G)$ is the subspace of the space $H^m(G)$.

By $\mathbf{L}^2(G)$ $(\mathbf{H}^m(G))$ we denote the r-fold Cartesian product of $L^2(G)$ $(H^m(G))$, respectively. We denote the scalar product and norms in the spaces $L^2(G)$, $\mathbf{L}^2(G)$ $(H^m(G), \mathbf{H}^m(G))$ by $(\cdot, \cdot)_{L^2}$, $(\cdot, \cdot)_{\mathbf{L}^2}$ $((\cdot, \cdot)_{H^m}, (\cdot, \cdot)_{\mathbf{H}^m})$ and $\|\cdot\|_{L^2}$, $\|\cdot\|_{\mathbf{L}^2}$ $(\|\cdot\|_{H^m}, \|\cdot\|_{\mathbf{H}^m})$, respectively.

In this paper we shall investigate the solvability of evolution problems using the Faedo–Galerkin method in the space $L^2(I, X)$, where $I = (0, \vartheta) \subset \mathbf{R}$ $(0 < \vartheta < \infty)$ is the time interval, X the Banach space with its norm denoted by $\|\cdot\|_X$ (cf. [8]).

By $L^p(I, X)$ we denote the space of (classes of) functions $t \to f(t)$ from $(0, \vartheta) \to X$ measurable for the measure dt such that

$$(2.8) \qquad \|u\|_{L^p(I,X)} = \Big(\int_0^\vartheta \|u(t)\|_X^p\, dt\Big)^{1/p}, \qquad 1 \leqslant p < \infty,$$

$$(2.9) \qquad \|u\|_{L^\infty(I,X)} = \operatorname*{ess\,sup}_{t \in X} \|u(t)\|_X, \qquad p = \infty.$$

This is a Banach space.

$W_2^k(I, X)$, $k \in \mathbf{N}$, denotes the space of the measurable functions $u \colon I \to X$, with $d^n u/dt^n \in L^2(I, X)$ for $0 \leqslant n \leqslant k$ (derivatives in the weak sense). The norm in $W_2^k(I, X)$ is given by:

$$(2.10) \qquad \|u\|^2_{W_2^k(I,X)} = \sum_{n=0}^{k} \int_0^\vartheta \|d^n u(t)/dt^n\|_X^2\, dt.$$

The space $W_2^k(I, X)$ is the Hilbert space (cf. [53], p. 168).

Let V and H be two Hilbert spaces over $\mathbf{R}$ with norms $\|\cdot\|_V$, $\|\cdot\|_H$, respectively, their scalar product in H being written $(\cdot, \cdot)_H$; we assume that $V \subset H$, V dense in H ([2]).

Identifying H with its dual ($H = H^*$) ([3]), H is then identified with a subspace of the dual V^* of V, whence

$$V \subset H \subset V^*. \tag{2.11}$$

The spaces V, H, V^* which have property (2.11) form the Gelfand triples (cf. [8], [53]).

In this paper we shall use the following inequalities:

1. The *Poincaré inequality* (cf. [12], p. 14)

$$\|u\|^2_{H^m} \leqslant C \sum_{|\alpha| \leqslant m} \int_G |\partial^\alpha u|^2 \, dx, \qquad \forall u \in H^m_0(G), \tag{2.12}$$

where $C = C(G, m)$.

2. The *Korn's second inequality* (cf. [8], p. 110)

$$\int_G \varepsilon_{ij}(u)\varepsilon_{ij}(u)\, dx + \int_G u_i u_i \, dx \geqslant C\|u\|^2_{H^1}, \qquad \forall u \in H^1(G), \tag{2.13}$$

where $\varepsilon_{ij}(u) = \frac{1}{2}(\partial_j u_i + \partial_i u_j)$ and $C = C(G)$, $C > 0$.

3. *Gronwall's inequality* (cf. [26], p. 298). Let g, ϱ be functions with the properties $g, \varrho \in C([0, \vartheta])$, $g, \varrho \geqslant 0$ and let g be a non-decreasing function. If ϱ satisfies the inequality

$$\varrho(t) \leqslant g(t) + C_0 \int_0^t \varrho(\sigma)\, d\sigma, \qquad 0 \leqslant t \leqslant \vartheta, \; C_0 = \text{const}, \tag{2.14}$$

then there exists a constant $C_1 = C_1(C_0, \vartheta)$ such that

$$\varrho(t) \leqslant C_1 g(t), \qquad \forall t \in [0, \vartheta]. \tag{2.15}$$

4. The *Gårding's inequality* (cf. [33], p. 192). Let A be a strong elliptic operator of order $2m$. Then there exist constant α_0, λ_0 ($\alpha_0 > 0$, $\lambda_0 > 0$) such that

$$(-1)^m \operatorname{Re}(Au, u) \geqslant \alpha_0 \|u\|^2_{H^m} - \lambda_0 \|u\|^2_{L^2} \quad \text{for } \forall u \in C_0^\infty(G). \tag{2.16}$$

Remark 2.1. The spaces used in our consideration form the Gelfand triples, for example in the case (cf. Theorem 4.2) of the first boundary-initial value problem we use the spaces $H^1_0(G)$, $L^2(G)$, $H^{-1}(G)$ which form the Gelfand triple.

3. Statment of the problems. In this introductory section we formulate

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([2]) Therefore, there exists a constant c such that $\|v\|_H \leqslant c\|v\|_V$, $\forall v \in V$.

([3]) By V^* we denote the dual space to the space V.

the initial-boundary value problems for the equations of thermal stresses of classical and generalized thermomechanics.

In the case of r-dimensional ($r = 1, 2, 3$) linear thermoelasticity theory, the equations of thermal stresses of classical thermomechanics for the inhomogeneous anisotropic medium covering bounded domain $G \subset E^r$ ($r = 1, 2, 3$) (cf. [23], [34], [35], [49]) have the following form:

$$\partial_t^2 u = A(x, \partial) u + B^+(x, \partial) T + F, \tag{3.1}$$

$$\partial_t T = a(x, \partial) T + q, \tag{3.2}$$

where $u = (u_1, \ldots, u_r)$ is the displacement vector field of the medium, T the temperature of the medium, $F = (F_1, \ldots, F_r)$ the body force, q intensity of heat sources. We denote by $A(x, \partial)$ the matrix differential operator $r \times r$ of the form [4]

$$A(x, \partial) = \big(\partial_h a_{jhkl}(x) \partial_l\big)_{j,k=1,\ldots,r} \tag{3.3}$$

which is a strong elliptic, self-adjoint (formally) operator. Its coefficients are continuously differentiable in a bounded domain G with smooth boundary ∂G (cf. [1], p. 63) and satisfy the following (cf. [34], [49]) conditions:

$$a_{jhkl}(x) = a_{hjkl}(x) = a_{jhlk}(x) = a_{kljh}(x), \qquad \forall x \in G. \tag{3.4}$$

The scalar differential operator $a(x, \partial)$ has the form

$$a(x, \partial) = \partial_h a_{lh}(x) \partial_l \tag{3.5}$$

and is strong elliptic (formally), self-adjoint. We denote by $B^+(x, \partial)$ the one-column matrix differential operator $r \times 1$ of the form

$$B^+(x, \partial) = \big(-\partial_l b_{lh}(x)\big)_{h=1,\ldots,r}. \tag{3.6}$$

The coefficients b_{lh} are bounded, continuously differentiable (cf. [49], p. 188) and satisfy the following conditions:

$$b_{lh}(x) = b_{hl}(x), \qquad \forall x \in G. \tag{3.7}$$

For equations (3.1), (3.2) we formulate (cf. [24], p. 55, 56, 600; [34], p. 69, [5]) five boundary-initial value problems in which the boundary conditions have the following forms:

$$\text{(I)} \qquad u|_{I \times \partial G} = U; \qquad T|_{I \times \partial G} = \Theta;$$

$$\text{(II)} \qquad S \cdot n|_{I \times \partial G} = S_R; \qquad \partial_\gamma T|_{I \times \partial G} = g;$$

$$\text{(III)} \qquad u|_{I \times \partial G} = U; \qquad \partial_\gamma T|_{I \times \partial G} = g;$$

$$\text{(IV)} \qquad S \cdot n|_{I \times \partial G} = S_R; \qquad T|_{I \times \partial G} = \Theta;$$

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(4) We adopt the summation convention.

