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Boundary value problems for Poisson equation in the domain E_n^+

1. The aim of this paper is to construct the solutions of certain boundary problems for Poisson equation

$$(1) \quad \Delta u(X) = f(X),$$

where $X = (x_1, \dots, x_n)$ denotes a point of n -dimensional Euclidean space E_n ($n \geq 2$) and $\Delta = \sum_{i=1}^n D_{x_i}^2$ denotes the Laplace operator, in the domain

$$E_n^+ = D = \{X: x_i > 0, i = 1, \dots, n\};$$

f is a given function defined for $X \in D$. Let

$$S_i^+ = \{X: x_i = 0, x_k > 0, k \in \{1, \dots, n\} \setminus \{i\}\}, \quad i = 1, \dots, n.$$

We shall look for the solutions of equation (1) which are regular or biregular in $D \cup S_1^+ \cup \dots \cup S_n^+$ and satisfy on subsets S_i^+ ($i = 1, \dots, n$) of the boundary of D the Dirichlet or Neumann conditions or the boundary value conditions of the third kind. To construct the solutions of those problems we shall use the suitable Green's functions. Green's functions for Poisson equation and boundary-value problem of the third kind in E_3^+ were presented in [2]. The boundary value problems of all three kinds for equation (1) were solved in [5] and [6] for E_3^+ and E_2^+ , respectively. The solutions of Dirichlet and Neumann problems for the Laplace equation in E_n^+ and the solution of Dirichlet problem for equation (1) in E_3^+ were given in [8]. Green's function for Laplace equation in E_3^+ with boundary conditions of the third kind on S_1^+ , S_2^+ and with the Dirichlet condition on S_3^+ was presented in [9]. The object of this paper is a generalization of results of [2], [5], [6], [8]. Following [3], [4], we introduce the operation \circ and we derive with its help the general formulae representing the Green's functions and certain boundary problems for (1) in E_n^+ . These general formulae may also be used for representation of particular Green's functions and boundary problems defined in [2], [5], [6], [8].

2. Let us consider the sets $N = \{1, \dots, n\}$, $W_i = \{0, 1, v_i\}$, where $v_i > 0$, $i \in N$. Let $A = W_1 \times \dots \times W_n$, $B = W \times \dots \times W$ (n -times). We shall consider the subsequences $(v_{n_1}, \dots, v_{n_k})$, $k \in N$, of the sequence (v_1, \dots, v_n) and the subsets A_{n_1, \dots, n_k} of the set $A \setminus B$ defined as follows:

$$A_{n_1, \dots, n_k} = C_1(n_1, \dots, n_k) \times \dots \times C_n(n_1, \dots, n_k),$$

where

$$C_i(n_1, \dots, n_k) = \begin{cases} \{v_i\} & \text{for } i \in \{n_1, \dots, n_k\}, \\ W & \text{for } i \in N \setminus \{n_1, \dots, n_k\}, \end{cases} \quad i \in N.$$

In virtue of the foregoing definitions we obtain the equality $\bigcup A_{n_1, \dots, n_k} = A \setminus B$, where we sum over all subsequences (n_1, \dots, n_k) of the sequence $(1, \dots, n)$. Let us denote by (e_1, \dots, e_n) the basis of the space E_n of the form $e_i = (e_{i1}, \dots, e_{in})$ with $e_{ii} = 1$ and $e_{ik} = 0$ for $i \neq k$ ($i, k \in N$). The elements $a = (a_1, \dots, a_n) \in A$ will be identified with the vectors $a_1 e_1 + \dots + a_n e_n$. Let $X \in E_n$, $X_a = (x_1^{a_1}, \dots, x_n^{a_n})$, where

$$x_i^{a_i} = \begin{cases} x_i & \text{for } a_i = 0, \\ -x_i & \text{for } a_i = 1, \\ -x_i - v_i & \text{for } a_i = v_i \end{cases}$$

and $\lim_{v_i \rightarrow 0^+} x_i^{v_i} = -x_i$, $i \in N$, $X_{(0, \dots, 0)} = X$. Let $Y = (y_1, \dots, y_n) \in E_n$ and let $r_a = |Y - X_a| = \left[\sum_{i=1}^n (y_i - x_i^{a_i})^2 \right]^{1/2}$ denote the distance of the points X_a and Y , $r_{(0, \dots, 0)} = r = |Y - X|$. Let $V(r)$ be a function defined for $r > 0$. We shall write V_a for $V(r_a)$ ($a \in A$).

Let $a \in A_{n_1, \dots, n_k}$ ($k \in N$) and let

$$(2) \quad I(V_a) = (2h)^k \int_{R_{n_1, \dots, n_k}^+} V_a \exp[h(v_{n_1} + \dots + v_{n_k})] dv_{n_1} \dots dv_{n_k},$$

where h is a fixed negative number, and

$$R_{n_1, \dots, n_k}^+ = \{(v_{n_1}, \dots, v_{n_k}) : v_{n_i} \geq 0 \ (i = 1, \dots, k)\} \quad (k \in N).$$

Let us assume that the functions V_a ($a \in A$), $I(V_a)$ ($a \in A \setminus B$) depending on variables X, Y, v_1, \dots, v_n are defined in some non-empty set $A \subset E_{2n} \times R_{1, \dots, n}^+$.

DEFINITION 1. In the set of all functions $V_a, I(V_a)$ where $a \in A, a' \in A \setminus B$, we define the operation \circ as follows:

$$1^\circ \quad V_a \circ V_{a'} = V_{a+a'} \text{ for } a, a', a+a' \in A,$$

$$2^\circ \quad V_a \circ I(V_{a'}) = I(V_{a+a'}) \text{ for } a \in B, a' \in A \setminus B, a+a' \in A \setminus B,$$

$$3^\circ I(V_a) \circ I(V_{a'}) = I(V_{a+a'}) \text{ for } a, a', a+a' \in A \setminus B,$$

In virtue of Definition 1, the operation \circ is commutative and associative and has a neutral element V . We shall assume that this operation is also distributive with respect to addition and that the fixed factors may be taken outside the operation and multiplied.

