ANNALES SOCIETATIS MATHEMATICAE POLONAE
Series I: COMMENTATIONES MATHEMATICAE XXVII (1987)
ROCZNIKI POLSKIEGO TOWARZYSTWA MATEMATYCZNEGO
Seria I: PRACE MATEMATYCZNE XXVII (1987)

LUDWIK BYSZEWSKI (Kraków)

Uniqueness criteria for the solutions of the linear iterated problems of parabolic type in arbitrary set

1. Introduction. The aim of the present paper is to give the uniqueness criterion for the solution of the linear mixed iterated problem of parabolic type with the operator

$$P = \sum_{i,j=1}^{n} a_{ij}(x, t) D_{x_i} D_{x_j} + \sum_{i=1}^{n} b_i(x, t) D_{x_i} + c(x, t) - D_t, \tag{1.1}$$

where $x = (x_1, \ldots, x_n)$, in an arbitrary $(n+1)$-dimensional time-space, Szarski's set. For this purpose the method of transforming of higher order problems to recurrent systems of the problems is used and the weak maximum principle for parabolic differential inequalities from [1] is applied.

We consider here only real functions.

2. Preliminaries. The notation, definitions and assumptions given in this section are valid throughout this paper. Some of them, useful to the first iteration, are similar to those given by J. Szarski in [3].

We use the following notation: $R = (-\infty, +\infty)$, $R^n = R \times \ldots \times R$ (n-times), $N_0 = \{0, 1, 2, \ldots\}$, $N_0^n = N_0 \times \ldots \times N_0$ (n-times), $x = (x_1, \ldots, x_n)$.

Let t_0 be a real finite number and let $0 < T < \infty$ or $T = \infty$. We mean by D a set contained in $\{(x, t)\colon x \in \boldsymbol{R}^n,\ t > t_0\}$ and satisfying the following conditions:

(a) The projection on the t-axis of the interior of the set D is the interval $(t_0, t_0 + T)$.

(b) For any $(\tilde{x}, \tilde{t}) \in D$ there exists $r > 0$ such that

$$\{(x, t)\colon (t - \tilde{t})^2 + \sum_{i=1}^{n} (x_i - \tilde{x}_i)^2 < r,\ t < \tilde{t}\} \subset D.$$

Let $\tilde{D} \subset \{(x, t)\colon x \in R^n,\ t \leqslant t_0 + T\}$ be an arbitrary set such that $\tilde{D} \supset \bar{D}$. We put $\partial_p D := \tilde{D} \setminus D$.

By Σ we denote a subset (possibly empty) of $(\bar{D} \setminus D) \cap (R^n \times (t_0, t_0 + T))$ with the property that for every $(x, t) \in \Sigma$ a direction $l(x, t)$ is given, such that

l is orthogonal to the t-axis and the interior of some segment starting at (x, t) of the straight half line from (x, t) in the direction l is contained in D.

Next, by Σ_* we denote an arbitrary fixed subset of Σ and by L we denote an arbitrary fixed positive constant.

Moreover, we define the set $(\Sigma_*)^L$ by the formula

$$(\Sigma_*)^L = \Sigma_* \cap \{(x, t)\colon\ |x| \leqslant L,\ -\infty < t \leqslant t_0 + T\}.$$

We put $|\alpha| := \sum_{i=1}^{n} \alpha_i$, where $\alpha = (\alpha_1, \ldots, \alpha_n) \in N_0^n$.

Finally, we assume that m is an arbitrary fixed natural number.

A function u is called *$(\Sigma_*)^L_{(m)}$-regular in D* if the derivatives $D_x^\alpha D_t^\beta u$ $\big(|\alpha| + 2\beta \leqslant 2(m-1)\big)$ are defined on $\tilde{D}$ and continuous in $\bar{D}$, the derivatives $D_x^\alpha D_t^\beta u$ $\big(2(m-1) < |\alpha| + 2\beta \leqslant 2m\big)$ are continuous in D, the derivatives $\frac{d}{dl} D_x^\alpha D_t^\beta u$ $\big(|\alpha| + 2\beta \leqslant 2(m-1)\big)$ are finite on $(\Sigma_*)^L$ and additionally for $m \geqslant 2$ the derivatives $D_x^\alpha D_t^\beta u$ $\big(|\alpha| + 2\beta \leqslant 2(m-1)\big)$ are finite on $(\Sigma_*)^L$.

A $(\Sigma_*)^L_{(1)}$-regular function in D is called *$(\Sigma_*)^L$-regular function in D*.

If a set Σ_* is bounded and a constant L is so large that the sets $(\Sigma_*)^L$ and Σ_* are identically equal, then the definitions and theorems which will be given in the sequel for $(\Sigma_*)^L_{(m)}$-regular functions in D are true for functions called *$(\Sigma_*)_{(m)}$-regular in D*.

If a set Σ_* is empty, then $(\Sigma_*)^L_{(m)}$-regular function in D is called *(m)-regular function* in D. Particularly, if the set Σ_* is empty, then $(\Sigma_*)^L$-regular function in D is called a *regular function* in D.

According to the definitions given above, if Σ_* is an empty set then the definitions and theorems which will be given in the sequel for $(\Sigma_*)^L_{(m)}$-regular functions in D are true for (m)-regular functions in D.