(V^a) $\quad u|_{I\times\partial G_1} = U; \quad S\cdot n|_{I\times\partial G_2} = S_R; \quad (\alpha\partial_\gamma T+\beta T)|_{I\times\partial G} = 0$ (5)

(V^b)

$$u|_{I\times\partial G_1} = 0; \quad (S\cdot n+ku)|_{I\times\partial G_1^c} = 0; \quad T|_{I\times\partial G_2} = 0; \quad (\alpha\partial_\gamma T+\beta T)|_{I\times\partial G_2^c} = 0;$$

where stress vector $S\cdot n$ and the transversal derivative $\partial_\gamma T$ are given by

$$S\cdot n = (n_j S_{ji})_{i=1,\dots,r} = \big(n_j a_{ijkl}(x)\,\partial_l u_k\big)_{i=1,\dots,r}, \tag{3.8}$$

$$\partial_\gamma T = \big(n_l a_{lh}(x)\,\partial_h T\big); \tag{3.9}$$

$n = (n_1, \dots, n_r)$ is the unit exterior normal to ∂G; U, S_R, θ, g are given functions, α, β, k are real positive constants,

$$\partial G_1^c = \partial G - \overline{\partial G_1}, \quad \partial G_2^c = \partial G - \overline{\partial G_2}, \quad I = (0, \vartheta) \quad (\vartheta < \infty).$$

With the system of conditions (I), ..., (V) we associate the following initial conditions:

$$u(+0) = u_0, \quad \partial_t u(+0) = u_1, \quad T(+0) = T_0. \tag{3.10}$$

In the case of r-dimensional ($r = 1, 2, 3$) linear generalized thermomechanics the equations of thermal stresses for the inhomogeneous anisotropic medium (cf. [45], p. 21, [49], p. 199) have the form (6)

$$\partial_t^2 u = A(x, \partial)u + B^+(x, \partial)\,T + F, \tag{3.11}$$

$$\tau_r\,\partial_t^2 T + \partial_t T = a(x, \partial)\,T + Q, \tag{3.12}$$

where $u = (u_1, \dots, u_r)$ is the displacement vector field of the medium, T the temperature of the medium, $F = (F_1, \dots, F_r)$ the body force, Q the intensity of heat source, τ_r the constant of relaxation (7). The operators $A(x, \partial)$, $a(x, \partial)$, $B^+(x, \partial)$ are designated by (3.3), ..., (3.6) (cf. formulas (3.3), ..., (3.6)). For equations (3.11), (3.12) the boundary conditions have the form

$(\tilde{\mathrm{I}})$ $\quad u|_{I\times\partial G} = U; \quad T|_{I\times\partial G} = \Theta;$

$(\tilde{\mathrm{II}})$ $\quad S\cdot n|_{I\times\partial G} = S_R; \quad \partial_\gamma T|_{I\times\partial G} = -lg;$

$(\tilde{\mathrm{III}})$ $\quad u|_{I\times\partial G} = U; \quad \partial_\gamma T|_{I\times\partial G} = -lg;$

$(\tilde{\mathrm{IV}})$ $\quad S\cdot n|_{I\times\partial G} = S_R; \quad T|_{I\times\partial G} = \Theta;$

$(\tilde{\mathrm{V}}^a)'$

$$u|_{I\times\partial G_1} = U; \quad S\cdot n|_{I\times\partial G_2} = S_R; \quad (\partial_\gamma T+\alpha_s T+\tau_r\alpha_s\,\partial_t T)|_{I\times\partial G} = 0$$ (8);

(5) The boundary condition $(\mathrm{V}^a)_3$ designates heat flux through the surface (cf. [34], p. 22) ∂G.

(6) Exactly $Q = lq$ (cf. [45], p. 21), where $l = 1+\tau_r\,\partial_t$.

(7) For metals $\tau_r = 10^{-11}$ [sec] (cf. [45], p. 7).

(8) The boundary condition $(\tilde{\mathrm{V}}^a)_3$ follows from the generalized heat low (cf. [45], p. 8).

$$(\tilde{\mathrm{V}^b}) \qquad u|_{I\times\partial G_1}=0; \quad (S\cdot n+ku)|_{I\times\partial G_1^c}=0;$$

$$T|_{I\times\partial G_2}=0; \quad (\partial_\gamma T+\alpha_s T+\tau_r\alpha_s\partial_t T)|_{I\times\partial G_2^c}=0;$$

where α_s is the coefficient of thermal expansion, k the constant (cf. [5]). With the boundary conditions $(\tilde{\mathrm{I}}), \ldots, (\tilde{\mathrm{V}})$ we associate the initial conditions:

$$(3.13)\quad u(+0)=u_0, \quad (\partial_t u)(+0)=u_1, \quad T(+0)=T_0, \quad (\partial_t T)(+0)=T_1,$$

where u_0, u_1, T_0, T_1 are given functions on G.

From now on, the boundary-initial value problems with boundary conditions (I), ..., (V) $((\tilde{\mathrm{I}}), \ldots, (\tilde{\mathrm{V}}))$ for equations (3.1), (3.2) ((3.11), (3.12)) and with initial conditions (3.10), (3.13) we shall call (I), ..., (V), $((\tilde{\mathrm{I}}), \ldots, (\tilde{\mathrm{V}}))$ problems of classical (generalized) linear thermomechanics.

Problem (V^a) is the most general problem for thermal stresses equations of the classical thermomechanics. Similarly, $(\tilde{\mathrm{V}^a})$ problem is the most general problem for thermal stresses equations of the generalized thermomechanics, because problems $(\tilde{\mathrm{I}}), \ldots, (\tilde{\mathrm{IV}})$ are its particular cases.

In the present paper, the existence and uniqueness of the weak solution of problem (V^a) and $(\tilde{\mathrm{V}^a})$ is proved using the Faedo–Galerkin method in Sobolev space $L^2(I, H^1(G))$. The proofs of these theorems imply the proofs of the theorems about existence and uniqueness of the weak solutions for problems (I), (II), (III), (IV), (V^b) and $(\tilde{\mathrm{I}})$, $(\tilde{\mathrm{II}})$, $(\tilde{\mathrm{III}})$, $(\tilde{\mathrm{IV}})$, $(\tilde{\mathrm{V}^b})$, respectively.

We shall describe the Faedo-Galerkin method in the next section (see the proof of Theorem 4.1).

4. Existence and uniqueness of the solutions of the boundary-initial value problems for thermal stresses equations of classical thermomechanics. In the present section we investigate the solvability of the boundary-initial value problems for thermal stresses equations of classical thermomechanics. At first, we study problem (V^a) because it is the most general problem for equations (3.1), (3.2). In order to do it, we start with the definition of the weak solution of this problem.

Definition 4.1 (a weak solution of problem (V^a)). The pair

$$(4.1)\qquad (u, T)\in L^2(I, V_0)\times L^2(I, V_1)$$

will be called a *weak solution of problem* (V^a) if (u, T) satisfies the following identities

$$(4.2)\qquad \big(\partial_t^2 u(t), w\big)+a_1\big(u(t), w\big)=\big(\Psi_F(t), w\big)+\big(B^+ T(t), w\big), \qquad \forall w\in V_0,$$

$$(4.3)\qquad \big(\partial_t T(t), v\big)+a_2\big(T(t), v\big)=-\frac{\beta}{\alpha}\int_{\partial G} T(t)\,v\,d\xi+\big(q(t), v\big), \qquad \forall v\in V_1,$$

with initial conditions

(4.4)
$$u(+0) = \hat{u}_0 = u_0 - \Phi(+0), \quad (\partial_t u)(+0) = \hat{u}_1 = u_1 - \partial_t \Phi(+0), \quad T(+0) = T_0,$$

where forms $a_1(\cdot, \cdot)$, $a_2(\cdot, \cdot)$ and functional $\Psi_F(\cdot)$ appearing in (4.2), (4.3) are given by:

$$a_1(u(t), w) = \int_G a_{jhkl}(x)\, \partial_l u_k\, \partial_h w_j\, dx, \tag{4.5}$$

$$a_2(T(t), v) = \int_G a_{lh}(x)\, \partial_l T \partial_h v dx, \tag{4.6}$$

$$(\Psi_F(t), w) = \int_{\partial G_2} S_R w d\xi + (F(t), w) - (\partial_t^2 \Phi(t), w) - a_1(\Phi(t), w), \tag{4.7}$$

where $\Phi(t) \in \boldsymbol{H}^1(G)$ with the property $\Phi(t)|_{\partial G_1} = U(t)$ and

$$\hat{u}_0 \in \boldsymbol{V}_0, \quad \hat{u}_1 \in \boldsymbol{L}^2(G), \quad T_0 \in V_1, \quad F \in \boldsymbol{L}^2(I, \boldsymbol{L}^2(G)),$$
$$q \in L^2(I, V_1^*), \quad S_R \in \boldsymbol{L}^2(I, \boldsymbol{L}^2(\partial G)). \tag{4.8}$$

By $\boldsymbol{V}_0$, V_1 we denote the spaces defined as follows:

$$\boldsymbol{V}_0 = \{w \in \boldsymbol{H}^1(G):\ w|_{\partial G_1} = 0\}, \tag{4.9}$$

$$V_1 = \{v \in H^1(G):\ (\alpha \partial_\gamma v + \beta v)|_{\partial G} = 0\}. \tag{4.10}$$

Let us notice that the spaces $\boldsymbol{V}_0$, $\boldsymbol{L}^2(G)$, $\boldsymbol{V}_0^*$ and V_1, $L^2(G)$, V_1^* form (cf. [52], [53]) the Gelfand triples. The symbol $(\cdot, \cdot)$ denotes the forms of duality on $(\boldsymbol{V}_0, \boldsymbol{V}_0^*)$ and (V_1, V_1^*), respectively, which on the Cartesian product $\boldsymbol{L}^2(G) \times \boldsymbol{L}^2(G)$ or on the product $L^2(G) \times L^2(G)$ becomes the scalar product in the spaces $\boldsymbol{L}^2(G)$ or $L^2(G)$, respectively.