3. Now let us consider the following sets:

$$D_i = \{X: x_i > 0\}, \quad S_i = \{X: x_i = 0\}, \quad i \in N; \quad \hat{D} = D \cup S_1^+ \cup \dots \cup S_n^+.$$

It is well known that the fundamental solution of Laplace's equation is of the form

$$(3) \quad \begin{aligned} U(r) &= (2\pi)^{-1} \ln(r^{-1}) && \text{for } n = 2, \\ U(r) &= [(n-2)Q_n]^{-1} r^{-n+2} && \text{for } n > 2, \end{aligned}$$

where Q_n denotes the measure of the surface of the unit n -dimensional ball. Let us consider the following integrals

$$(4) \quad H(D_X^\alpha U_a) = (2h)^k \int_{R_{n_1, \dots, n_k}^+} D_X^\alpha U_a \exp[h(v_{n_1} + \dots + v_{n_k})] dv_{n_1} \dots dv_{n_k},$$

where $a \in A_{n_1, \dots, n_k}$, $k \in N$, α denotes a multiindex $(\alpha_1, \dots, \alpha_n)$, D_X^α denotes the derivative $D_{x_1}^{\alpha_1} \dots D_{x_n}^{\alpha_n}$ of the order $|\alpha| = \alpha_1 + \dots + \alpha_n$, $|\alpha| = 0, 1, 2, \dots$

Let $N_a = \{i \in N: a_i \neq 0\}$. We shall prove the following

LEMMA 1. If $i \in N_a$, then

1° the integrals $H(D_X^\alpha U_a)$ given in (4) are locally uniformly convergent at every point $(X, Y) \in D_i \times (D_i \cup S_i) [(D_i \cup S_i) \times D_i]$,

2° $D_X^\alpha I(U_a) = H(D_X^\alpha U_a)$ for $(X, Y) \in D_i \times (D_i \cup S_i)$, $[(D_i \cup S_i) \times D_i]$.

Proof. Let $K(\bar{X}, \eta)$ denote the ball with centre $\bar{X} = (\bar{x}_1, \dots, \bar{x}_n)$ and radius $\eta > 0$, $K(\bar{X}, \eta) \subset D_i$, $i \in N_a$, and let $K(\bar{Y}, \eta_1)$ denote the ball with centre $\bar{Y} = (\bar{y}_1, \dots, \bar{y}_n) \in D_i \cup S_i$ and radius $\eta_1 > 0$.

While

$$(5) \quad \begin{aligned} r_a &\geq \bar{x}_i - \eta && \text{for } X \in K(\bar{X}, \eta), Y \in D_i \cup S_i, (v_{n_1}, \dots, v_{n_k}) \in R_{n_1, \dots, n_k}^+, \\ r_a &\leq M(v_{n_1} + \dots + v_{n_k}) && \text{for } X \in K(\bar{X}, \eta), Y \in K(\bar{Y}, \eta_1) \cap (D_i \cup S_i), \\ v_{n_s} &\geq M_1 && (s = 1, \dots, k), \end{aligned}$$

where M, M_1 are suitably chosen positive constants, then due to (3) and (5) we obtain

$$(6) \quad \begin{aligned} |U_a| &\leq M_2 && \text{if } n \geq 3, \\ |D_X^\alpha U_a| &\leq M_2 && \text{if } n \geq 2, |\alpha| \geq 1, \end{aligned}$$

for $X \in K(\bar{X}, \eta)$, $Y \in D_i \cup S_i$, $(v_{n_1}, \dots, v_{n_k}) \in R_{n_1, \dots, n_k}^+$, and

$$(7) \quad \begin{aligned} |U_a| &\leq M_3(v_{n_1} + \dots + v_{n_k}) \quad \text{for } n = 2, Y \in K(\bar{Y}, \eta) \cap (D_i \cup S_i), \\ v_{n_s} &\geq M_1, \quad s = 1, \dots, k, X \in K(\bar{X}, \eta), \end{aligned}$$

M_i ($i = 2, 3$) being convenient positive constants.

From (6), (7) it follows that the integrals (4) are locally uniformly convergent at the point $(\bar{X}, \bar{Y}) \in D_i \times (D_i \cup S_i)$, $i \in N_a$.

The proof of local uniform convergence of the integrals (4) at any point $(X, Y) \in (D_i \cup S_i) \times D_i$, $i \in N_a$ is analogous.

2° is a consequence of 1°.

Let us consider the following functions:

$$(8) \quad \begin{aligned} G_{1e_i} &= G_{1e_i}(X, Y) = U + U_{e_i}, \\ G_{2e_i} &= G_{2e_i}(X, Y) = U - U_{e_i}, \\ G_{3e_i} &= G_{3e_i}(X, Y) = U + U_{e_i} + I(U_{v_i e_i}) \end{aligned}$$

for $i \in N$; the function U is given in (3).

We shall prove the following

LEMMA 2. *The functions G_{ke_i} ($k = 1, 2, 3$; $i \in N$), given by (8) have the following properties:*

1° G_{ke_i} are defined and of class C^∞ for $X \neq Y$,

$$(X, Y) \in D_i \times (D_i \cup S_i) [(D_i \cup S_i) \times D_i].$$

2° G_{ke_i} satisfy Laplace equation with respect to $X \in D_i$, $X \neq Y \in D_i \cup S_i$,

3° (a) $D_{x_i} G_{1e_i} \rightarrow 0$, (b) $G_{2e_i} \rightarrow 0$, (c) $(D_{x_i} + h)G_{3e_i} \rightarrow 0$ as $X \rightarrow X_i \in S_i$, $X \in D_i$, Y is a fixed point belonging to the set S_i , $i \in N$, $Y \neq X$.

4° (a) $D_{y_i} G_{1e_i} \rightarrow 0$, (b) $G_{2e_i} \rightarrow 0$, (c) $(D_{y_i} + h)G_{3e_i} \rightarrow 0$ as $Y \rightarrow Y_i \in S_i$, $Y \in D_i$, X is a fixed point of the set D_i , $Y \neq X$, $i \in N$.