The definition of $(\Sigma_*)^L_{(m)}$-regular function in D is a modified generalization to the case $m \geqslant 1$ of the definition of Σ-regular function in D given in [4] and [1].

We mean the partial derivatives with respect to the variable t in the sense of left-sided derivatives. Instead, we mean the partial derivatives with respect to the spatial variables in the usual sense.

For given functions a_{ij} $(= a_{ji})$, b_i $(i, j = 1, 2, \ldots, n)$, c we denote by P the operator given by formula (1.1).

ASSUMPTION $\mathscr{A}$. Let $i, j = 1, 2, \ldots, n$. We assume that the coefficients a_{ij}, b_i, c of the operator P have the following properties:

1° If $m = 1$, the coefficients a_{ij}, b_i, c are defined on D.

2° If $m \geqslant 2$; the coefficients a_{ij}, b_i, c are $(\Sigma_*)^L_{(m-1)}$-regular functions in D.

By $\mathscr{P}_m$ we denote the class of systems of functions $u_1, \ldots, u_m$ such that u_k $(k = 1, 2, \ldots, m)$ are, respectively, $(\Sigma_*)^L_{(k)}$-regular in D and such that they satisfy, respectively, the equations of the form $u_k(x, t) = P^{m-k} u(x, t)$ $(k$

$= 1, \ldots, m)$ for $(x, t) \in \tilde{D}$, where the function u depends on the system $u_1, \ldots, u_m$.

3. Mixed problems and regular solutions. Given the functions a and b defined and positive on $(\Sigma_*)^L$ and given the functions f, f_i, g_i $(i = 0, 1, \ldots, m-1)$ defined respectively on D, $\partial_p D \backslash (\Sigma_*)^L$, $(\Sigma_*)^L$, the mixed iterated problem of type (P^m) in D consists in finding $(\Sigma_*)^L_{(m)}$-regular function u in D, bounded together with $P^i u$ $(i = 1, 2, \ldots, m-1)$ in D, satisfying the equation

$$(3.1)\qquad P^m u(x, t) = f(x, t) \quad \text{for } (x, t) \in D$$

and initial-boundary conditions

$$(3.2)\qquad P^i u(x, t) = f_i(x, t) \quad \text{for } (x, t) \in \partial_p D \backslash (\Sigma_*)^L \ (i = 0, 1, \ldots, m-1)$$

and

$$(3.3)\qquad b(x, t) P^i u(x, t) - a(x, t) \frac{dP^i u(x, t)}{dl} = g_i(x, t)$$
$$\text{for } (x, t) \in (\Sigma_*)^L \ (i = 0, 1, \ldots, m-1).$$

A function u with the foregoing properties is called $(\Sigma_*)^L_{(m)}$*-regular solution in* D of the above problem.

Given the functions a and b defined and positive on $(\Sigma_*)^L$ and given the functions u_0, φ_k, ψ_k $(k = 1, 2, \ldots, m)$ defined respectively on D, $\partial_p D \backslash (\Sigma_*)^L$, $(\Sigma_*)^L$, the system of functions $u_1, \ldots, u_m$ is called $(\Sigma_*)^L_{(m)}$*-regular solution in* D of the recurrent system

$$(3.4)\qquad P u_k(x, t) = u_{k-1}(x, t) \quad \text{for } (x, t) \in D \ (k = 1, 2, \ldots, m),$$

$$(3.5)\qquad u_k(x, t) = \varphi_k(x, t) \quad \text{for } (x, t) \in \partial_p D \backslash (\Sigma_*)^L \ (k = 1, 2, \ldots, m),$$

$$(3.6)\qquad b(x, t) u_k(x, t) - a(x, t) \frac{du_k(x, t)}{dl} = \psi_k(x, t)$$
$$\text{for } (x, t) \in (\Sigma_*)^L \ (k = 1, 2, \ldots, m)$$

if the functions u_k $(k = 1, 2, \ldots, m)$ are bounded in D and respectively $(\Sigma_*)^L_{(k)}$-regular in D and if they satisfy the each mixed problem of this recurrent system, respectively.

4. Relation between the solutions of the mixed iterated problem and the solutions of the recurrent system of the mixed problems. For a function w and for the coefficients a_{ij}, b_i $(i, j = 1, 2, \ldots, n)$, c of the operator P we put the following denotation (under the assumption that this denotation is not meaningless)

$$(4.1)\quad D_x^p D_t^q [abc] D_x^r D_t^s w$$

$$:= \Big(\prod_{i,j=1}^{n} D_x^{p^{ij}} D_t^{q^{ij}} a_{ij}\Big)\Big(\prod_{i=1}^{n} D_x^{p^i} D_t^{q^i} b_i\Big)(D_x^{\tilde{p}} D_t^{\tilde{q}} c)(D_x^r D_t^s w),$$

where

$$p = (p^{11}, p^{12}, \ldots, p^{nn}, p^1, p^2, \ldots, p^n, \tilde{p}),$$

$$q = (q^{11}, q^{12}, \ldots, q^{nn}, q^1, q^2, \ldots, q^n, \tilde{q}),$$

$$p^{ij} \in N_0^n, \quad p^i \in N_0^n \quad (i, j = 1, 2, \ldots, n),\ \tilde{p} \in N_0^n,\ r \in N_0^n,$$

$$q^{ij} \in N_0, \quad q^i \in N_0 \quad (i, j = 1, 2, \ldots, n),\ \tilde{q} \in N_0,\ s \in N_0.$$