THEOREM 4.1. *Let Ψ_F, q satisfy*

$$\Psi_F \in W_2^1(I, \boldsymbol{V}_0^*), \quad q \in L^2(I, V_1^*). \tag{4.11}$$

Then there exists a unique weak solution (u, T) of problem (V^a), *with the properties*

$$\partial_t u \in \boldsymbol{L}^2(I, \boldsymbol{L}^2(G)), \quad \partial_t^2 u \in \boldsymbol{L}^2(I, \boldsymbol{V}_0^*), \quad \partial_t T \in L^2(I, V_1^*), \tag{4.12}$$

and it depends on the given functions u_0, u_1, T_0, U, S_R, F, q, continuously.

Remark 4.1. In order that the functional Ψ_F given by (4.7) ought to satisfy condition (4.11), it is sufficient that $F \in W_2^1(I, \boldsymbol{L}^2(G))$ and the extension $\tilde{S}_R$, $\tilde{U}$ (cf. [20]) to $I \times G$ of the functions S_R and U have the properties:

$$\tilde{S}_R|_{I \times \partial G} \in W_2^1(I, \boldsymbol{L}^2(\partial G)), \quad \tilde{U}|_{I \times \partial G} \in W_2^3(I, \boldsymbol{H}^{1/2}(\partial G))\ (^9).$$

(9) The definition of the space $H^{1/2}(\partial G)$ may be found in [26], p. 48–53.

Outline of the proof of Theorem 4.1. The proof is divided into two parts. In the first part we proof the existence of the solution of problem (V^a) using the Faedo–Galerkin method and show the continuous dependence of the solution on given data.

In the second part we proof the uniqueness of the solution of above problem.

I. Let $\{w^m\colon m \in \boldsymbol{N}\}$ be a linear, independent and complete system in V_0 and let $\{v_m\colon m \in \boldsymbol{N}\}$ be a linearly independent and complete system in V_1 ([10]).

We define the Galerkin approximations of the solution (u, T) by

$$u^m(t) = \sum_{j=1}^{m} g_j^m(t)\, w^j; \qquad T_m(t) = \sum_{j=1}^{m} h_{mj}(t)\, v_j, \tag{4.13}$$

where $g_j^m(\cdot)$, $h_{mj}(\cdot)$ are chosen in such a way that they satisfy (cf. [20]) the following system of equations:

$$\big(\partial_t^2 u^m(t), w^l\big)_{L^2} + a_1\big(u^m(t), w^l\big) = \big(\Psi_F(t), w^l\big)_{L^2} + \big(B^+ T_m(t), w^l\big)_{L^2}, \qquad 1 \leqslant l \leqslant m, \tag{4.14}$$

$$\big(\partial_t T(t), v_l\big)_{L^2} + a_2\big(T_m(t), v_l\big) = -\frac{\beta}{\alpha} \int_{\partial G} T_m(t)\, v_l\, d\xi + \big(q(t), v_l\big), \qquad 1 \leqslant l \leqslant m, \tag{4.15}$$

with initial conditions

$$\begin{gathered} u^m(+0) = \hat{u}_0^m = \sum_{j=1}^{m} \gamma_{0j}^m w_j; \qquad (\partial_t u^m)(+0) = \hat{u}_1^m = \sum_{j=1}^{m} \gamma_{1j}^m w^j; \\ T_m(+0) = T_0^m = \sum_{j=1}^{m} \eta_{0j}^m v_j; \end{gathered} \tag{4.16}$$

where

$$\begin{gathered} \sum_{j=1}^{m} \gamma_{0j}^m w^j \to \hat{u}_0 \quad \text{in } V_0, \qquad \sum_{j=1}^{m} \gamma_{1j}^m w^j \to \hat{u}_1 \quad \text{in } L^2(G), \\ \sum_{j=1}^{m} \eta_{0j}^m v_j \to T_0 \quad \text{in } V_1 \quad \text{if } m \to \infty. \end{gathered} \tag{4.17}$$

The system of equations (4.14), (4.15) with initial conditions (4.16) is a system (cf. [20]) of ordinary linear differential equations and has a global solution on interval $I = (0, \vartheta)$ $(\vartheta < \infty)$. Thus, the Galerkin approximation

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([10]) The spaces V_0 and V_1 (cf. [26]) are separable.

sequences $(u^m)_{m\in N}$ and $(T_m)_{m\in N}$ are uniquely determined by system (4.14), (4.15).

By multiplying equations (4.14) and (4.15) by $(\partial_t g_l^m)(t)$ and $h_{ml}(t)$, respectively, and taking the sum over l for $1 \leqslant l \leqslant m$, we obtain:

(4.18)
$$\big(\partial_t^2 u^m(t),\, \partial_t u^m(t)\big)_{L^2} + a_1\big(u^m(t),\, \partial_t u^m(t)\big) = \big(\Psi_F(t),\, \partial_t u^m(t)\big)_{L^2} + \big(B^+ T_m(t),\, \partial_t u^m(t)\big)_{L^2},$$

(4.19)
$$\big(\partial_t T_m(t),\, T_m(t)\big)_{L^2} + a_2\big(T_m(t),\, T_m(t)\big) = -\frac{\beta}{\alpha}\int_{\partial G}\big(T_m(t)\big)^2\, d\xi + \big(q(t),\, T_m(t)\big).$$

Using the simple transformation and integration on the interval $(0, t)$ $(t \leqslant \vartheta)$ and applying Korn's second inequality (cf. [8], p. 110), Schwarz's inequality and taking into account the inequalities (cf. [8], p. 99)

$$\int_{\partial G} |v|^2\, d\xi \leqslant \varepsilon\, \|v\|_{H^1}^2 + c_\varepsilon\, \|v\|_{L^2}^2, \qquad \forall\, v \in H^1(G),$$

and

$$2|ab| \leqslant \varepsilon a^2 + \frac{1}{\varepsilon} b^2, \qquad \forall\, \varepsilon > 0,$$

we get the following estimates:

(4.20)
$$\begin{aligned} &\|\partial_t u^m(t)\|_{L^2}^2 + \tfrac{1}{2}\alpha_1\, \|u^m(t)\|_{V_0}^2 \\ &\leqslant C_{01}\Big(\|\hat{u}_0^m\|_{V_0}^2 + \|\hat{u}_1^m\|_{L^2}^2 + \|\Psi_F(0)\|_{V_0^*}^2 + \|\Psi_F(t)\|_{V_0^*}^2 + \int_0^t \|\partial s\, \Psi_F(t)\|_{V_0^*}^2\, ds\Big) \\ &\quad + C_1 \int_0^t \big(\|\partial_s u^m(s)\|_{L^2}^2 + \|u^m(s)\|_{V_0}^2\big)\, ds + C\int_0^t \|T_m(s)\|_{V_1}^2\, ds, \end{aligned}$$

(4.21)
$$\begin{aligned} &\|T_m(t)\|_{L^2}^2 + \int_0^t \|T_m(s)\|_{V_1}^2\, ds \\ &\leqslant C_{02}\Big(\|T_0^m\|_{V_1}^2 + \int_0^t \|q(s)\|_{V_1^*}^2\, ds\Big) + C_2 \int_0^t \|T_m(s)\|_{L_2}^2\, ds, \end{aligned}$$

where α_1, C_{01}, C_1, C_{02}, C_2 are various positive constants independent of m. After applying Gronwall's inequality (cf. [26], p. 298) to inequalites (4.21), (4.20) we get:

(4.22) $\qquad \|T_m(t)\|_{L^2}^2 \leqslant \bar{C}_{02}\, \tilde{C}_2 \quad$ for any $m \in N$ and $t \in [0, \vartheta]$,

(4.23) $\qquad \|\partial_t u^m(t)\|_{L^2}^2 \leqslant \bar{C}_{01}\, \tilde{C}_1 \quad$ for any $m \in N$ and $t \in [0, \vartheta]$.