Proof. We shall prove assertions 1°–4° only for the function G_{3e_i} because the proof for the other functions is similar. While the functions U , U_{e_i} , $U_{v_i e_i}$ satisfy the Laplace equation as the functions of $X \in D_i$ with fixed $Y \in D_i$, $X \neq Y$, then in virtue of Lemma 1 the function G_{3e_i} has properties 1°, 2°. We shall prove that the function G_{3e_i} satisfies the boundary condition 3°. By Lemma 1 and the rule of integration by parts for the integral $D_{x_i} I(U_{v_i e_i})$ we obtain

$$(9) \quad (D_{x_i} + h)G_{3e_i} = D_{x_i} G_{1e_i} + hG_{1e_i} - 2hU_{e_i} \quad \text{for } X, Y \in D_i, X \neq Y.$$

In virtue of (8), (9), 3°(a), we get 3°(c). The proof of 4°(c) is analogous. That completes the proof.

Let $\Phi = \{1, 2, 3\} \times \dots \times \{1, 2, 3\}$ (n -times) and let $\varphi = (\varphi_1, \dots, \varphi_n) = \varphi_1 e_1 + \dots + \varphi_n e_n$ be a fixed point of the set Φ . Consider the function G_φ of the form

$$(10) \quad G_\varphi = G_\varphi(X, Y) = G_{\varphi_1 e_1}(X, Y) \circ \dots \circ G_{\varphi_n e_n}(X, Y),$$

where $G_{\varphi_i e_i}$, $i \in N$, is given by (8).

LEMMA 3. *The functions G_φ given by (10), where $\varphi = \varphi_1 e_1 + \dots + \varphi_n e_n \in \Phi$ have the following properties:*

1° G_φ are defined and of class C^∞ for $(X, Y) \in D \times \hat{D}$

$$[\hat{D} \times D, (D \cup S_i^+) \times (\hat{D} \setminus S_i^+), (\hat{D} \setminus S_i^+) \times (D \cup S_i^+)], \quad X \neq Y, i \in N;$$

2° G_φ satisfy Laplace equation as the functions of $X \in D$ with fixed $Y \in \hat{D}$, $Y \neq X$;

3° (a) for $\varphi_i = 1$, $D_{x_i} G_\varphi \rightarrow 0$, (b) for $\varphi_i = 2$, $G_\varphi \rightarrow 0$; (c) for $\varphi_i = 3$, $(D_{x_i} + h) G_\varphi \rightarrow 0$ as $X \rightarrow X_i \in S_i^+$, $X \in D$, $Y \in \hat{D} \setminus S_i^+$, $X \neq Y$, $i \in N$;

4° (a) for $\varphi_i = 1$, $D_{y_i} G_\varphi \rightarrow 0$, (b) for $\varphi_i = 2$, $G_\varphi \rightarrow 0$, (c) for $\varphi_i = 3$, $(D_{y_i} + h) G_\varphi \rightarrow 0$, as $Y \rightarrow Y_i \in S_i^+$, $Y \in D$, $X \in \hat{D} \setminus S_i^+$, $Y \neq X$, $i \in N$.

PROOF. By (8), (10) and Definition 1 of the operation \circ the function G_φ ($\varphi \in \Phi$) is a linear combination of the functions U_a and $I(U_{a'})$, where $a \in B$, $a' \in A \setminus B$. Then in virtue of Lemma 1 and the fact that U_a ($a \in A$) satisfies the Laplace equation as the function of $X \in D$, $X \neq Y$ with fixed $Y \in D$, we obtain thesis 1° and 2° of Lemma 3. To prove 3° we shall show only that the function G_φ satisfies boundary conditions 3° in the case $i = 1$. The proof that G_φ satisfies boundary conditions 3° for $i \in N \setminus \{1\}$ is analogous. The function G_φ is a linear combination of the functions $G_{\varphi_1 e_1} \circ U_b$ and $G_{\varphi_1 e_1} \circ I(U_{b'})$, where $b = (0, b_2, \dots, b_n) \in B$,

$$b' = (0, b'_2, \dots, b'_n) \in A_{n_1, \dots, n_k}, \quad \{n_1, \dots, n_k\} \subset N \setminus \{1\}.$$

By Definition 1 and Lemma 1, the following equalities hold:

$$(11) \quad \begin{aligned} G_{\varphi_1 e_1} \circ U_b &= G_{\varphi_1 e_1}|_{x_i = x_i^{b_i}}, \quad i \in N \setminus \{1\}, \\ G_{\varphi_1 e_1} \circ I(U_{b'}) &= [(2h)^k \int_{R_{n_1, \dots, n_k}^+} G_{\varphi_1 e_1} \exp[h(v_{n_1} + \dots + v_{n_k})]|_{x_i = x_i^{b_i}, i \in N \setminus \{1\}} dv_{n_1} \dots dv_{n_k} \end{aligned}$$

for $X \in D \cup S_1^+$, $Y \in \hat{D} \setminus S_1^+$.

Thesis 3° of Lemma 3 follows now from Lemma 1, 3° of Lemma 2 and formula (9). The proof of 4° is analogous. That completes the proof.

4. To construct the solution of the boundary-value problems we shall avail the function G_φ given in (10) and their properties proved in Lemma 3.

Let $y^i = (y_1, \dots, y_{i-1}, y_{i+1}, \dots, y_n)$ denote the projection of the point Y

on the plane $y_i = 0$ ($i \in N$), if we identify this plane with the space E_{n-1} . We shall write Z_i for the set

$$Z_i = \{y^i: y_k > 0, k \in N \setminus \{i\}\}, \quad i \in N.$$

Let $\varphi = (\varphi_1, \dots, \varphi_n)$ be a fixed point of Φ . Consider the functions

$$(12) \quad u_\varphi^j(X) = (-1)^{\varphi_j} \int_{Z_j} f_j^{\varphi_j}(y^j) D_{y_j}^{p_{\varphi_j}} G_\varphi(X, Y)|_{y_j=0} dy^j,$$

where $p_{\varphi_j} = 0$ for $\varphi_j = 1, 3$; $p_{\varphi_j} = 1$ for $\varphi_j = 2$; $f_j^{\varphi_j}(y^j)$ being a given function defined on Z_j ($j \in N$), and the function

$$(13) \quad z_\varphi(X) = - \int_D f(X) G_\varphi(X, Y) dY.$$

We shall prove that under suitable assumptions on the functions $f_j^{\varphi_j}$, $j \in N$, f , that the function

$$(14) \quad w_\varphi(X) = z_\varphi(X) + u_\varphi(X),$$

where

$$(15) \quad u_\varphi(X) = \sum_{i=1}^n u_\varphi^i(X),$$

is a solution of equation (1) satisfying the following boundary conditions:

$$(16) \quad \begin{aligned} D_{x_j} w_\varphi(X) &= f_j(x^j) & \text{as } \varphi_j = 1; \\ w_\varphi(X) &= f_j^2(x^j) & \text{as } \varphi_j = 2, \\ (D_{x_j} + h) w_\varphi(X) &= f_j^3(x^j) & \text{as } \varphi_j = 3 \end{aligned}$$

for $X \in S_j^+$, $j \in N$.