Moreover, let

$$|p| := \sum_{i,j=1}^{n} |p^{ij}| + \sum_{i=1}^{n} |p^i| + |\tilde{p}|, \qquad |q| := \sum_{i,j=1}^{n} q^{ij} + \sum_{i=1}^{n} q^i + \tilde{q}.$$

We say that the product of the derivatives $D_x^p D_t^q [abc] D_x^r D_t^s w$, given by formula (4.1), is of the $[2(k-1), 2k]$ *order* (k is a fixed natural number) if the inequalities

$$|p| + 2|q| \leqslant 2(k-1), \qquad |r| + 2s \leqslant 2k, \qquad |p| + |r| + 2(|q| + s) \leqslant 2k$$

are true.

Put

$$(4.2)\quad u_k(x, t) = P^{m-k} u(x, t) \quad \text{for } (x, t) \in \tilde{D}\ (k = 1, 2, \ldots, m),$$

where P^0 denotes the identity operator, and put

$$(4.3)\quad u_0(x, t) = f(x, t) \quad \text{for } (x, t) \in D.$$

Finally, let

$$(4.4)\quad \varphi_k(x, t) = f_{m-k}(x, t) \quad \text{for } (x, t) \in \partial_p D \setminus (\Sigma_*)^L\ (k = 1, 2, \ldots, m)$$

and

$$(4.5)\quad \psi_k(x, t) = g_{m-k}(x, t) \quad \text{for } (x, t) \in (\Sigma_*)^L\ (k = 1, 2, \ldots, m).$$

Now we shall prove the following lemma.

LEMMA 4.1. *Assume that the coefficients* a_{ij}, b_i ($i, j = 1, 2, \ldots, n$), c *of the operator* P *satisfy Assumption* $\mathscr{R}$. *Then the function* u *is* $(\Sigma_*)^L_{(m)}$*-regular in* D *if and only if the functions* u_k ($k = 1, 2, \ldots, m$), *given by formulae* (4.2), *are respectively* $(\Sigma_*)^L_{(k)}$*-regular in* D.

Proof. We shall consider two cases: I. $m = 1$, II. $m \geqslant 2$.

In case I, Lemma 4.1 is the consequence of the definition of the function u_m.

Consider case II. Assume for this purpose that the function u is $(\Sigma_*)^L_{(m)}$-regular in D. Hence, in order to prove Lemma 4.1 in this case in one direction, it is sufficient to show that the functions u_k $(k = 1, 2, \ldots, m-1)$ are respectively $(\Sigma_*)^L_{(k)}$-regular in D.

And so, the functions u_k $(k = 1, 2, \ldots, m-1)$, as linear combinations of products of derivatives $D^p_x D^q_t [abc] D^r_x D^s_t u$ respectively of order at most $[2(m-k-1), 2(m-k)]$, are defined on $\tilde{D}$ and continuous in $\bar{D}$. Therefore, the derivatives $D^\alpha_x D^\beta_t u_k$ $\left(|\alpha| + 2\beta \leqslant 2(k-1),\ k = 1, 2, \ldots, m-1\right)$ are linear combinations of products of derivatives $D^p_x D^q_t [abc] \cdot D^r_x D^s_t u$ of order at most $[2(m-2), 2(m-1)]$, and since these products are defined on $\tilde{D}$ and continuous in $\bar{D}$, the derivatives $D^\alpha_x D^\beta_t u_k$ $\left(|\alpha| + 2\beta \leqslant 2(k-1),\ k = 1, 2, \ldots, m-1\right)$ also have these properties.

Next, the derivatives $D^\alpha_x D^\beta_t u_k$ $\left(2(k-1) < |\alpha| + 2\beta \leqslant 2k,\ k = 1, 2, \ldots, m-1\right)$ are linear combinations of products of derivatives $D^p_x D^q_t [abc] D^r_x D^s_t u$ of order at most $[2(m-1), 2m]$, which are continuous in D. Then the derivatives $D^\alpha_x D^\beta_t u_k$ $\left(2(k-1) < |\alpha| + 2\beta \leqslant 2k,\ k = 1, 2, \ldots, m-1\right)$ are also continuous in D.

At last, the derivatives $\frac{d}{dl} D^\alpha_x D^\beta_t u_k$ $\left(|\alpha| + 2\beta \leqslant 2(k-1),\ k = 1, 2, \ldots, m-1\right)$ are finite on $(\Sigma_*)^L$ as linear combinations of products of derivatives $\frac{d}{dl}\{D^p_x D^q_t [abc]\} D^r_x D^s_t u$, $D^p_x D^q_t [abc] \frac{d}{dl} D^r_x D^s_t u$ (1) of order at most $[2(m-2), 2(m-1)]$, having finite values on $(\Sigma_*)^L$.

So the functions u_k $(k = 1, 2, \ldots, m-1)$ are respectively $(\Sigma_*)^L_{(k)}$-regular in D.

The truth of Lemma 4.1 in case II in the second direction is obvious.

Now, using Lemma 4.1 and the method of transforming higher order problems to recurrent systems of problems, we shall prove Theorem 4.1. This method was applied a bit otherwise for some iterated problems of the elliptic and parabolic type in [5] and [2].