The estimates (4.22), (4.23), (4.20), (4.21) imply:

$$
(4.24)\qquad \begin{array}{lll}
(u^m)_{m\in N} & \text{bounded in} & L^2(I, V_0),\\
(\partial_t u^m)_{m\in N} & \text{bounded in} & L^2(I, L^2(G)),\\
(T_m)_{m\in N} & \text{bounded in} & L^2(I, V_1) \text{ and in } L^2(I, L^2(G)).
\end{array}
$$

Consequently, there exist weakly convergent subsequences (u^{m_n}), $(\partial_t u^{m_n})$, (T_{m_n}) of the sequences $(u^m)_{m\in N}$, $(\partial_t u^m)_{m\in N}$, $(T_m)_{m\in N}$ (they will be denoted by the same symbols as the Galerkin sequences, i.e, $(u^n)_{n\in N}$, $(\partial_t u^n)_{n\in N}$, $(T_n)_{n\in N}$). Without loss of generality we may assume that:

$$
(4.25)\qquad \begin{array}{l}
(u^n) \rightharpoonup z \text{ (weakly) in } L^2(I, V_0),\\
(\partial_t u^n) \rightharpoonup z' \text{ (weakly) in } L^2(I, L^2(G)),\\
(T_n) \rightharpoonup Z \text{ (weakly) in } L^2(I, V_1) \text{ and in } L^2(I, L^2(G))
\end{array}
$$

for $n\to\infty$. Obviously, $z' = \partial_t z$ (cf. [53]) and since $u^n(+0)\to z(0)$ in V_0 we get $z(0) = u_0$. Let $\Phi\in C^\infty(I)$ such that $\Phi(\vartheta) = 0$. We put $\Phi^l(\cdot) = \Phi(\cdot)w^l$, $\Phi_l(\cdot) = \Phi(\cdot)v_l$.

Multiplying (4.14), (4.15) by $\Phi(\cdot)$, taking $m = n \geqslant l$ and integrating by parts on the interval $(0, \vartheta)$ we have:

$$
(4.26)\qquad -\int_0^\vartheta \big(\partial_t u^n(t), \partial_t \Phi^l(t)\big)_{L^2} dt + \int_0^\vartheta a_1\big(u^n(t), \Phi^l(t)\big) dt
$$
$$
= \int_0^\vartheta \big(\Psi_F(t), \Phi^l(t)\big)_{L^2} dt + \big(\partial_t u^n(+0), \Phi^l(0)\big)_{L^2} + \int_0^\vartheta \big(B^+ T_n(t), \Phi^l(t)\big)_{L^2} dt,
$$

$$
(4.27)\qquad -\int_0^\vartheta \big(T_n(t), \partial_t \Phi_l(t)\big)_{L^2} dt + \int_0^\vartheta a_2\big(T_n(t), \Phi_l(t)\big) dt
$$
$$
= \int_0^\vartheta \Big[-\frac{\beta}{\alpha}\int_{\partial G} T_n(t)\,\Phi_l(t)\, d\xi\Big] dt + \int_0^\vartheta \big(q(t), \Phi_l(t)\big) dt + \big(T_n(0), \Phi_l(0)\big)_{L^2}.
$$

In view of (4.25), taking $n\to\infty$ in (4.26), (4.27) we get:

$$
(4.28)\qquad -\int_0^\vartheta \big(\partial_t z(t), \partial_t \Phi^l(t)\big) dt + \int_0^\vartheta a_1\big(z(t), \Phi^l(t)\big) dt
$$
$$
= \big(\hat{u}_1, \Phi^l(0)\big) + \int_0^\vartheta \big(\Psi_F(t), \Phi^l(t)\big) dt + \int_0^\vartheta \big(B^+ Z(t), \Phi^l(t)\big) dt,
$$

$$
(4.29)\qquad -\int_0^\vartheta \big(Z(t), \partial_t \Phi_l(t)\big) dt + \int_0^\vartheta a_2\big(Z(t), \Phi_l(t)\big) dt
$$
$$
= \int_0^\vartheta \Big[-\frac{\beta}{\alpha}\int_{\partial G} Z(t)\,\Phi_l(t)\, d\xi\Big] dt + \int_0^\vartheta \big(q(t), \Phi_l(t)\big) dt + \big(T_0, \Phi_l(0)\big).
$$

In particular, the above equations are true for any $\Phi \in C_0^\infty(I)$. Thus it follows from (4.26)–(4.29) that:

$$(4.30)\qquad \left(\partial_t^2 z(t), w^l\right) + a_1\left(z(t), w^l\right) = \left(\Psi_F(t), w^l\right) + \left(B^+ Z(t), w^l\right),$$

$$(4.31)\qquad \left(\partial_t Z(t), v_l\right) + a_2\left(Z(t), v_l\right) = -\frac{\beta}{\alpha}\int\limits_{\partial G} Z(t) v_l \, d\xi + \left(q(t), v_l\right)$$

for arbitrary $w^l \in \{w^m\colon m \in \boldsymbol{N}\}$, $v_l \in \{v_m\colon m \in \boldsymbol{N}\}$.

From (4.28), (4.29) after integrating by parts we obtain:

$$(4.32)\qquad \left(\partial_t z(0), w^l\right)\Phi(0) = \left(\hat{u}_1, w^l\right)\Phi(0),$$

$$(4.33)\qquad \left(Z(0), v_l\right)\Phi(0) = \left(T_0, v_l\right)\Phi(0)$$

for any w^l, v_l; so $\partial_t z(0) = \hat{u}_1$, $Z(0) = T_0$.

Therefore the pair (z, Z) is the weak solution of problem (V^a) in the meaning of Definition 1. Under the foregoing assumptions it can be proved (cf. [26], Chapter 1) that:

$$z \in C(I, V_0), \qquad \partial_t z \in C\left(I, L^2(G)\right), \qquad Z \in C\left(I, L^2(G)\right).$$

From inequality (4.21), after taking $n \to \infty$ and using a simple transformation, we obtain:

$$(4.34)\qquad \int\limits_0^{\vartheta} \|Z(t)\|_{V_1}^2 \, dt + \int\limits_0^{\vartheta} \|\partial_t Z(t)\|_{V_1^*}^2 \, dt \leqslant C_2\Big[\|T_0\|_{V_1}^2 + \int\limits_0^{\vartheta} \|q(t)\|_{V_1^*}^2 \, dt\Big],$$

where C_2 is a constant independent of m.

Similarly, taking $m \to \infty$ in inequality (4.20) and using Gronwall's inequality, we get:

$$(4.35)\qquad \|\partial_t z(t)\|_{L^2}^2 + \|z(t)\|_{V_0}^2$$
$$\leqslant \bar{C}_1\Big[\|\hat{u}_1\|_{L^2}^2 + \|\hat{u}_0\|_{V_0}^2 + \|\Psi_F(0)\|_{V_0^*}^2 + \int\limits_0^{\vartheta} \|\Psi_F(s)\|_{V_0^*}^2 \, ds + \int\limits_0^{\vartheta} \|\partial_s \Psi_F(s)\|_{V_0^*}^2 \, ds$$
$$+ \|T_0\|_{V_1}^2 + \int\limits_0^{\vartheta} \|q(t)\|_{V_1^*}^2 \, dt\Big].$$

Integrating (4.35) on the interval $(0, t)$, using the definition of the functional $\Psi_F(\cdot)$ and the trace theorem (cf. [26], [21] and [3], p. 376, formula (7.17)), we obtain estimates, which show that the solution (u, T) depends continuously on given data.

II. The proof of the uniqueness of the solution to the above problem is performed (cf. [20]) classicaly and is based on Korn's second inequality (cf. [8], p. 110, [12], p. 75, (2.13)).

Now we introduce the existence-uniqueness theorems for the boundary-initial problems (I), (II), (III), (IV), (V^b).

Problem (I) with homogeneous boundary conditions is a particular case of ($\partial G_2 = \emptyset$, $\beta \to \infty$) problem (V^a). Therefore, we get the definition of the solution of this problem from Definition 4.1 putting

$$V_0 = H_0^1(G), \quad V_1 = H_0^1(G), \quad \Phi(t) = 0, \quad \Psi_F(t) = F(t) \tag{4.36}$$

and substituting the forms $a_1(\cdot, \cdot)$, $a_2(\cdot, \cdot)$ by forms $\big(A(\cdot, \partial)\cdot, \cdot\big)$ $\big(a(\cdot, \partial)\cdot, \cdot\big)$ and neglecting the integral which is on the right-hand side of (4.3). The following (cf. [15], [19]) theorem is true.