5. In this section we deal with the functions u_φ^j ($j \in N$) and the function u_φ given in formulae (12), (15). We shall prove under suitable assumptions on the functions $f_j^{\varphi_j}$ ($j \in N$) that the function u_φ is a solution of Laplace equation satisfying the boundary conditions (16).

Let us consider the functions

$$(17) \quad \begin{aligned} D_{x_i} D_{y_j}^{p_{\varphi_j}} G_\varphi(X, Y) & \quad \text{as } \varphi_i = 1, \\ D_{y_j}^{p_{\varphi_j}} G_\varphi(X, Y) & \quad \text{as } \varphi_i = 2, \\ (D_{x_i} + h) D_{y_j}^{p_{\varphi_j}} G_\varphi(X, Y) & \quad \text{as } \varphi_i = 3, \end{aligned}$$

where $\varphi = (\varphi_1, \dots, \varphi_n) \in \Phi$, $(X, Y) \in D \times S_j^+$, $j \in N \setminus \{i\}$, $i \in N$.

We shall prove the following.

LEMMA 4. *The functions defined in formula (17) tend to zero as $X \rightarrow X_i \in S_i^+$, $X \in D$, $j \in N \setminus \{i\}$, $i \in N$.*

Proof. We shall prove Lemma 4 only for the case $j \in N \setminus \{1\}$ because for $j \in N \setminus \{i\}$, $i = 2, \dots, n$, the proof is similar. When $\varphi_j = 1, 3$; $j \in N \setminus \{1\}$, the thesis of our lemma follows from Lemma 3, immediately. Now let $\varphi_j = 2$, $j \in N \setminus \{1\}$. While the functions G_φ defined in (10) are linear combinations of the functions $G_{\varphi_1 e_1} \circ U_b$ and $G_{\varphi_1 e_1} \circ I(U_{b'})$ given in (11), then it is sufficient to show that the functions

$$(18) \quad D_{x_1} D_{y_j} (G_{1e_1} \circ U_b), \quad D_{y_j} (G_{2e_1} \circ U_b), \quad (D_{x_1} + h) D_{y_j} (G_{3e_1} \circ U_b)$$

and the functions

$$(19) \quad D_{x_1} D_{y_j} (G_{1e_1} \circ I(U_{b'})), \quad D_{y_j} (G_{2e_1} \circ I(U_{b'})), \quad (D_{x_1} + h) D_{y_j} (G_{3e_1} \circ I(U_{b'})),$$

where $(X, Y) \in D \times S_j^+$, $b = (0, b_2, \dots, b_n) \in B$, $b' = (0, b'_2, \dots, b'_n) \in A_{n_1, \dots, n_k}$, $\{n_1, \dots, n_k\} \subset N \setminus \{1, j\}$, $j \in N \setminus \{1\}$ tend to zero as $X \rightarrow X_1 \in S_1^+$, $X \in D$.

Let $M(r) = r^{-1} D_r U(r)$, $\tilde{G}_{1e_1} = M + M_{e_1}$, $\tilde{G}_{2e_1} = M - M_{e_1}$, $\tilde{G}_{3e_1} = M + M_{e_1} + I(M_{v_{1e_1}})$. By Lemma 1 and Definition 1, we obtain the equalities

$$(20) \quad D_{y_i} (G_{\varphi_1 e_1} \circ U_b) = (-1)^{b_j+1} x_j (\tilde{G}_{\varphi_1 e_1} \circ M_b) \\ = (-1)^{b_j+1} x_j \tilde{G}_{\varphi_1 e_1} \Big|_{x_s = x_s^{b_s}}, \quad s \in N \setminus \{1\}$$

for $(X, Y) \in D \times S_j^+$, and

$$(21) \quad D_{y_j} (G_{\varphi_1 e_1} \circ I(U_{b'})) \\ = (-1)^{b_j+1} x_j (2h)^k \int_{R_{n_1, \dots, n_k}^+} \tilde{G}_{\varphi_1 e_1} \exp[h(v_{n_1} + \dots + v_{n_k})] \Big|_{x_s = x_s^{b_s}, s \in N \setminus \{1\}} dv_{n_1} \dots dv_{n_k}$$

for $(X, Y) \in D \times S_j^+$.

In a similar way as in the proof of thesis 3° of Lemma 2 (for the function $G_{\varphi_1 e_1}$) we may prove that the functions $D_{x_1} \tilde{G}_{1e_1}$, \tilde{G}_2 , $(D_{x_1} + h) \tilde{G}_{3e_1}$ tend to zero as $X \rightarrow X_1 \in S_1$; $X, Y \in D_1$, $X \neq Y$. Hence, by (20), (21), the functions defined in (18), (19) tend to zero as $X \rightarrow X_1 \in S_1^+$, $X \in D$. That completes the proof.

Let $|y^j|$ denote the distance from the point y^j ($j \in N$) to the point $(0, \dots, 0) \in E_{n-1}$.