THEOREM 4.1. *Let the coefficients* a_{ij}, b_i $(i, j = 1, 2, \ldots, n)$, c *of the operator* P *satisfy Assumption* $\mathscr{R}$, *and let formulae* (4.3)–(4.5) *hold. Then*

(a) *The function* u *is* $(\Sigma_*)^L_{(m)}$*-regular solution of the mixed iterated problem* (3.1)–(3.3) *of type* (P^m) *in* D *if and only if the system of functions* $u_1, \ldots, u_m$, *given by formulae* (4.2), *is a* $(\Sigma_*)^L_{(m)}$*-regular solution of the recurrent system* (3.4)–(3.6) *in* D.

(b) *The function* u *is the only one* $(\Sigma_*)^L_{(m)}$*-regular solution of the mixed*

---

(1) We apply here the equation $\frac{d}{dl}(vw) = \frac{dv}{dl}w + v\frac{dw}{dl}$.

iterated problem (3.1)–(3.3) *of type* (P^m) *in* D *if and only if the system of functions* $u_1, \ldots, u_m$, *given by formulae* (4.2), *is the only one, in the class* $\mathscr{P}_m$, $(\Sigma_*)^L_{(m)}$*-regular solution of the recurrent system* (3.4)–(3.6) *in* D.

Proof. (a) Assume that the function u is $(\Sigma_*)^L_{(m)}$-regular solution of the iterated problem (3.1)–(3.3) in D. Then, according to the definition of the functions u_k $(k = 1, 2, \ldots, m)$, these functions are bounded in D. Besides, from Lemma 4.1, the functions u_k $(k = 1, 2, \ldots, m)$ are respectively $(\Sigma_*)^L_{(k)}$-regular in D. At last, respectively from formulae (4.2), (4.3); (4.2), (3.2), (4.4); (4.2), (3.3), (4.5) we obtain that the functions u_k $(k = 1, 2, \ldots, m)$ satisfy the following equations

$$Pu_k(x, t) = P\big(P^{m-k} u(x, t)\big) = P^{m-(k-1)} u(x, t) = u_{k-1}(x, t)$$
$$\text{for } (x, t) \in D \quad (k = 1, 2, \ldots, m),$$

$$u_k(x, t) = P^{m-k} u(x, t) = f_{m-k}(x, t) = \varphi_k(x, t)$$
$$\text{for } (x, t) \in \partial_p D \setminus (\Sigma_*)^L \quad (k = 1, 2, \ldots, m)$$

and

$$\begin{aligned} b(x, t) u_k(x, t) - a(x, t) \frac{du_k(x, t)}{dl} &= b(x, t) P^{m-k} u(x, t) - a(x, t) \frac{dP^{m-k} u(x, t)}{dl} \\ &= g_{m-k}(x, t) = \psi_k(x, t) \quad \text{for } (x, t) \in (\Sigma_*)^L \ (k = 1, 2, \ldots, m). \end{aligned}$$

Assume now that the system of functions $u_1, \ldots, u_m$, given by formulae (4.2), is $(\Sigma_*)^L_{(m)}$-regular solution of the recurrent system (3.4)–(3.6) in D. Hence the function u is bounded in D and $(\Sigma_*)^L_{(m)}$-regular in D. Next, from (4.3) and (3.4) we get

$$\begin{aligned} P^i u_i(x, t) &= f(x, t) && \text{for } (x, t) \in D, \\ Pu_{i+1}(x, t) &= u_i(x, t) && \text{for } (x, t) \in D, \\ &\cdots\cdots\cdots\cdots \\ Pu_m(x, t) &= u_{m-1}(x, t) && \text{for } (x, t) \in D \end{aligned}$$

for $i = 1, 2, \ldots, m$.

Putting $i = m$ in the above equations, we obtain that the function u satisfies equation (3.1). Next, respectively from formulae (4.2), (3.5), (4.4); and (4.2), (3.6), (4.5) we obtain

$$P^i u(x, t) = u_{m-i}(x, t) = \varphi_{m-i}(x, t) = f_i(x, t)$$
$$\text{for } (x, t) \in \partial_p D \setminus (\Sigma_*)^L \quad (i = 0, 1, \ldots, m-1)$$

and

$$b(x,t)\,P^i u(x,t)-a(x,t)\frac{dP^i u(x,t)}{dl}=b(x,t)\,u_{m-i}(x,t)-a(x,t)\frac{du_{m-i}(x,t)}{dl}$$

$$=\psi_{m-i}(x,t)=g_i(x,t)\quad \text{for } (x,t)\in(\Sigma_*)^L\ (i=0,1,\dots,m-1).$$

This ends the proof of assertion (a) of Theorem 4.1.

Now, we shall prove assertion (b). Since this assertion is obvious for $m=1$, we shall take into account in the proof only the case $m\geqslant 2$.