THEOREM 4.2. *Let* u_0, u_1, T_0, F, q *satisfy*

$$\begin{gathered} u_0 \in H_0^1(G), \quad u_1 \in L^2(G), \quad T_0 \in H_0^1(G), \quad F \in L^2\big(I, L^2(G)\big), \\ q \in L^2\big(I, H^{-1}(G)\big). \end{gathered} \tag{4.37}$$

Then problem (I) *has a unique solution* (u, T)

$$(u, T) \in L^2\big(I, H_0^1(G)\big) \times \big(L^2\big(I, H_0^1(G)\big) \cap L^2\big(I, L^2(G)\big)\big)$$

with properties

$$\partial_t u \in L^2\big(I, L^2(G)\big), \quad \partial_t^2 u \in L^2\big(I, H^{-1}(G)\big), \quad \partial_t T \in L^2\big(I, H^{-1}(G)\big) \tag{4.38}$$

and it depends continuously on given data u_0, u_1, T_0, F, q.

The proof of Theorem 4.2 runs similarly to the proof of Theorem 4.1. In order to obtain the suitable estimations of sequences of the Galerkin approximation we apply to the strong elliptic operators $A(\cdot, \partial)$ and $a(\cdot, \partial)$ Gårding's inequality (cf. [33], p. 192, (2.16)).

Problem (II) with homogeneous boundary conditions $(\mathrm{II})_2$ is the specific case ($\partial G_1 = \emptyset$, $\alpha \to \infty$) of problem (V^a).

Putting in Definition 4.1

$$\begin{gathered} V_0 = H^1(G), \quad \Phi(t) = 0, \quad \big(\Psi_F(t), w\big) = \int_{\partial G_2} S_R w \, d\xi + \big(F(t), w\big), \\ V_1 = \{v \in H^1(G);\ \partial_\gamma v|_{\partial G} = 0\} \end{gathered} \tag{4.39}$$

and omiting the integral on the right-hand side of (4.3), we have the definition of the solution of problem (II). For this problem the following theorem is true:

THEOREM 4.3. *Suppose that*

$$u_0 \in H^1(G), \quad u_1 \in L^2(G), \quad T_0 \in V_1, \quad \Psi_F \in W_2^1\big(I, (H^1(G))^*\big), \quad q \in L^2(I, V_1^*).$$

Then there exists a solution (u, T) *of problem* (II)

$$(u, T) \in L^2\big(I, H^1(G)\big) \times L^2(I, V_1) \tag{4.40}$$

with properties

$$\partial_t u \in L^2(I, L^2(G)), \quad \partial_t^2 u \in L^2(I, H^1(G)^*), \quad \partial_t T \in L^2(I, V_1^*), \tag{4.41}$$

where u is designated with occuracy to the rigid displacement.

The proof of Theorem 4.3 follows from the first part of the proof of Theorem 4.1.

Problem (III) with homogeneous boundary conditions (III) is a particular case ($\partial G_2 = \emptyset$, $\alpha \to \infty$) of problem (V^a). So, in that case putting in Definition 4.1

$$V_0 = H_0^1(G), \quad V_1 = \{v \in H^1(G),\ \partial_\gamma v|_{\partial G} = 0\}, \quad \Phi(t) = 0, \quad \Psi_F(t) = F(t) \tag{4.42}$$

and substituting the form $a(\cdot, \cdot)$ by the form $(A(\cdot, \partial)\cdot, \cdot)$ and neglecting the integral occurring on the right-side of (4.3) we obtain the definition of the solution of problem (III).

THEOREM 4.4. *Let u_0, u_1, T_0, F, q satisfy*

$$u_0 \in H_0^1(G), \quad u_1 \in L^2(G), \quad T_0 \in V_1, \quad F \in L^2(I, L^2(G)), \quad q \in L^2(I, V_1^*). \tag{4.43}$$

Then problem (III) *has a unique solution* (u, T)

$$(u, T) \in L^2(I, H_0^1(G)) \times L^2(I, V_1) \tag{4.44}$$

with properties

$$\partial_t u \in L^2(I, L^2(G)), \quad \partial_t^2 u \in L^2(I, H^{-1}(G)), \quad \partial_t T \in L^2(I, V_1^*) \tag{4.45}$$

and it depends continuously on given data u_0, u_1, T_0, F, q.

The proof of Theorem 4.4 runs similarly to the proof of Theorem 4.1. The difference lies in the fact that in order to get the suitable estimations the sequences of the Galerkin approximations we apply Gårding's (cf. [20]) inequality to the strong elliptic operator $A(\cdot, \partial)$.

The definition of the solution of problem (IV) with homogeneous boundary conditions $(\mathrm{IV})_2$ as a particular case ($\partial G_1 = \emptyset$, $\beta \to \infty$) of problem (V^a) is obtained from Definition 4.1 taking

$$\begin{aligned} &V_0 = H^1(G), \quad V_1 = H_0^1(G), \quad \Phi(t) = 0, \\ &(\Psi_F(t), w) = \int_{\partial G_2} S_R w \, d\xi + (F(t), w), \end{aligned} \tag{4.46}$$

exchanging the form $a_2(\cdot, \cdot)$ by the form $(a(\cdot, \partial)\cdot, \cdot)$ and neglecting the integral occurring on the right-hand side of (4.3).

THEOREM 4.5. *Suppose that*

$$u_0 \in H^1(G),\ u_1 \in L^2(G),\ T_0 \in H_0^1(G),\ \Psi_F \in W_2^1\big(I, (H^1(G))^*\big),\ q \in L^2(I, H^{-1}(G)).$$

Then there exists a solution (u, T) *of problem* (IV)

$$(u, T) \in L^2(I, H^1(G)) \times L^2(I, H_0^1(G)) \tag{4.47}$$

with properties

$$\partial_t u \in L^2(I, L^2(G)), \qquad \partial_t^2 u \in L^2\big(I, (H^1(G))^*\big), \qquad \partial_t T \in L^2(I, H^{-1}(G)), \tag{4.48}$$

where u *is designated with occuracy to the rigid displacement.*

The proof of Theorem 4.5 follows from the first part of the proof of Theorem 4.1. The definition of problem (V^b) is a modification of Definition 4.1.

Putting in Definition 4.1

$$\begin{aligned} V_0 &= \{w\colon w \in H^1(G)\colon w|_{\partial G_1} = 0\}, \qquad \Phi(t) = 0, \\ V_1 &= \{v\colon v \in H^1(G)\colon v|_{\partial G_2} = 0\}, \qquad \Psi_F(t) = 0, \end{aligned} \tag{4.49}$$

and adding on the right-hand side of (4.2) the term of the form $-k \int_{\partial G_1^c} uw d\xi$ and exchanging ∂G by ∂G_2^c in the integral taking place in formula (4.3) we get the definition of problem (V^b). For this problem the following theorem is true:

THEOREM 4.6. *Let* u_0, u_1, T_0, F, q *satisfy*

$$u_0 \in V_0, \qquad u_1 \in L^2(G), \qquad T_0 \in V_1, \qquad F \in W_2^1(I, L^2(G)), \qquad q \in L^2(I, V_1^*). \tag{4.50}$$

Then problem (V^b) *has a unique solution* (u, T)

$$(u, T) \in L^2(I, V_0) \times L^2(I, V_1) \tag{4.51}$$

with properties

$$\partial_t u \in L^2(I, L^2(G)), \qquad \partial_t^2 u \in L^2(I, V_0^*), \qquad \partial_t T \in L^2(I, V_1^*) \tag{4.52}$$

and it depends continuously on given data u_0, u_1, T_0, F, q.

The proof of Theorem 4.6 runs similarly as the proof of Theorem 4.1.

5. Existence and uniqueness of the solution of the boundary-initial value problems for thermal stresses equations of generalized thermomechanics. In this section we formulate the existence-uniqueness theorems of the boundary-initial value problems for the thermal stresses equations of generalized thermomechanics. At first, we consider problem ($\tilde{\mathrm{V}}^a$) because it is the most general problem for equations (3.11), (3.12). Now, we proceede to the definition of a weak solution of problem ($\tilde{\mathrm{V}}^a$).