LEMMA 5. If the function $f_j(y^j)$ is measurable and bounded on Z_j ($j \in N$) and

$$\int_{Z_j} |f_j(y^j)| dy^j < \infty \quad \text{for } j \in N = \{1, 2, \dots, n\},$$

if $n = 3, 4, 5, \dots$ and $\int_{Z_j} |f_j(y^j)| |y^j| dy^j < \infty$ for $j \in N = \{1, 2\}$, and, moreover, $\varphi = (\varphi_1, \dots, \varphi_n) \in \Phi$, then

1° the integrals

$$\int_{Z_j} f_j(y^j) D_Y^\beta D_X^\alpha G_\varphi(X, Y)|_{y_j=0} dy^j,$$

where $|\alpha|, |\beta| = 0, 1, 2, \dots$, are locally uniformly convergent at any point $X \in \hat{D} \setminus S_j^+$ ($j \in N$),

$$\begin{aligned} 2^\circ \quad D_X^\alpha \int_{Z_j} f_j(y^j) D_Y^\beta G_\varphi(X, Y)|_{y_j=0} dy^j \\ = \int_{Z_j} f_j(y^j) D_Y^\beta D_X^\alpha G_\varphi(X, Y)|_{y_j=0} dy^j \quad (j \in N). \end{aligned}$$

Proof. We shall prove thesis 1° only for the case $j = 1$ while the proof for the other cases ($j \neq 1$) is similar. Observe that the definitions of the function G_φ and of the operation \circ (cf. Definition 1) yield that it is sufficient to prove local uniform convergence of the following integrals I_i , $i = 1, 2$ of the form

$$I_1(X) = \int_{Z_1} f_1(y^1) D_Y^\beta D_X^\alpha U_a|_{y_1=0} dy^1 \quad \text{for } a \in B,$$

$$I_2(X) = \int_{Z_1} f_1(y^1) D_Y^\beta D_X^\alpha I(U_{a'})|_{y_1=0} dy^1 \quad \text{for } a' \in A_{n_1, \dots, n_k}, k \in N$$

at any point $X \in \hat{D} \setminus S_1^+$. By Lemma 1, we have

$$D_Y^\beta D_X^\alpha I(U_{a'}) = (2h)^k \int_{R_{n_1, \dots, n_k}^+} D_X^\alpha D_Y^\beta U_{a'} \exp[h(v_{n_1} + \dots + v_{n_k})] dv_{n_1} \dots dv_{n_k}$$

for $a' \in A_{n_1, \dots, n_k}$, $X \in \hat{D} \setminus S_1^+$, $Y \in D \cup S_1^+$.

Let $K(\bar{X}, \eta)$ be a ball with centre $\bar{X} = (\bar{x}_1, \dots, \bar{x}_n) \in \hat{D} \setminus S_1^+$ and radius $\eta > 0$, $K(\bar{X}, \eta) \subset D_1$. Then the following inequalities hold:

$$(22) \quad r_a \geq \bar{x}_1 - \eta > 0 \quad \text{for } a \in B, X \in K(\bar{X}, \eta), Y \in S_1^+,$$

$$r_a \leq M|y^1| \quad \text{for } a \in B, X \in K(\bar{X}, \eta), |y^1| \geq M_1 > 1, y_1 = 0$$

and

$$r_{a'} \geq \bar{x}_1 - \eta > 0 \quad \text{for } a' \in A_{n_1, \dots, n_k}, X \in K(\bar{X}, \eta), Y \in S_1^+,$$

$$(23) \quad r_{a'} \leq M_2(|y^1| + v_{n_1} + \dots + v_{n_k}) \quad \text{for } a' \in A_{n_1, \dots, n_k},$$

$$X \in K(\bar{X}, \eta), |y^1| \geq M_3 > 1, y_1 = 0, v_{n_i} \geq M_4 \quad (i = 1, \dots, k),$$

where M, M_i ($i = 1, 2, 3, 4$) are suitable positive constants. By the assumptions of our lemma and formulas (3), (22), (23) it follows that the

integrals I_1, I_2 , are locally uniformly convergent at any point $\bar{X} \in \hat{D} \setminus S_1^+$. Thesis 2° is a consequence of 1°. That completes the proof.

In virtue of Lemmas 4, 5 and 3 (thesis 2°) we obtain the following

LEMMA 6. Let $\varphi = (\varphi_1, \dots, \varphi_n) \in \Phi$ and let the function $f_j^{\varphi_j}(y^j)$ be measurable and bounded on Z_j ($j \in N$). Furthermore, let

$$\begin{aligned} \int_{Z_j} |f_j^{\varphi_j}(y^j)| |y^j| dy^j < \infty & \quad \text{for } \varphi_j = 1, 3, j \in N = \{1, 2\}, \\ \int_{Z_j} |f_j^{\varphi_j}(y^j)| dy^j < \infty & \quad \text{for } \varphi_j = 2, j \in N = \{1, 2\}, \\ \int_{Z_j} |f_j^{\varphi_j}(y^j)| dy^j < \infty & \quad \text{for } j \in N = \{1, 2, 3, \dots, n\} \text{ if } n \geq 3. \end{aligned}$$

Then

1° the functions u_φ^j for $j \in N$ given in (12) satisfy the Laplace equation in the domain D ,

2° the functions u_φ^j for $j \in N$ satisfy the following boundary conditions:

$$D_{x_i} u_\varphi^j(X) = 0 \quad \text{for } X \in S_i^+ \text{ if } \varphi_i = 1,$$

$$u_\varphi^j(X) = 0 \quad \text{for } X \in S_i^+ \text{ if } \varphi_i = 2,$$

$$(D_{x_i} + h) u_\varphi^j(X) = 0 \quad \text{for } X \in S_i^+ \text{ if } \varphi_i = 3, \text{ for } i \in N \setminus \{j\}, j \in N,$$

We cite the following lemma from [1].

LEMMA 7. If the function $f(y^j)$ is measurable and bounded on Z_j and continuous at the point $\tilde{x}^j = (\tilde{x}_1, \dots, \tilde{x}_{j-1}, \tilde{x}_{j+1}, \dots, \tilde{x}_n) \in Z_j$ and, moreover, $\int_{Z_j} |f_j(y^j)| dy^j < \infty$, $j \in N$, then the function

$$L_j(X) = 2 \int_{Z_j} f_j(y^j) D_{y_j} U(r)|_{y_j=0} dy^j$$

tends to the function $f_j(\tilde{x}^j)$ as $X \rightarrow \tilde{X}_j$, where $\tilde{X}_j = (\tilde{x}_1, \dots, \tilde{x}_{j-1}, 0, \tilde{x}_{j+1}, \dots, \tilde{x}_n) \in S_j^+$, $j \in N$.