Suppose that the function u is the only one $(\Sigma_*)^L_{(m)}$-regular solution of the iterated problem (3.1)–(3.3) in D. Then, by assertion (a) of Theorem 4.1, the system of functions $u_1,\dots,u_m$, with u_k $(k=1,2,\dots,m)$ given by formulae (4.2), is $(\Sigma_*)^L_{(m)}$-regular solution of the recurrent system (3.4)–(3.6) in D, where the functions u_0, φ_k, ψ_k $(k=1,2,\dots,m)$ are defined, respectively, by formulae (4.3)–(4.5). To prove the uniqueness of the solution $u_1,\dots,u_m$ in the class $\mathscr{P}_m$, let us assume that there exists a second $(\Sigma_*)^L_{(m)}$-regular solution $v_1,\dots,v_m$ of the recurrent system (3.4)–(3.6) in D (with the functions u_0, φ_k, ψ_k $(k=1,2,\dots,m)$ possessing the above properties) such that

$$v_k(x,t)=P^{m-k}v(x,t)\quad \text{for } (x,t)\in\tilde{D}\ (k=1,2,\dots,m), \tag{4.6}$$

where v is $(\Sigma_*)^L_{(m)}$-regular function in D. Hence the following equations hold:

$$\begin{aligned} &Pu_k(x,t)=u_{k-1}(x,t) \quad \text{for } (x,t)\in D\ (k=1,2,\dots,m),\\ &u_k(x,t)=\varphi_k(x,t)\quad \text{for } (x,t)\in\partial_p D\setminus(\Sigma_*)^L\ (k=1,2,\dots,m),\\ &b(x,t)\,u_k(x,t)-a(x,t)\frac{du_k(x,t)}{dl}=\psi_k(x,t)\\ &\qquad\qquad \text{for } (x,t)\in(\Sigma_*)^L\ (k=1,2,\dots,m)\end{aligned} \tag{4.7}$$

and

$$\begin{aligned} &Pv_k(x,t)=v_{k-1}(x,t)\ ^{(2)} \quad \text{for } (x,t)\in D\ (k=1,2,\dots,m),\\ &v_k(x,t)=\varphi_k(x,t)\quad \text{for } (x,t)\in\partial_p D\setminus(\Sigma_*)^L\ (k=1,2,\dots,m),\\ &b(x,t)\,v_k(x,t)-a(x,t)\frac{dv_k(x,t)}{dl}=\psi_k(x,t)\\ &\qquad\qquad \text{for } (x,t)\in(\Sigma_*)^L\ (k=1,2,\dots,m).\end{aligned} \tag{4.8}$$

Since the first equations for $k=2,3,\dots,m$ in the recurrent systems (4.7) and

---

(2) Since $v_1,\dots,v_m$ is $(\Sigma_*)^L_{(m)}$-regular solution of the recurrent system (3.4)–(3.6) in D, where $u_0(x,t)=f(x,t)$ for $(x,t)\in D$, we have $v_0(x,t)=f(x,t)$ for $(x,t)\in D$.

(4.8) are logic tautologies, these systems are equivalent to the following system:

$$Pu_1(x,t) = Pv_1(x,t) = f(x,t) \quad \text{for } (x,t)\in D,$$

$$u_k(x,t) = v_k(x,t) = \varphi_k(x,t) \quad \text{for } (x,t)\in \partial_p D\backslash(\Sigma_*)^L \ (k=1,2,\ldots,m),$$

$$b(x,t)u_k(x,t) - a(x,t)\frac{du_k(x,t)}{dl} = b(x,t)v_k(x,t) - a(x,t)\frac{dv_k(x,t)}{dl} = \psi_k(x,t) \quad \text{for } (x,t)\in(\Sigma_*)^L \ (k=1,2,\ldots,m).$$

Taking into account formulae (4.2), (4.6), (4.4) and (4.5), we have

$$P^m u(x,t) = P^m v(x,t) = f(x,t) \quad \text{for } (x,t)\in D,$$

$$P^{m-k}u(x,t) = P^{m-k}v(x,t) = f_{m-k}(x,t) \quad \text{for } (x,t)\in \partial_p D\backslash(\Sigma_*)^L \ (k=1,2,\ldots,m),$$

$$b(x,t)P^{m-k}u(x,t) - a(x,t)\frac{dP^{m-k}u(x,t)}{dl} = b(x,t)P^{m-k}v(x,t) - a(x,t)\frac{dP^{m-k}v(x,t)}{dl} = g_{m-k}(x,t) \quad \text{for } (x,t)\in(\Sigma_*)^L \ (k=1,2,\ldots,m).$$

Since u is the only one $(\Sigma_*)^L_{(m)}$-regular solution of the iterated problem (3.1)–(3.3) in D, then from the above equations we get

$$u(x,t) = v(x,t) \quad \text{for } (x,t)\in\tilde{D}$$

and consequently, by (4.2) and (4.6), we obtain

$$u_k(x,t) = v_k(x,t) \quad \text{for } (x,t)\in\tilde{D} \ (k=1,2,\ldots,m).$$

Assume now that the system of functions $u_1, \ldots, u_m$, given by formulae (4.2), is the only one $(\Sigma_*)^L_{(m)}$-regular solution of the recurrent system (3.4)–(3.6) in D, where the functions u_0, φ_k, ψ_k $(k=1,2,\ldots,m)$ are defined, respectively, by formulae (4.3)–(4.5). Hence, by assertion (a) of Theorem 4.1, u is $(\Sigma_*)^L_{(m)}$-regular solution of the iterated problem (3.1)–(3.3) in D. To prove the uniqueness of the solution u, suppose that there exists a second $(\Sigma_*)^L_{(m)}$-regular solution v of the iterated problem (3.1)–(3.3) in D. Then, by the above argumentation, by (4.2) and by assertion (a) of Theorem 4.1 applied to the function v,