DEFINITION 5.1. (a weak solution of problem $(\tilde{V^a})$). The pair

$$(u, T) \in L^2(I, V_0) \times L^2(I, V_1) \tag{5.1}$$

will be called a *weak solution of problem* $(\tilde{V^a})$ if (u, T) satisfies the following identities:

$$(\partial_t^2 u(t), w) + a_1(u(t), w) = (\Psi_F(t), w) + (B^+ T(t), w), \qquad \forall w \in V_0, \tag{5.2}$$

$$\begin{aligned} \tau_r(\partial_t^2 T(t), v) + (\partial_t T(t), v) + a_2(T(t), v) \\ + \tau_r \alpha_s \int_{\partial G} \partial_t T(t) v d\xi + \alpha_s \int_{\partial G} T(t) v d\xi = (Q(t), v), \qquad \forall v \in V_1 \end{aligned} \tag{5.3}$$

with initial conditions

$$\begin{aligned} & u(+0) = \hat{u}_0 = u_0 - \Phi(0); \quad (\partial_t u)(+0) = \hat{u}_1 = u_1 - \partial_t \Phi(0), \\ & T(+0) = T_0, \quad (\partial_t T)(+0) = T_1, \end{aligned} \tag{5.4}$$

where forms $a_1(\cdot, \cdot)$, $a_2(\cdot, \cdot)$ and functional $\Psi_F(t)$ occurring in (5.2), (5.3) are given by (4.5), (4.6), (4.7) and the spaces V_0 and V_1 are defined as follows:

$$V_0 = \{w \in H^1(G);\ w|_{\partial G_1} = 0\}, \tag{5.5}$$

$$V_1 = \{v:\ v \in H^1(G)\} \tag{5.6}$$

and

$$\begin{aligned} & \hat{u}_0 \in V_0, \quad \hat{u}_1 \in L^2(G), \quad T_0 \in V_1, \quad T_1 \in L^2(G), \\ & F \in L^2(I, L^2(G)), \quad Q \in L^2(I, L^2(G)), \quad S_R \in L^2(I, L^2(\partial G)), \end{aligned} \tag{5.7}$$

where $\Phi(t) \in H^1(G)$ with the property (cf. [8]) $\Phi(t)|_{\partial G_1} = U(t)$.

Let us notice first that in this case the spaces V_0, $L^2(G)$, V_0^*, V_1, $L^2(G)$, V_1^* also form (cf. [52], [53]) Gelfand triples.

In this case (cf. [20], [19]) the following theorem is true.

THEOREM 5.1. *Let* T_0, T_1, Ψ_F, Q *satisfy*

$$a(\cdot, \partial) T_0 \in L^2(G), \qquad T_1 \in V_1, \qquad \Psi_F \in W_2^1(I, V_0^*), \qquad Q \in W_2^1(I, L^2(G)). \tag{5.8}$$

Then there exists a unique weak solution (u, T) *of problem* $(\tilde{V^a})$ *with the properties*

$$\begin{aligned} & \partial_t u \in L^2(I, L^2(G)), \quad \partial_t^2 u \in L^2(I, V_0^*), \\ & \partial_t T \in L^2(I, V_1), \quad \partial_t^2 \in L^2(I, L^2(G)), \end{aligned} \tag{5.9}$$

and it depends continuously on given data u_0, u_1, T_0, T_1, U, S_R, F, Q.

Remark 5.1. In order the functional Ψ_F to satisfy conditions (5.9), the functions F, S_R, U ought to satisfy the conditions mentioned in Remark 4.1.

The proof of Theorem 5.1 is (cf. [20]) analogous to the proof of Theorem 4.1.

Now we introduce the existence-uniqueness theorems for problems ($\tilde{\mathrm{I}}$), ($\tilde{\mathrm{II}}$), ($\tilde{\mathrm{III}}$), ($\tilde{\mathrm{IV}}$), ($\tilde{\mathrm{V}}^b$). We obtain the definitions of these problems from Definition 5.1 using a similar consideration as in Section 4. We restrict ourselves to the formulation of the theorems for these boundary-initial value problems.

THEOREM 5.2. *If we assume that*

$$u_0 \in \boldsymbol{H}_0^1(G), \quad u_1 \in \boldsymbol{L}^2(G), \quad T_0 \in H_0^1(G), \quad T_1 \in L^2(G),$$

then problem ($\tilde{\mathrm{I}}$) *with homogeneous boundary conditions* ($\tilde{\mathrm{I}}$) *possesses a unique solution* (u, T)

$$(u, T) \in L^2(I, \boldsymbol{H}_0^1(G)) \times L^2(I, H_0^1(G)) \tag{5.10}$$

with properties

$$\begin{aligned} &\partial_t u \in L^2(I, \boldsymbol{L}^2(G)), \quad \partial_t^2 u \in L^2(I, \boldsymbol{H}^{-1}(G)),\\ &\partial_t T \in L^2(I, L^2(G)), \quad \partial_t^2 T \in L^2(I, H^{-1}(G)), \end{aligned} \tag{5.11}$$

and it depends continuously on given data u_0, u_1, T_0, T_1, F, Q.

The proof of Theorem 5.2 follows from (cf. [16]) the proof of Theorem 5.1.

THEOREM 5.3. *Suppose that*

$$\begin{aligned} &u_0 \in \boldsymbol{H}^1(G), \quad u_1 \in \boldsymbol{L}^2(G), \quad T_0 \in H^1(G), \quad T_1 \in L^2(G),\\ &\Psi_F \in W_2^1(I, (\boldsymbol{H}^1(G))^*), \quad Q \in W_2^1(I, \check{L}^2(G)), \quad g \in W_2^1(I, L^2(\partial G)). \end{aligned} \tag{5.12}$$

Then there exists a solution (u, T) *of problem* ($\tilde{\mathrm{II}}$)

$$(u, T) \in L^2(I, \boldsymbol{H}^1(G)) \times L^2(I, H^1(G)) \tag{5.13}$$

with properties

$$\begin{aligned} &\partial_t u \in L^2(I, \boldsymbol{L}^2(G)), \quad \partial_t^2 u \in L^2(I, (\boldsymbol{H}^1(G))^*),\\ &\partial_t T \in L^2(I, L^2(G)), \quad \partial_t^2 T \in L^2(I, (H^1(G))^*), \end{aligned} \tag{5.14}$$

where u *is designated with accuracy to the rigid displacement* (cf. [34], [8]).

The proof of Theorem 5.3 follows from (cf. [20]) the first part of proof of Theorem 5.1.

THEOREM 5.4. *Let* u_0, u_1, T_0, T_1, F, Q, g *satisfy*

$$\begin{aligned} &u_0 \in \boldsymbol{H}_0^1(G), \quad u_1 \in \boldsymbol{L}^2(G), \quad T_0 \in H^1(G), \quad T_1 \in L^2(G),\\ &F \in L^2(I, \boldsymbol{L}^2(G)), \quad Q \in W_2^1(I, L^2(G)), \quad g \in W_2^1(I, L^2(\partial G)). \end{aligned}$$

Then problem ($\tilde{\mathrm{III}}$) *with homogeneous boundary condition* ($\tilde{\mathrm{III}}_1$) *has a unique solution* (u, T)

$$(u, T) \in L^2(I, \boldsymbol{H}_0^1(G)) \times L^2(I, H^1(G)) \tag{5.16}$$

with properties

$$\text{(5.17)}\qquad \begin{aligned} &\partial_t u \in L^2(I, L^2(G)), \quad \partial_t^2 u \in L^2(I, H^{-1}(G)),\\ &\partial_t T \in L^2(I, L^2(G)), \quad \partial_t^2 T \in L^2\big(I, (H^1(G))^*\big), \end{aligned}$$

and it depends continuously on given data u_0, u_1, T_0, T_1, F, Q.

The proof of Theorem 5.4 runs similarly as the proof of Theorem 5.1.

THEOREM 5.5. *Suppose that*

$$\text{(5.18)}\qquad \begin{aligned} &u_0 \in H^1(G), \quad u_1 \in L^2(G), \quad T_0 \in H_0^1(G), \quad T_1 \in L^2(G),\\ &\Psi_F \in W_2^1\big(I, (H^1(G))^*\big), \quad Q \in L^2\big(I, L^2(G)\big). \end{aligned}$$

Then there exists a solution (u, T) of problem $(\widetilde{\mathrm{IV}})$ with homogeneous boundary condition $(\widetilde{\mathrm{IV}})$

$$\text{(5.19)}\qquad (u, T) \in L^2\big(I, H^1(G)\big) \times L^2\big(I, H_0^1(G)\big)$$

which has the following properties

$$\text{(5.20)}\qquad \begin{aligned} &\partial_t u \in I^2(I, L^2(G)), \quad \partial_t^2 u \in L^2\big(I, (H^1(G))^*\big),\\ &\partial_t T \in L^2(I, L^2(G)), \quad \partial_t^2 T \in L^2(I, H^{-1}(G)), \end{aligned}$$

where u is designated with occuracy to the rigid displacement (cf. [34], [8]).