Now we shall prove the following

LEMMA 8. If the function $f_i(y^i)$ is measurable and bounded on Z_i and $\int_{Z_i} |f_i(y^i)| dy^i < \infty$ ($i \in N$), then $\int_{Z_i} f_i(y^i) D_{y_i} U_b|_{y_i=0} dy^i \rightarrow 0$ and

$$\int_{Z_i} f_i(y^i) dy^i \int_{R_{n_1, \dots, n_k}^+} D_{y_i} U_b|_{y_i=0} \exp[h(v_{n_1} + \dots + v_{n_k})] dv_{n_1} \dots dv_{n_k} \rightarrow 0$$

as $X \rightarrow \tilde{X}_i \in S_i^+$, $X \in D$, for $b \in B \setminus \{(0, \dots, 0), e_i\}$, $b' \in A_{n_1, \dots, n_k}$, $\{n_1, \dots, n_k\} \subset N \setminus \{i\}$, $i \in N$.

Proof. Let $K(\tilde{X}_i, \delta)$ be a ball with centre $\tilde{X}_i \in S_i^+$ and radius $\delta > 0$ such

that the projection of $K(\tilde{X}_i, \delta)$ on S_i is contained in S_i^+ , $i \in N$. Then there exist such positive number δ_1 that the following inequalities hold:

$$(24) \quad \begin{aligned} r_b &\geq \delta_1 && \text{for } X \in K(\tilde{X}_i, \delta) \cap (D \cup S_i^+), Y \in S_i^+, i \in N, \\ r_{b'} &\geq \delta_1 && \text{for } X \in K(\tilde{X}_i, \delta) \cap (D \cup S_i^+), Y \in S_i^+, \\ &&& (v_{n_1}, \dots, v_{n_k}) \in R_{n_1, \dots, n_k}^+. \end{aligned}$$

By (3) and (24) we obtain the following inequalities

$$(25) \quad |D_{y_i} U_b| \leq Mx_i, \quad |D_{y_i} U_{b'}| \leq Mx_i$$

for $X \in K(\tilde{X}_i, \delta) \cap (D \cup S_i^+)$, $Y \in S_i^+$, where M is a suitable positive constant. In virtue of (25) and assumptions of our lemma we obtain the thesis of Lemma 8. That completes the proof.

Now we shall prove the following.

LEMMA 9. *If $\varphi = (\varphi_1, \dots, \varphi_n) \in \Phi$ and the function $f_j^{\varphi_j}(y^j)$ is continuous and bounded on Z_j , and $\int_{Z_j} |f_j^{\varphi_j}(y^j)| dy^j < \infty$, $j \in N$, then*

$$\begin{aligned} - \int_{Z_j} f_j^1(y^j) D_{x_j} G_{\varphi}|_{y_j=0} dy^j &\rightarrow f_j^1(x^j) && \text{if } \varphi_j = 1, \\ \int_{Z_j} f_j^2(y^j) D_{y_j} G_{\varphi}|_{y_j=0} dy^j &\rightarrow f_j^2(x^j) && \text{if } \varphi_j = 2, \\ - \int_{Z_j} f_j^3(y^j) (D_{x_j} + h) G_{\varphi}|_{y_j=0} dy^j &\rightarrow f_j^3(x^j) && \text{if } \varphi_j = 3 \end{aligned}$$

as $X \rightarrow X_j \in S_j^+$, $j \in N$, $X \in D$.

Proof. We shall prove only the case $j = 1$ while for the other values of $j \neq 1$ the proof is analogous. Observe that the function G_{φ} is a linear combination of the functions $G_{\varphi_1 e_1} \circ U_b$, $G_{\varphi_1 e_1} \circ I(U_{b'})$, where $b = (0, b_2, \dots, b_n) \in B$, $b' = (0, b'_2, \dots, b'_n) \in A_{n_1, \dots, n_k}$, $\{n_1, \dots, n_k\} \subset N \setminus \{1\}$.

The derivatives of those functions are given in the following formulas

$$(26) \quad \begin{aligned} D_{x_1} (G_{1e_1} \circ U_b) &= -D_{y_1} (G_{2e_1} \circ U_b) \\ &= (D_{x_1} + h) (G_{3e_1} \circ U_b) = 2D_{x_1} U_b = -2D_{y_1} U_b && \text{for } X \in D, Y \in S_1^+, \end{aligned}$$

$$(27) \quad \begin{aligned} D_{x_1} (G_{1e_1} \circ I(U_{b'})) &= -D_{y_1} (G_{2e_1} \circ I(U_{b'})) \\ &= (D_{x_1} + h) [G_{3e_1} \circ I(U_{b'})] \\ &= 2(2h)^k \int_{R_{n_1, \dots, n_k}^+} D_{x_1} U_{b'} \exp[h(v_{n_1} + \dots + v_{n_k})] dv_{n_1} \dots dv_{n_k} \\ &= -2(2h)^k \int_{R_{n_1, \dots, n_k}^+} D_{y_1} U_{b'} \exp[h(v_{n_1} + \dots + v_{n_k})] dv_{n_1} \dots dv_{n_k} \end{aligned}$$

for $X \in D$, $Y \in S_1^+$. Formula (27) is obtained due to Lemma 1. By (26), (27) and Lemmas 7, 8 we obtain the thesis of our lemma. That completes the proof.

From Lemmas 5, 6, 9 follows

THEOREM 1. *Let $\varphi = (\varphi_1, \dots, \varphi_n)$ be a fixed point of the set Φ . Let the function $f_j^{\varphi_j}(y^j)$ be continuous and bounded on Z_j and*

$$\begin{aligned} \int_{Z_j} |f_j^{\varphi_j}(y^j)| dy^j < \infty & \quad \text{for } j \in N = \{1, 2, 3, \dots, n\}, \quad n = 3; 4, \dots, \\ \int_{Z_j} |f_j^{\varphi_j}(y^j)| y^j dy^j < \infty & \quad \text{for } \varphi_j = 1, 3, \quad j \in N = \{1, 2\}, \\ \int_{Z_j} |f_j^{\varphi_j}(y^j)| dy^j < \infty & \quad \text{for } \varphi_j = 2, \quad j \in N = \{1, 2\}. \end{aligned}$$

Then the function u_φ given by (12) and (15) satisfies Laplace equation in D and boundary conditions (16).

