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Bounds for periodic solutions of difference and differential equations

Introduction. In this note we consider the bounds for periods of periodic solutions of difference equations in normed linear spaces with Lipschitz continuous right-hand sides. As a consequence of an obtained result we get a theorem on bounds for periods of periodic solutions of differential equations in Banach spaces, giving thus a generalization of the result which has been obtained in a different way by Lasota and Yorke in [2].

An approach applied in the note, consisting in considering first the discrete cases and then passing to the limit, has been used previously in similar situations by Ky Fan, Taussky and Todd in [1].

The existence of a lower bound for the periods of periodic solutions of differential equations has also been studied by Li [4], Vidossich [5] and Yorke [6].

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1. Notations. N, Z, R will denote, respectively, the sets of all natural, integer and real numbers. E will denote a real normed linear space with the norm $|\ |$. Let $G = Z$ or R. A map $x\colon G \to E$ is called *p-periodic* if $x(g+p) = x(g)$ for each $g \in G$. If $G = Z$ we will write x_i instead of $x(i)$.

For $p \in N$, set

$$\delta(p) = \begin{cases} 1 & \text{if } p \text{ is odd,} \\ 0 & \text{if } p \text{ is even,} \end{cases} \qquad w(p) = \frac{p+\delta(p)}{2} - 1.$$

Clearly, $w(p)$ is natural.

2. The discrete case. The results of this paper come as the consequence of the following fundamental lemma:

LEMMA. *If $v\colon Z \to E$ is a p-periodic and non-constant map satisfying the condition*

$$\sum_{i=0}^{p-1} v_i = 0 \tag{1}$$

then

$$4 \leqslant \left(p-\frac{\delta(p)}{p}\right)\frac{\alpha}{\beta}, \tag{2}$$

where $\alpha = \sum_{i=0}^{p-1} |v_{i+1}-v_i|$, $\beta = \sum_{i=0}^{p-1} |v_i|$.

Proof. In the proof we use ideas similar to those used in [1], § 5.

For $k, l \in Z$, $k \geqslant l$, set $a_l^k = \sum_{j=l}^{k} |v_{j+1}-v_j|$. Since v is p-periodic, $\alpha = \sum_{j=0}^{p-1} |v_{t+j+1}-v_{t+j}|$ for any $t \in Z$, which implies that

$$\sum_{k=0}^{p-1} a_{k+q}^{k+r} = \alpha(r-q+1) \quad \text{for all } r, q \in Z,\ r \geqslant q. \tag{3}$$

From the inequality $|v_s-v_i| \leqslant \sum_{j=i}^{s-1} |v_{j+1}-v_j|$ we get for $s, i \in Z$, $s > i$

$$|v_s-v_i| \leqslant a_i^{s-1}. \tag{4}$$

To simplify notations we will write w, δ instead of $w(p)$, $\delta(p)$. Let $k \in N$ and $k < p$. Using the identities

$$0 = \sum_{j=-w}^{w} v_{k+j} + (1-\delta)\, v_{k+w+1}, \qquad pv_k = \sum_{j=-w}^{w} v_k + (1-\delta)\, v_k$$

and (4), we obtain

$$p|v_k| = |pv_k| \leqslant \sum_{j=-w}^{w} |v_{k+j}-v_k| + (1-\delta)|v_{k+w+1}-v_k|$$

$$\leqslant \sum_{j=1}^{w} (a_k^{k+j-1} + a_{k-j}^{k-1}) + (1-\delta)\, a_k^{k+w}.$$

Summing it over $k = 0, \ldots, p-1$ and applying (3), we have

$$\beta p = \sum_{k=0}^{p-1} p|v_k| \leqslant \sum_{j=1}^{w} 2\alpha j + (1-\delta)\alpha(w+1) = \alpha(w+1)(w+1-\delta)$$

$$= \alpha\frac{p^2-\delta}{4}.$$

Since v is non-constant, $\beta \neq 0$. Consequently,

$$4 \leqslant \left(p-\frac{\delta(p)}{p}\right)\frac{\alpha}{\beta}. \qquad \text{Q.E.D.}$$

Remark. The estimate in the statement of Lemma is the best possible.