The proof of Theorem 5.5 follows from the first part of the proof of Theorem 5.1. In the case of problem $(\widetilde{\mathrm{V}^b})$ the following theorem is true.

THEOREM 5.6. *Let u_0, u_1, T_0, T_1, F, Q satisfy*

$$\begin{aligned} &u_0 \in V_0, \quad u_1 \in L^2(G), \quad T_0 \in V_1, \quad T_1 \in L^2(G),\\ &a(\cdot, \cdot)\, T_0 \in L^2(G), \quad F \in W_2^1\big(I, L^2(G)\big), \quad Q \in W_2^1\big(I, L^2(G)\big). \end{aligned}$$

Then problem $(\widetilde{\mathrm{V}^b})$ has a unique solution (u, T),

$$\text{(5.22)}\qquad (u, T) \in L^2(I, V_0) \times L^2(I, V_1),$$

with properties

$$\text{(5.23)}\qquad \begin{aligned} &\partial_t u \in L^2(I, L^2(G)), \quad \partial_t^2 u \in L^2(I, V_0^*),\\ &\partial_t T \in L^2(I, V_1), \quad \partial_t^2 T \in L^2\big(I, L^2(G)\big), \end{aligned}$$

and it depends continuously on given data u_0, u_1, T_0, T_1, F, Q.

Remark 5.2. The spaces V_0 and V_1 appearing in Theorem 5.6 are given as follows: V_0 by formula (5.5) and $V_1 = \{v\colon v \in H^1(G)\colon v|_{\partial G_2} = 0\}$.

The proof of Theorem 5.6 runs similarly (cf. [20]) as the proof of Theorem 5.1.

6. Regularity of the solution of the boundary-initial value problems for thermal stresses equations of classical thermomechanics. In this section we introduce the theorems about the regularity of the solutions for the boundary-initial value problems for thermal stresses equations of classical thermomechanics.

At first, we formulate two theorems about regularity of the solution to problem (I).

THEOREM 6.1. (regularity with respect to t). *If the following supplementary conditions are satisfied*

$$F \in W_2^{k-1}(I, \boldsymbol{L}^2(G)), \quad q \in W_2^{k-1}(I, H^{-1}(G)), \quad k \geqslant 1, \tag{6.1}$$

$$\partial_t^l u(0) \in \boldsymbol{H}_0^1(G) \quad \textit{for } l = 0, \ldots, k-1; \ \partial_t^k u(0) \in \boldsymbol{L}^2(G); \tag{6.2}$$

$$\partial_t^l T(0) \in H_0^1(G) \quad \textit{for } l = 0, \ldots, k-1, \tag{6.3}$$

then under these additional hypothesis the solution (u, T) *of problem* (I) *has the additional regularity*

$$\begin{gathered}(u, T) \in W_2^{k-1}(I, \boldsymbol{H}_0^1(G)) \times W_2^{k-1}(I, H_0^1(G)),\\ \partial_t^k u \in L^2(I, \boldsymbol{L}^2(G)), \quad \partial_t^{k+1} u \in L^2(I, \boldsymbol{H}^{-1}(G)), \quad \partial_t^k T \in L^2(I, H^{-1}(G)).\end{gathered} \tag{6.4}$$

Remark 6.1. Conditions (6.1)–(6.3) are conditions of the regularity for u_0, u_1, T_0, $F(0)$, $q(0)$.

The proof of Theorem 6.1 is carried out using the principle of mathematical induction and basing (cf. [17], [18]) on Theorem 4.2.

THEOREM 6.2. (regularity with respect to x). *Let* u_0, u_1, T_0, F, q *be so regular that the solution* (u, T) *of problem* (I) *satisfies the condition*

$$(u, T) \in W_2^k(I, \boldsymbol{H}_0^1(G)) \times W_2^k(I, H_0^1(G)). \tag{6.5}$$

Moreover, it is assumed that

$$F \in W_2^{k-2}(I, \boldsymbol{H}^m(G)), \quad q \in W_2^{k-1}(I, H^m(G)), \quad m \geqslant 1. \tag{6.6}$$

Then under this additional hypothesis the solution (u, T) *of problem* (I) *has the additional regularity*

$$(u, T) \in W_2^{k-2l}(I, \boldsymbol{H}^{2l}(G) \cap \boldsymbol{H}_0^1(G)) \times W_2^{k-l}(I, H^{2l+1}(G) \cap H_0^1(G)) \tag{6.7}$$

for $l \in N$ *satisfying the conditions* $2l \leqslant k$ *and* $2l-1 \leqslant m$.

The above theorem follows from regularity theorems for elliptic (cf. [53], [32]) differential operator.

EXAMPLE 6.1. From Theorems 6.1, 6.2 and Sobolev's imbedding theorem ([11]) (cf. [48], p. 77–78, [53], p. 184) it follows that if the following

([11]) $W_2^l((0, \vartheta), W_2^l(G)) \subsetneq C^{l'}([0, \vartheta]) \times \bar{G})$ if $l-l' > (r+1)/2$, conditions are satisfied:

$$F \in W_2^{12}(I, L^2(G)) \cap W_2^{10}(I, H^5(G)),$$

$$q = W_2^{11}(I, H^5(G)), \quad \partial_t^{12} q \in L^2(I, H^{-1}(G)), \tag{6.8}$$

$$u_0 \in H_0^{12}(G), \quad u_1 \in H^{12}(G) \cap H_0^{11}(G), \quad T_0 \in H^{23}(G) \cap H_0^{25}(G),$$

$$F(0) \in H_0^{11}(G), \quad \partial_t F(0) \in H^{10}(G) \cap H_0^9(G), \quad \partial_t^2 F(0) \in H_0^9(G),$$

$$\partial_t^3 F(0) \in H^8(G) \cap H_0^7(G), \quad \partial_t^4 F(0) \in H_0^7(G), \quad \partial_t^5 F(0) \in H^6(G) \cap H_0^3(G),$$

$$\partial_t^6 F(0) \in H_0^3(G), \ \partial_t^7 F(0) \in H^4(G) \cap H_0^3(G), \ \partial_t^8 F(0) \in H_0^3(G),$$

$$\partial_t^9 F(0) \in H^2(G) \cap H_0^1(G),$$

$$\partial_t^{10} F(0) \in H_0^1(G), \quad \partial_t^{11} F(0) \in \mathscr{L}^2(G), \quad q(0) \in H^{21}(G) \cap H_0^{23}(G),$$

$$\partial_t q(0) \in H^{19}(G) \cap H_0^{21}(G), \quad \partial_t^2 q(0) \in H^{17}(G) \cap H_0^{19}(G),$$

$$\partial_t^3 q(0) \in H^{15}(G) \cap H_0^{17} G, \quad \partial_t^4 q(0) \in H^{13}(G) \cap H_0^{15}(G),$$

$$\partial_t^5 q(0) \in H^{11}(G) \cap H_0^{13}(G),$$

$$\partial_t^6 q(0) \in H^9(G) \cap H_0^{11}(G), \quad \partial_t^7 q(0) \in H^7(G) \cap H_0^9(G), \quad \partial_t^8 q(0) \in H^5(G) \cap H_0^7(G),$$

$$\partial_t^9 q(0) \in H^3(G) \cap H_0^5(G), \quad \partial_t^{10} q(0) \in H^1(G) \cap H_0^3(G), \quad \partial_t^{11} q(0) \in H_0^1(G).$$

Then the solution (u, T) of problem (I) for homogeneous izotropic medium $(r = 3)$ ([12]) has the regularity

$$(u, T) \in C^2([0, \vartheta] \times \bar{G}) \times C^4([0, \vartheta] \times \bar{G}). \tag{6.10}$$

Now, we mention the regularity theorems for problem (V^a).

THEOREM 6.3 (regularity with respect to t). *If the following supplementary conditions are satisfied*

$$\Psi_F \in W_2^k(I, V_0^*), \quad q \in W_2^{k-1}(I, V_1^*), \quad k \geqslant 1. \tag{6.11}$$

$$\begin{aligned} &\partial_t^l u(0) \in V_0 \quad \text{for } l = 0, \ldots, k-1; \quad \partial_t^k u(0) \in L^2(G), \\ &\partial_t^l T(0) \in V_1 \quad \text{for } l = 0, \ldots, k-1, \end{aligned} \tag{6.12}$$

then the solution (u, T) *of problem* (V^a) *has the additional regularity*

$$\begin{gathered} (u, T) \in W_2^{k-1}(I, V_0) \times W_2^{k-1}(I, V_1), \\ \partial_t^k u \in L^2(I, L^2(G)), \quad \partial_t^{k+1} u \in L^2(I, V_0^*), \quad \partial_t^k T \in L^2(I, V_1^*). \end{gathered} \tag{6.13}$$

([12]) In this case $A(\cdot, \partial) = A(\partial) = (\lambda + \mu)\,\mathrm{grad\,div} + \mu\Delta$, $B^+(\cdot, \partial) = B^+(\partial) = \gamma\,\mathrm{grad}$, $a(\cdot, \partial) = a(\partial) = \Delta$, where $\Delta = \partial_{jj}^2$ (cf. [34], [44]), $j = 1, \ldots, r$.