6. In this chapter we shall consider the function $z_\varphi(X)$ given by (13). We shall prove that under suitable assumptions on the function f the function z_φ satisfies equation (1) in D and the homogeneous boundary conditions of the form

$$(28) \quad \begin{aligned} D_{x_i} z_\varphi(X) &= 0 & \text{for } \varphi_i = 1, \\ z_\varphi(X) &= 0 & \text{for } \varphi_i = 2, \\ (D_{x_i} + h) z_\varphi(X) &= 0 & \text{for } \varphi_i = 3 \end{aligned}$$

if $X \in S_i^+$ ($i \in N$).

Now we shall prove two lemmas on the function $m(X)$ defined as follows

$$(29) \quad m(X) = - \int_D f(Y) U(r) dY,$$

where the function $U(r)$ is given by (3).

LEMMA 10. *If the function f is measurable and bounded in D and $\int_D |f(Y)| dY < \infty$ for $n = 3, 4, \dots$ and $\int_D |f(Y)| |Y| dY < \infty$ for $n = 2$, then*

1° the integral $m(X)$ and the integrals

$$m_i(X) = \int_D f(Y) D_{x_i} U(r) dY, \quad i \in N,$$

are locally uniformly convergent at every point $X \in \hat{D}$,

2° the function $m(X)$ is of class C^1 in D and

$$D_{x_i} m(X) = m_i(X) \quad \text{for } X \in \hat{D}, \quad i \in N.$$

Proof. By the definition of the function $U(r)$ for $n = 2$ it follows that there exist positive constants R, M such that

$$(30) \quad \begin{aligned} |U(r)| &\leq Mr^{-s} && \text{for } n = 2, 0 < r \leq 4R, 0 < s < 1, \\ |U(r)| &\leq r && \text{for } n = 2, r \geq 2R, \end{aligned}$$

Let $K(\bar{X}, 3R)$ be the ball with centre $\bar{X} \in \hat{D}$ and radius $3R$. Then the integrals $m(X)$ and $m_i(X)$ may be written in the following form:

$$\begin{aligned} m(X) &= - \int_{D \cap K(\bar{X}, 3R)} f(Y) U(r) dY - \int_{D \setminus K(\bar{X}, 3R)} f(Y) U(r) dY, \\ m_i(X) &= - \int_{D \cap K(\bar{X}, 3R)} f(Y) D_{x_i} U(r) dY - \int_{D \setminus K(\bar{X}, 3R)} f(Y) D_{x_i} U(r) dY. \end{aligned}$$

Let $X \in K(\bar{X}, R)$. For $Y \in K(\bar{X}, 3R)$ we have $r < R + 3R = 4R$, for $Y \in D \setminus K(\bar{X}, 3R)$ we have $r \geq 2R$. By (30) and the definition of the function $U(r)$ for $n = 3, 4, \dots$ we obtain

$$\begin{aligned} |m(X)| &\leq M_1 \int_{K(\bar{X}, 3R)} r^{-s} dY + M_2 \int_D |f(Y)| |Y| dY && \text{for } n = 2, \\ |m(X)| &\leq M_1 \int_{K(\bar{X}, 3R)} r^{-n+2} dY + M_2 \int_D |f(Y)| dY && \text{for } n \geq 3, \\ |m_i(X)| &\leq M_1 \int_{K(\bar{X}, 3R)} r^{-n+1} dY + M_2 \int_D |f(Y)| dY && \text{for } n \geq 2, \end{aligned}$$

$i \in N$, for $X \in K(\bar{X}, R)$ M_i ($i = 1, 2$) are suitable positive constants that yield that the integrals $m(X), m_i(X), i \in N$, are locally uniformly convergent at every point $\bar{X} \in \hat{D}$. Thesis 2° is a consequence of 1°.

That completes the proof.

LEMMA 11. Let the functions $f(Y), D_{y_i} f(Y), i \in N$, be continuous and bounded in D and

$$\int_D |f(Y)| |Y| dY < \infty \quad \text{for } n = 2; \quad \int_D |f(Y)| dY < \infty \quad \text{for } n = 3, 4, \dots$$

Then the function $m(X)$ is of class C^2 in D and satisfies equation (1) in D .

PROOF. Let $K(\bar{X}, 3R)$ be the ball with centre $\bar{X} \in D$ and radius $3R > 0$ contained with its boundary in D . Let $m(X) = l_1(X) + l_2(X)$, where

$$l_1(X) = - \int_{K(\bar{X}, 3R)} f(Y) U(r) dY; \quad l_2(X) = - \int_{D \setminus K(\bar{X}, 3R)} f(Y) U(r) dY.$$

Let $X \in K(\bar{X}, R)$. For $Y \in D \setminus K(\bar{X}, 3R), r \geq 2R$. By the definition of $U(r)$, the function $l_2(X)$ is of class C^2 in $K(\bar{X}, R)$ and the order of derivation and integration may be changed. While the function $U(r)$ satisfies Laplace equation with respect to the point X ($X \neq Y$), then the function $l_2(X)$ satisfies Laplace equation for $X \in K(\bar{X}, R)$. Due to Poisson theorem ([7], p.

193), the function $l_1(X)$ is of class C^2 in the ball $K(\bar{X}, R)$ and satisfies equation (1) for $X \in K(\bar{X}, R)$. That completes the proof.