In fact, for any $b \in E$, $b \neq 0$, the p-periodic map $v\colon Z \to E$ defined by the formula

$$v_i = \begin{cases} (w+1-p)\,b & \text{for } i = 0, \ldots, w, \\ (w+1)\,b & \text{for } i = w+1, \ldots, p-1, \end{cases}$$

where w is defined as above, satisfies (1) and (2) with equality sign.

THEOREM 1. *Let $f\colon E \times E \times Z \to E$ satisfy the Lipschitz condition with constants K, L, M*

$$|f(a, b, n) - f(c, d, k)| \leqslant K\,|a-c| + L|b-d| + M\,|n-k| \tag{5}$$

for all (a, b, n), $(c, d, k) \in E \times E \times Z$.

If $x\colon Z \to E$ is a p-periodic and non-constant solution of a difference equation

$$x_{n+1} - x_n = f(x_n, x_{n-1}, n) \quad \textit{for each } n \in Z, \tag{6}$$

then

$$4 \leqslant (p - \delta(p)/p)\cdot(K + L + M\,p/\beta), \qquad \textit{where } \beta = \sum_{i=0}^{p-1} |x_{i+1} - x_i|.$$

Proof. For $n \in Z$, define $v_n = x_{n+1} - x_n$ and set $\alpha = \sum_{i=0}^{p-1} |v_{i+1} - v_i|$. Note that since x is p-periodic, v is p-periodic, too, and $\sum_{i=0}^{p-1} v_i = 0$. Moreover, $\sum_{i=0}^{p-1} |v_i| = \sum_{i=0}^{p-1} |v_{i-1}| = \beta$. Let $i \in Z$, by (5) and (6),

$$\begin{aligned} |v_{i+1} - v_i| &= |f(x_{i+1}, x_i, i+1) - f(x_i, x_{i-1}, i)| \\ &\leqslant K\,|v_i| + L|v_{i-1}| + M. \end{aligned}$$

Hence from the above remarks and inequality it follows that

$$\alpha \leqslant K\beta + L\beta + Mp. \tag{7}$$

Since x is non-constant, $\beta \neq 0$. Applying Lemma and (7) we get

$$4 \leqslant (p - \delta(p)/p)\cdot(K + L + Mp/\beta) \qquad \text{Q.E.D.}$$

It is clear that similar result can be obtained for equations

$$x_{n+1} - x_n = f(x_n, x_{n-1}, \ldots, x_{n-k}, n)$$

with f Lipschitz continuous with respect to all variables.

3. The continuous case. As a consequence of Lemma we get the extension of Theorem 4 of Lasota and Yorke [2].

THEOREM 2. *Let B be a Banach space with the norm $|\ |$. Let $F\colon B\times B\times R\to B$ be Lipschitz continuous*

$$|F(y,z,t)-F(w,u,s)|\leqslant K|y-w|+L|z-u|+M|t-s| \tag{8}$$

for all $(y,z,t),(w,u,s)\in B\times B\times R$, where K, L, M are constants. Let $\tau\colon R\to R$ be a C^1 map. If $x\colon R\to B$ is a p-periodic and non-constant solution of a delay differential equation

$$x'(t)=F\big(x(t),x(\tau(t)),t\big)\quad \textit{for each } t\in R, \tag{9}$$

then

$$4\leqslant p(K+L\gamma/\beta+Mp/\beta), \tag{10}$$

where $\beta=\int_0^p|x'(s)|\,ds$, $\gamma=\int_0^p|(x\circ\tau)'(s)|\,ds$.