$$Pu_1(x, t) = P\big(P^{m-1}u(x, t)\big) = P^m u(x, t) = P^m v(x, t) = P\big(P^{m-1}v(x, t)\big)$$

$$= f(x, t) \quad \text{for } (x, t)\in D,$$

$$u_1(x, t) = v_1(x, t) = \varphi_1(x, t) \quad \text{for } (x, t)\in \partial_p D\setminus(\Sigma_*)^L,$$

$$b(x, t)u_1(x, t)-a(x, t)\frac{du_1(x, t)}{dl} = b(x, t)v_1(x, t)-a(x, t)\frac{dv_1(x, t)}{dl}$$

$$= \psi_1(x, t) \quad \text{for } (x, t)\in(\Sigma_*)^L.$$

Since, according to the assumption, u_1 is the only one bounded in D and $(\Sigma_*)^L_{(1)}$-regular function in D satisfying the above problem, we have

$$u_1(x, t) = P^{m-1}u(x, t) = P^{m-1}v(x, t) \quad \text{for } (x, t)\in\tilde{D}. \tag{4.9}$$

Next, by (3.4), (4.2), (4.9) and by assertion (a) of Theorem 4.1 applied to the function v,

$$Pu_2(x, t) = P\big(P^{m-2}u(x, t)\big) = P^{m-1}u(x, t) = P^{m-1}v(x, t) = P\big(P^{m-2}v(x, t)\big)$$

$$= u_1(x, t) \quad \text{for } (x, t)\in D,$$

$$u_2(x, t) = v_2(x, t) = \varphi_2(x, t) \quad \text{for } (x, t)\in \partial_p D\setminus(\Sigma_*)^L,$$

$$b(x, t)u_2(x, t)-a(x, t)\frac{du_2(x, t)}{dl} = b(x, t)v_2(x, t)-a(x, t)\frac{dv_2(x, t)}{dl}$$

$$= \psi_2(x, t) \quad \text{for } (x, t)\in(\Sigma_*)^L.$$

Since, from assumption, u_2 is the only one bounded in D and $(\Sigma_*)^L_{(2)}$-regular function in D satisfying the above problem, we have

$$u_2(x, t) = P^{m-2}u(x, t) = P^{m-2}v(x, t) \quad \text{for } (x, t)\in\tilde{D}.$$

Repeating this argument recurrently, we obtain

$$u_{m-1}(x, t) = Pu(x, t) = Pv(x, t) \quad \text{for } (x, t)\in\tilde{D} \tag{4.10}$$

and we get finally, by (3.5), (3.6), (4.10) and by assertion (a) of Theorem 4.1 applied to the function v,

$$Pu(x, t) = Pv(x, t) = u_{m-1}(x, t) \quad \text{for } (x, t)\in D,$$

$$u(x, t) = v(x, t) = \varphi_m(x, t) \quad \text{for } (x, t)\in \partial_p D\setminus(\Sigma_*)^L,$$

$$b(x, t)u(x, t)-a(x, t)\frac{du(x, t)}{dl} = b(x, t)v(x, t)-a(x, t)\frac{dv(x, t)}{dl} = \psi_m(x, t)$$

$$\text{for } (x, t)\in(\Sigma_*)^L.$$

Since, according to the assumption, $u = u_m$ for $(x, t) \in \tilde{D}$ is the only one bounded in D and $(\Sigma_*)^L_{(m)}$-regular function in D satisfying the above problem, we infer.

$$u(x, t) = v(x, t) \quad \text{for } (x, t) \in \tilde{D}.$$

This ends the proof of Theorem 4.1.

5. Uniqueness criterion for the solution of the mixed iterated problem.

THEOREM 5.1. *Suppose that*

1° *The coefficients* a_{ij}, b_i $(i, j = 1, 2, \ldots, n)$, c *of the operator* P *satisfy Assumption* $\mathscr{R}$.

2° *The inequalities*

$$\sum_{i,j=1}^{n} |a_{ij}(x, t)| \leqslant L|x|^2, \qquad \sum_{i=1}^{n} |b_i(x, t)| \leqslant L|x|, \qquad c(x, t) \leqslant L$$

are satisfied for $(x, t) \in D$, $|x| > L$ *and the inequality*

$$c(x, t) \leqslant L$$

is satisfied for $(x, t) \in D$, $|x| \leqslant L$.

3° *The real quadratic form* $\sum_{i,j=1}^{n} a_{ij}(x, t)\lambda_i\lambda_j$ *is non-negative for every* $(x, t) \in D$.

Then the mixed iterated problem (3.1)–(3.3) *of type* (P^m) *admits at most one* $(\Sigma_*)^L_{(m)}$*-regular solution in* D.

Proof. Assume that u is a $(\Sigma_*)^L_{(m)}$-regular solution of problem (3.1)–(3.3) in D. By Theorem 4.1 the system of functions $u_1, \ldots, u_m$, given by formulae (4.2), is $(\Sigma_*)^L_{(m)}$-regular solution of the recurrent system (3.4)–(3.6) in D, where the functions u_0, φ_k, ψ_k $(k = 1, 2, \ldots, m)$ are defined, respectively, by formulae (4.3)–(4.5).