LEMMA 12. *If the function f is measurable and bounded in D and*

$$\int_D |f(Y)| |Y| dY < \infty \quad \text{for } n = 2; \quad \int_D |f(Y)| dY < \infty \quad \text{for } n \geq 3,$$

then

1° the integrals $T_\varphi^\alpha(X) = - \int_D f(Y) D_X^\alpha (G_\varphi - U) dY$, $0 \leq |\alpha|$ are locally uniformly convergent at every point $X \in D$,

2° the function $T_\varphi^{(0, \dots, 0)}(X) = T_\varphi(X) = - \int_D f(Y) [G_\varphi - U] dY$ is of class C^∞ in D and $D_X^\alpha T_\varphi(X) = T_\varphi^\alpha(X)$ for $X \in D$,

3° the function $T_\varphi(X)$ satisfies Laplace equation in D .

Proof. The function $G_\varphi - U$ is a linear combination of the functions U_a and $U_{a'}$, $a \in B \setminus \{(0, \dots, 0)\}$, $a' \in A_{n_1, \dots, n_k}$ ($k \in N$). To prove 1° and 2° it is sufficient to show the local uniform convergence at every point $X \in D$ of the integrals

$$I_1^\alpha(X) = \int_D f(Y) D_X^\alpha U_a dY$$

and

$$I_2^\alpha(X) = \int_D f(Y) D_X^\alpha I(U_{a'}) dY \\ = (2h)^k \int_D f(Y) dY \int_{R_{n_1, \dots, n_k}} D_X^\alpha U_{a'} \exp[h(v_{n_1} + \dots + v_{n_k})] dv_{n_1} \dots dv_{n_k}.$$

Let $K(\bar{X}, \eta) \subset D$. Then there exist a positive constant $\delta > 0$ such that $r_a \geq \delta$, $r_{a'} \geq \delta$ for $X \in K(\bar{X}, \eta)$, $Y \in D$, $(v_{n_1, \dots, n_k}) \in R_{n_1, \dots, n_k}^+$ and positive constants M, M_1 such that

$$r_{a'} \leq M(|Y| + v_{n_1} + \dots + v_{n_k}) \quad r_a \leq M|Y|,$$

$$\text{for } X \in K(\bar{X}, \eta), |Y| \geq M_1, v_{n_s} \geq M_1 \quad (n = 1, \dots, k).$$

The local uniform convergence of the integrals I_i^α ($i = 1, 2$) at the point $\bar{X} \in D$ follows from the foregoing inequalities, definition of the function U and assumptions of our lemma. Thesis 2° is a consequence of 1°. While $G_\varphi - U$ satisfies Laplace equation with respect to $X \in D$ for fixed $Y \in D$, $X \neq Y$ and by 2° we obtain

$$\Delta T_\varphi(X) = - \int_D f(Y) \Delta(G_\varphi - U) dY = 0 \quad \text{for } X \in D.$$

That completes the proof.

Let us consider the function $z_\varphi(X)$, $\varphi = (\varphi_1, \dots, \varphi_n) \in \Phi$ given by (13). We shall prove the following lemma

LEMMA 13. *If the assumptions of Lemma 12 are fulfilled, then*
1° *the integrals*

$$\int_D f(Y) D_{x_i} G_\varphi|_{\varphi_i=1} dY, \quad \int_D f(Y) G_\varphi|_{\varphi_i=2} dY, \quad \int_D f(Y) (D_{x_i} + h) G_\varphi|_{\varphi_i=3} dY$$

are locally uniformly convergent at every point $X \in S_i^+$, $i \in N$,

2° *the function $z_\varphi(X)$ given by (13) satisfies the boundary conditions (28).*

Proof. While the function G_φ is a linear combination of the functions U_a and $U_{a'}$, $a \in B$, $a' \in A \setminus B$, then due to Lemmas 10 and 12 to prove 1° it is sufficient to show that the integrals

$$(31) \quad \int_D f(Y) D_X^\alpha U_a dY, \\ \int_D f(Y) dY \int_{R_{n_1, \dots, n_k}^+} D_X^\alpha U_{a'} \exp[h(v_{n_1} + \dots + v_{n_k})] dv_{n_1} \dots dv_{n_k},$$

where $|\alpha| = 0, 1$, $a \in B \setminus \{(0, \dots, 0), e_i\}$, $a' \in A_{n_1, \dots, n_k}$, $\{n_1, \dots, n_k\} \subset N \setminus \{i\}$, are locally uniformly convergent at every point $X \in S_i^+$ ($i \in N$). Let $K(\bar{X}, \eta)$ be the ball with centre $\bar{X} \in S_i^+$ and such a radius $\eta > 0$ that its projection on S_i is contained in S_i^+ . Then there exists positive numbers δ, M, M_1 such that the following inequalities hold: $r_a \geq \delta$, $r_{a'} \geq \delta$ for $X \in K(\bar{X}, \eta) \cap (D \cup S_i^+)$, $Y \in D$, $(v_{n_1}, \dots, v_{n_k}) \in R_{n_1, \dots, n_k}^+$.

Then

$$r_a \leq M|Y|, \quad r_{a'} \leq M(|Y| + v_{n_1} + \dots + v_{n_k})$$

for $X \in K(\bar{X}, \eta) \cap (D \cup S_i^+)$, $|Y| \geq M_1$, $v_n \geq M_1$, $s = 1, \dots, k$. From the foregoing and the assumptions of our lemma it follows that the integrals (31) are locally uniformly convergent at the point $\bar{X} \in S_i^+$, $i \in N$. By 1° of Lemma 13 and 2° of Lemma 12 we obtain thesis 2° of Lemma 13.

That completes the proof.

In virtue of Lemmas 11, 12, 13 we obtain the following theorem.

THEOREM 2. *Let $\varphi = (\varphi_1, \dots, \varphi_n)$ be a fixed point of Φ , let the functions $f(Y)$, $D_{y_j} f(Y)$ ($j \in N$) be continuous and bounded in D and let*

$$\int_D |f(Y)| |Y| dY < \infty \quad \text{for } n = 2; \quad \int_D |f(Y)| dY < \infty \quad \text{for } n \geq 3.$$

Then the function $z_\varphi(X)$ given by (13) satisfies equation (1) in D and the boundary conditions (28).

By the Theorems 1 and 2 it follows the fundamental

THEOREM 3. *If the assumptions of Theorems 1 and 2 are fulfilled, then the function $w_\varphi(X)$ given by formulas (12)–(14) satisfies equation (1) in D and boundary conditions (16).*

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