Proof. For each $k\in N$ define a map $v_k\colon Z\to B$ by the formula

$$v_k(n)=x\left(\frac{(n+1)p}{k}\right)-x\left(\frac{np}{k}\right)\quad \text{for } n\in Z.$$

Since x is p-periodic, it follows that v_k is k-periodic and $\sum_{j=0}^{k-1}v_k(j)=0$ for each $k\in N$. From (8) and (9) it follows that

$$\begin{aligned}
|v_k(n+1)-v_k(n)| &= \left|x\left(\frac{(n+2)p}{k}\right)-2\cdot x\left(\frac{(n+1)p}{k}\right)+x\left(\frac{np}{k}\right)\right| \\
&= \left|\int_{(n+1)p/k}^{(n+2)p/k}x'(s)\,ds-\int_{np/k}^{(n+1)p/k}x'(s)\,ds\right| \\
&= \left|\int_{np/k}^{(n+1)p/k}\big[F\big(x(s+p/k),x(\tau(s+p/k)),s+p/k\big)-\right. \\
&\quad \left.-F\big(x(s),x(\tau(s)),s\big)\big]\,ds\right| \\
&\leqslant K\int_{np/k}^{(n+1)p/k}|x(s+p/k)-x(s)|\,ds+L\int_{np/k}^{(n+1)p/k}\big|x(\tau(s+p/k))- \\
&\quad -x(\tau(s))\big|\,ds+M\cdot p^2/k^2 \quad \text{for all } k\in N,\ n\in Z.
\end{aligned}$$

Summing it over $n = 0, \ldots, k-1$ and using the lemma, we get

$$(11)\qquad 4 \leqslant \left(k - \frac{\delta(k)}{k}\right) \times$$

$$\times \frac{\int_0^p K\,|x(s+p/k) - x(s)|\,ds + \int_0^p L\,|x(\tau(s+p/k)) - x(\tau(s))|\,ds + M \cdot p^2/k}{\sum_{n=0}^{k-1} |x((n+1)\,p/k) - x(np/k)|}$$

$$= p(1 - \delta(k)/k^2) \times$$

$$\times \frac{K \int_0^p |x(s+p/k) - x(s)|/(p/k)\,ds + L \int_0^p |x(\tau(s+p/k)) - x(\tau(s))|/(p/k)\,ds + Mp}{\sum_{n=0}^{k-1} |x((n+1)\,p/k) - x(np/k)|}$$

for each $k \in N$. From Lebesgue dominated convergence theorem [3], p. 487, it follows immediately that

$$(12)\qquad \int_0^p |x(s+p/k) - x(s)|/(p/k)\,ds \to \beta \qquad (k \to \infty),$$

$$(13)\qquad \int_0^p |x(\tau(s+p/k)) - x(\tau(s))|/(p/k)\,ds \to \gamma \qquad (k \to \infty).$$

From the formula for the length of a C^1-curve in Banach space [3], p. 618–625, we get

$$(14)\qquad \sum_{n=0}^{k-1} |x((n+1)\,p/k) - x(np/k)| \to \beta \qquad (k \to \infty).$$

Clearly $\beta \neq 0$, since x is non-constant.

Passing in (11) to the limit for $k \to \infty$ and applying (12), (13) and (14) we obtain finally

$$4 \leqslant p(K + L\gamma/\beta + Mp/\beta). \qquad \text{Q.E.D.}$$

Remark 1. The right-hand side of estimate (10) in Theorem 2 depends on the solution of (9) which is usually not known. But making some assumptions about the functions F and τ we can express the estimate of the period p in terms connected exclusively with these functions.

For instance, if there exist $b, c, d \in R$ such that

$$0 < c \leqslant |F(y, z, t)| \leqslant d \qquad \text{for all } (y, z, t) \in B \times B \times R$$

and

$$|\tau'(t)| \leqslant b \qquad \text{for each } t \in R,$$

then

$$4 \leqslant p(K+L\cdot db/c+M\cdot 1/c).$$

Remark 2. Consider an autonomous delay differential equation

$$x'(t) = F\big(x(t),\, x\big(t-g(t)\big)\big) \quad \text{for each } t\in R, \tag{15}$$

where the function $g\colon R\to R$ is positive and bounded.

If the assumptions of Theorem 2 are satisfied, we obtain from Theorem 1 of Li [4] the estimate for the period of the solution of (15)

$$\frac{4}{K+L} \leqslant p$$

which may be better than estimate (10) if $\gamma/\beta > 1$. However, if g is constant, then our estimate is the same as the estimate of Li.

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