Now, we shall prove that the system of functions $u_1, \ldots, u_m$ is the only one in the class $\mathscr{P}_m$, $(\Sigma_*)^L_{(m)}$-regular solution of the recurrent system (3.4)–(3.6) in D. To this purpose suppose that there exists, in the class $\mathscr{P}_m$, a second $(\Sigma_*)^L_{(m)}$-regular solution $v_1, \ldots, v_m$ of the recurrent system (3.4)–(3.6) in D. Particularly, this means for $k = 1$ that the functions u_1 and v_1 are two $(\Sigma_*)^L$-regular solutions in D of the after-mentioned problem

$$Pu_1(x, t) = f(x, t) \quad \text{for } (x, t) \in D,$$

$$u_1(x, t) = \varphi_1(x, t) \quad \text{for } (x, t) \in \partial_p D \backslash (\Sigma_*)^L,$$

$$b(x, t)u_1(x, t) - a(x, t)\frac{du_1(x, t)}{dl} = \psi_1(x, t) \quad \text{for } (x, t) \in (\Sigma_*)^L.$$

Then

$$Pu_1(x,t) = Pv_1(x,t) \quad \text{for } (x,t)\in D,$$

$$u_1(x,t) = v_1(x,t) \quad \text{for } (x,t)\in \partial_p D\backslash(\Sigma_*)^L,$$

$$b(x,t)u_1(x,t)-a(x,t)\frac{du_1(x,t)}{dl} = b(x,t)v_1(x,t)-a(x,t)\frac{dv_1(x,t)}{dl} \quad \text{for } (x,t)\in(\Sigma_*)^L.$$

Consequently, from the linear version of the proper modification of the non-linear maximum principle, given in [1], we obtain

$$u_1(x,t) = v_1(x,t) \quad \text{for } (x,t)\in\tilde{D}.$$

Next, by the above equation and by the assumption that $v_1, \ldots, v_m$ is a second, in the class $\mathscr{P}_m$, $(\Sigma_*)^L_{(m)}$-regular solution of the recurrent system (3.4)–(3.6) in D, the functions u_2 and v_2 are two $(\Sigma_*)^L$-regular solutions in D of the following problem:

$$Pu_2(x,t) = u_1(x,t) \quad \text{for } (x,t)\in D,$$

$$u_2(x,t) = \varphi_2(x,t) \quad \text{for } (x,t)\in \partial_p D\backslash(\Sigma_*)^L,$$

$$b(x,t)u_2(x,t)-a(x,t)\frac{du_2(x,t)}{dl} = \psi_2(x,t) \quad \text{for } (x,t)\in(\Sigma_*)^L.$$

Hence

$$Pu_2(x,t) = Pv_2(x,t) \quad \text{for } (x,t)\in D,$$

$$u_2(x,t) = v_2(x,t) \quad \text{for } (x,t)\in \partial_p D\backslash(\Sigma_*)^L,$$

$$b(x,t)u_2(x,t)-a(x,t)\frac{du_2(x,t)}{dl} = b(x,t)v_2(x,t)-a(x,t)\frac{dv_2(x,t)}{dl} \quad \text{for } (x,t)\in(\Sigma_*)^L$$

and, by the maximum principle,

$$u_2(x,t) = v_2(x,t) \quad \text{for } (x,t)\in\tilde{D}.$$

Repeating this argumentation recurrently, we get

$$u_k(x,t) = v_k(x,t) \quad \text{for } (x,t)\in\tilde{D},\ k = 3, 4, \ldots, m.$$

Therefore, the system of functions $u_1, \ldots, u_m$ is the only one, in the class $\mathscr{P}_m$, $(\Sigma_*)^L_{(m)}$-regular solution of the recurrent system (3.4)–(3.6) in D. Then, by assertion (b) of Theorem 4.1, u is also the only one $(\Sigma_*)^L_{(m)}$-regular solution of the iterated problem (3.1)–(3.3) in D.

In this way, Theorem 5.1 is proved.

6. Uniqueness criteria for the solutions of the Fourier's first and second iterated problems. Given the functions f, f_i, g_i $(i = 0, 1, \ldots, m-1)$ defined respectively on D, $\partial_p D \setminus (\Sigma_*)^L$, $(\Sigma_*)^L$, mixed iterated reduced problem of type (P^m) in D consists in finding $(\Sigma_*)^L_{(m)}$-regular function u in D, bounded together with $P^i u$ $(i = 1, 2, \ldots, m-1)$ in D, satisfying (3.1), (3.2) and boundary conditions

$$(6.1) \qquad \frac{dP^i u(x, t)}{dl} = g_i(x, t) \quad \text{for } (x, t) \in (\Sigma_*)^L \ (i = 0, 1, \ldots, m-1)\ (^3).$$

A function u with the foregoing properties is called *$(\Sigma_*)^L_{(m)}$-regular solution* in D of the above problem.

Applying an analogous argument as in the proof of Theorem 5.1, we get

THEOREM 6.1. *Under the assumptions of Theorem* 5.1, *the mixed iterated reduced problem of type (P^m) in D admits at most one $(\Sigma_*)^L_{(m)}$-regular solution in D.*

If $\Sigma_* = \emptyset$ $[\Sigma_* = \Sigma]$ in the mixed iterated reduced problem of type (P^m), then this problem is called *Fourier's first* [*second*] *iterated problem of type* (P^m).