Remark 6.2. In order that the functional $\Psi_F(\cdot)$ to satisfy conditions (6.11) it is sufficient that $F \in W_2^k(I, L^2(G))$ and the extensions $\tilde{U}$, $\tilde{S}_R$ (cf. [20]) to $I \times G$ of the functions U and S_R have the properties

$$\tilde{U}|_{I \times \partial G} \in W_2^{k+2}(I, H^{1/2}(\partial G)), \quad \tilde{S}|_{I \times \partial G} \in W_2^k(I, L^2(\partial G)), \quad k \geqslant 1.$$

The proof of Theorem 6.3 is similar to the proof of Theorem 6.1.

THEOREM 6.4. (regularity with respect to x). *Let u_0, u_1, T_0, Ψ_F, q be so regular that the solution (u, T) of problem (V^a) satisfies the following conditions*:

$$(u, T) \in W_2^k(I, V_0) \times W_2^k(I, V_1). \tag{6.14}$$

Moreover, we assume that

$$\Psi_F \in W_2^{k-2}(I, H^m(G)), \quad q \in W_2^{k-1}(I, H^m(G)), \quad m \geqslant 1. \tag{6.15}$$

Then under this additional hypothesis the solution of problem (V^a) has the additional regularity

$$(u, T) \in W_2^{k-2l}(I, H^{2l}(\underset{\sim}{G})) \times W_2^{k-l}(I, H^{2l+1}(\underset{\sim}{G})) \tag{6.16}$$

for $l \in N$ satisfying the conditions $2l \leqslant k$, $2l-1 \leqslant m$, where $\bar{\underset{\sim}{G}} \subset G$.

The above theorem follows from the theorem of internal smoothness for the elliptic operators (cf. [12], p. 24, [32], p. 235).

Remark 6.3. The proofs of the theorems on regularity with respect to t of the solution of problems (II), (III), (IV), (V^b) run similiarly to the proofs of Theorems 6.1 and 6.3. The proofs of the theorems on regularity with respect to x of the above problems are analogous to the proof of the Theorems 6.2, 6.4.

7. Regularity of the solution of the boundary-initial value problems for thermal stresses equations of generalized thermomechanics. Below we introduce theorems about the regularity of the solution of the boundary-initial value problems for thermal stresses equations of generalized thermomechanics.

In the case of problem $(\tilde{\mathrm{I}})$ with homogeneous boundary conditions $(\tilde{\mathrm{I}})$ the following theorems (cf. [19]) are true:

THEOREM 7.1 (regularity with respect to t). *If the following supplementary conditions are satisfied*

$$F \in W_2^{k-1}(I, L^2(G)), \quad Q \in W_2^{k-1}(I, L^2(G)), \quad k \geqslant 1, \tag{7.1}$$

$$\partial_t^l u(0) \in H_0^1(G) \quad \text{for } l = 0, \ldots, k-1; \ \partial_t^k u(0) \in L^2(G), \tag{7.2}$$

$$\partial_t^l T(0) \in H_0^1(G) \quad \text{for } l = 0, \ldots, k-1; \ \partial_t^k T(0) \in L^2(G), \tag{7.3}$$

then under this additional hypothesis the solution (u, T) of problem $(\tilde{\mathrm{I}})$ has the *additional regularity*

$$(7.4)\qquad \begin{gathered}(u,T)\in W_2^{k-1}(I,H_0^1(G))\times W_2^{k-1}(I,H_0^1(G)),\\ \partial_t^k u\in L^2(I,L^2(G)),\quad \partial_t^{k+1}u\in L^2(I,H^{-1}(G)),\\ \partial_t^k T\in L^2(I,L^2(G)),\quad \partial_t^{k+1}T\in L^2(I,H^{-1}(G)).\end{gathered}$$

The proof of the above theorem runs in the same way as the proof of Theorem 6.1.

THEOREM 7.2. (regularity with respect to x). *Let u_0, u_1, T_0, T_1, F, Q be so regular that the solution (u, T) of problem $(\tilde{\mathrm{I}})$ satisfies the following conditions*

$$(7.5)\qquad (u,T)\in W_2^k(I,H_0^1(G))\times W_2^k(I,H_0^1(G)).$$

Moreover, we assume

$$(7.6)\qquad F\in W_2^{k-2}(I,H^m(G)),\quad Q\in W_2^{k-2}(I,H^m(G)),\quad m\geqslant 1.$$

Then under this additional hypothesis the solution (u, T) of problem $(\tilde{\mathrm{I}})$ has the additional regularity

$$(7.7)\quad (u,T)\in W_2^{k-2l}(I,H^{2l}(G)\cap H_0^1(G))\times W_2^{k-2l}(I,H^{2l+1}(G)\cap H_0^1(G))$$

for $l\in N$ satisfying the conditions: $2l\leqslant k$ *and* $2l-1\leqslant m$.

The proof of Theorem 7.2 is similar to the proof of Theorem 6.2.

Now we formulate the regularity theorems for problem $(\tilde{\mathrm{V}}^a)$.

THEOREM 7.3 (regularity with respect to t). *If the following supplementary conditions are satisfied*

$$(7.8)\qquad \Psi_F\in W_2^k(I,V_0^*),\quad Q\in W_2^k(I,L^2(G)),\quad k\geqslant 1,$$

$$(7.9)\qquad \partial_t^l u(0)\in V_0\quad \text{for } l=0,\ldots,k-1,\quad \partial_t^k u(0)\in L^2(G),$$

$$(7.10)\qquad \partial_t^l T(0)\in V_1\quad \text{for } l=0,\ldots,k-1,\ \partial_t^k T(0)\in L^2(G),$$

then the solution (u, T) of problem $(\tilde{\mathrm{V}}^a)$ has the additional regularity

$$(7.11)\qquad \begin{gathered}(u,T)\in W_2^{k-1}(I,V_0)\times W_2^{k-1}(I,V_1),\\ \partial_t^k u\in L^2(I,L^2(G)),\quad \partial_t^{k+1}u\in L^2(I,V_0^*),\\ \partial_t^k T\in L^2(I,L^2(G)),\quad \partial_t^{k+1}T\in L^2(I,V_1^*).\end{gathered}$$

Remark 7.1. In order that the functional $\Psi_F(\cdot)$ should satisfy condition (7.8) it is sufficient that F, S_R, U satisfy the same conditions as in Remark 6.1.

The proof of Theorem 7.3 is similar to the proof of Theorem 6.3.

THEOREM 7.4. (regularity with respect to x). *Let u_0, u_1, T_0, T_1, F, Q be* *so regular that the solution* (u, T) *of problem* $(\widetilde{\mathrm{V}}^a)$ *satisfies the conditions*

$$(7.12) \qquad (u, T) \in W_2^k(I, V_0) \times W_2^k(I, V_1).$$

Moreover, we assume

$$(7.13) \qquad \Psi_F \in W_2^{k-2}(I, \boldsymbol{H}^m(G)), \qquad Q \in W_2^{k-2}(I, H^m(G)), \qquad m \geqslant 1.$$

Then under this additional hypothesis the solution of problem $(\widetilde{\mathrm{V}}^a)$ *has the additional regularity*

$$(7.14) \qquad (u, T) \in W_2^{k-2l}(I, \boldsymbol{H}^{2l}(\underset{\sim}{G})) \times W_2^{k-2l}(I, H^{2l+1}(\underset{\sim}{G}))$$

for $l \in N$ *satisfying the conditions* $2l \leqslant k$, $2l-1 \leqslant m$, *where* $\bar{\underset{\sim}{G}} \subset G$.

The above theorems follow from the theorem on internal smoothness for the elliptic operators (cf. [12], p. 24, [32], p. 235).

Remark 7.2. The proofs of the theorems of regularity with respect to t of the solution of problems $(\widetilde{\mathrm{II}})$, $(\widetilde{\mathrm{III}})$, $(\widetilde{\mathrm{IV}})$, $(\widetilde{\mathrm{V}}^b)$ run similiarly to the proofs of Theorems 7.1 and 7.3.

The proofs of the theorems of regularity with respect to x of the above problems are analogous to the proofs of the Theorems 7.3 and 7.4.

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