From Theorem 6.1 we obtain

THEOREM 6.2. *Under the assumptions of Theorem* 5.1 *the Fourier's first* [*second*] *iterated problem of type (P^m) admits at most one (m)-regular $[(\Sigma_*)^L_{(m)}$-regular] solution in D.*

7. Uniqueness criteria for the solutions of the iterated modified problems. In this section we always put $\tilde{D} = \bar{D}$.

ASSUMPTION $\bar{\mathscr{R}}$. Let $i, j = 1, 2, \ldots, n$. We assume that the coefficients a_{ij}, b_i, c of the operator P have the following properties:

1° If $m = 1$, the coefficients a_{ij}, b_i, c are defined on D.

2° If $m \geqslant 2$, the derivatives $D_x^\alpha D_t^\beta a_{ij}$, $D_x^\alpha D_t^\beta b_i$, $D_x^\alpha D_t^\beta c$ $\big(|\alpha| + \beta \leqslant 2(m-1)\big)$ are continuous in $\bar{D}$ and the derivatives $D_x^\alpha D_t^\beta a_{ij}$, $\frac{d}{dl} D_x^\alpha D_t^\beta a_{ij}$, $D_x^\alpha D_t^\beta b_i$, $\frac{d}{dl} D_x^\alpha D_t^\beta b_i$, $D_x^\alpha D_t^\beta c$, $\frac{d}{dl} D_x^\alpha D_t^\beta c$ $\big(|\alpha| + \beta \leqslant 2(m-2)\big)$ are finite on $(\Sigma_*)^L$.

It is seen that the considerations from the above sections, given for $[(\Sigma_*)^L_{(m)}$-regular] (m)-regular functions in D, are true for the functions belonging to $[C^{2m}(\bar{D})$ and possessing the finite derivatives $\frac{d}{dl} D_x^\alpha D_t^\beta u$ $\big(|\alpha| + \beta \leqslant 2(m-1)\big)$ on $(\Sigma_*)^L]$ $C^{2m}(\bar{D})$. Particularly, we have

---

[3] This problem is not a particular case of the mixed iterated problem of type (P^m).

THEOREM 7.1. *Let the coefficients* a_{ij}, b_i $(i, j = 1, 2, \ldots, n)$, c *of the operator* P *satisfy Assumption* $\bar{\mathscr{R}}$, *and let assumptions* 2° *and* 3° *of Theorem* 5.1 *hold. Then for the given functions* f, f_i, g_i $(i = 0, 1, \ldots, m-1)$ *defined respectively on* D, $\partial_p D \setminus (\Sigma_*)^L$, $(\Sigma_*)^L$, *there exists at most one function* $u \in C^{2m}(\bar{D})$ *possessing finite derivatives* $\frac{d}{dl} D_x^\alpha D_t^\beta u \big(|\alpha| + \beta \leqslant 2(m-1)\big)$ *on* $(\Sigma_*)^L$, *bounded together with* $P^i u$ $(i = 1, 2, \ldots, m-1)$ *on* D *and satisfying formulae* (3.1), (3.2) *and* (6.1).

8. Remark. Suppose that P is an operator with constant coefficients of the form

$$P = \sum_{i,j=1}^{n} a_{ij} D_{x_i} D_{x_j} + \sum_{i=1}^{n} b_i D_{x_i} + c - D_t.$$

Since

$$\frac{P^i du(x, t)}{dl} = \frac{dP^i u(x, t)}{dl} \quad \text{for } (x, t) \in (\Sigma_*)^L \ (i = 0, 1, \ldots, m-1)$$

in the class of $(\Sigma_*)^L_{(m)}$-regular functions u in D having additionally continuous derivatives $D_x^\alpha D_t^\beta u \big(|\alpha| + 2\beta \leqslant 2m-1,\ \alpha \neq 0\big)$ on $(\Sigma_*)^L$, it follows that all the considerations from Sections 5 and 6 are true also, in the above class of functions u, for the iterated problems of type (P^m) with conditions (3.3) replaced by the following conditions:

$$b(x, t) P^i u(x, t) - a(x, t) \frac{P^i du(x, t)}{dl} = g_i(x, t)$$

$$\text{for } (x, t) \in (\Sigma_*)^L \ (i = 0, 1, \ldots, m-1).$$

An analogous remark is true for the iterated problems in the formulation of Section 7.

References

[1] L. Byszewski, *Strong maximum principle for implicit non-linear parabolic functional-differential inequalities in arbitrary domains*, Universitatis Iagellonicae Acta Math. 24 (1984), 327–339.

[2] J. Milewski, *On the limit problems for certain class of partial differential equations of higher order* (in Polish), Zeszyty Naukowe Politechniki Krakowskiej, Podstawowe Nauki Techniczne 17 (1981), 3–66.

[3] J. Szarski, *Differential inequalities*, Polish Scientific Publishers, Warszawa 1967.

[4] —, *Uniqueness of the solution to a mixed problem for parabolic functional-differential*

equations in arbitrary domains, Bull. Acad. Polon. Sci., Sér. sci math., astr. et phys. 24 (1976), 481–489.

[5] E. Wachnicki, *On boundary value problems for some partial differential equations of higher order*, Comment. Math. 20 (1977), 215–233.

INSTYTUT MATEMATYKI
POLITECHNIKI KRAKOWSKIEJ
INSTITUTE OF MATHEMATICS
TECHNICAL UNIVERSITY OF CRACOW