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On functions of bounded essential variation

1. Introduction and definitions. The notion of essential variation of functions was introduced in [5] to obtain some function theoretic properties in an abstract space. This idea was used in [6] to obtain some properties of derivatives and integrals relative to such a function in addition to a suitable generation of signed measure induced by such a functional or its ancillary. In this paper our purpose is to prove some fundamental properties of functions of bounded essential variation. We also construct certain pseudometric space of the class of such functions in the light of papers [1], [4]. Property A* as used in Theorem 3.1 first appeared in [3]. The idea of using polygonal functions to the theory of functions of bounded variation appeared first in [2] and was used in [1] and subsequently in [4] to obtain separability of the spaces of functions of bounded variation as constructed in [1] and [4] in two different contexts.

We quote below some lemmas and definitions some of which are new and others are borrowed from [5].

DEFINITION 1.1 ([5]; § 2). Let $f(x)$ be defined in the closed interval $[a, b]$ and let E be any subset of $[a, b]$ with Lebesgue measure, $mE = b-a$. Consider a subdivision D: $x_0 < x_1 < \ldots < x_n$ of $[a, b]$ with $x_i \in E$ and define

$$V[f; E] = \sup_D \sum_{i=1}^{n} |f(x_i)-f(x_{i-1})|.$$

The essential variation of the function f in the interval $[a, b]$ is defined by $\inf\{V[f; E]: E \subset [a, b],\ mE = b-a\}$ and is denoted by $V^*[f; a, b]$. If $V^*[f; a, b] < +\infty$, we say that f is of *bounded essential variation in* $[a, b]$ and we write $f \in BV^*[a, b]$.

LEMMA 1.1 (cf. [5]; Theorem 2.2). *For every $f(x)$ defined in $[a, b]$ there is a subset A of $[a, b]$ with $mA = b-a$ such that $V[f; A] = V^*[f; a, b]$.*

In future we shall use such a set A without further explicit reference.

Note 1.1. Without any loss of generality it may be assumed that $[a, b]\setminus A$ is everywhere dense in $[a, b]$, because if $B \subset A$, $mB = mA$, then $V^*[f; a, b] = V[f; B]$.

LEMMA 1.2. *If* $f \in BV^*[a, b]$, *then* $\lim_{v_i \to x+} f(v_i)$, $v_i \in A$, $a \leqslant x < b$ *and* $\lim_{v_i \to b-} f(v_i)$, $v_i \in A$ *exist. Further* f *is continuous for* $x \in A$ *with respect to* A *except at most a countable set of points.*

DEFINITION 1.2 (cf. [5]; Theorem 2.2). Let $f \in BV^*[a, b]$. Define $f^*(x)$ in $[a, b]$ as follows:

$$f^*(x) = \lim_{v_i \to x+} f(v_i), \qquad v_i \in A,\ a \leqslant x < b,$$

$$f^*(b) = \lim_{v_i \to b-} f(v_i), \qquad v_i \in A,$$

and call $f^*(x)$ the *reduced function* of f in $[a, b]$.

LEMMA 1.3 ([5]; Theorem 2.2). *If* $f \in BV^*[a,b]$, *then the reduced function* f^* *is right-hand side continuous for* $a \leqslant x < b$ *and left-hand side continuous for* $x = b$. *Further* $f^*(x)$ *equals* f *almost everywhere in* $[a, b]$.

LEMMA 1.4 ([5]; § 2, Corollary 1). *If* $f \in BV^*[a, b]$, $g \in BV^*[a, b]$ *and* $f(x) = g(x)$ *almost everywhere, then* $f^*(x) = g^*(x)$ *for* $a \leqslant x \leqslant b$ *and* $V^*[f; a, b] = V^*[g; a, b] = V[f^*; a, b] = V[g^*; a, b]$.

From the definition of reduced function and Lemma 1.4 we obtain

LEMMA 1.5. *If* $f \in BV^*[a, b]$ *and* f *is right continuous in* $[a, b)$ *and left continuous at* b, *then* $V^*[f; a, b] = V[f; a, b]$.

2. Some results on bounded essential variation. If $f \in BV[a, b]$, then clearly $f \in BV^*[a, b]$. But the converse is not true as may be seen by considering the well-known Dirichlet's function $\psi(x)$ which equals 1 for rational x and 0 for irrational x in $[0, 1]$.

We first prove the following lemma.

LEMMA 2.1. *For every function* $F(x) \in BV[a, b]$ *there exists a function* $f(x) \in BV^*[a, b] - BV[a, b]$ *such that* $f(x)$ *equals* $F(x)$ *except for a countable set of points and* $V^*[f; a, b] = V^*[F; a, b]$.

Proof. Let $F(x) \in BV[a, b]$ and let $\sum v_n$ be a divergent series of positive terms with $v_n \to 0$ as $n \to \infty$. Let $\{u_n\}$ be a strictly increasing sequence contained in an infinite subset G of $[a, b]$ with $mG = 0$. Define $f(x)$ in $[a, b]$ as follows:

$$\begin{aligned} f(x) &= F(x) + v_n \quad \text{for } x = u_{2n}\ (n = 1, 2, \ldots) \\ &= F(x) \qquad\quad \text{elsewhere.} \end{aligned}$$

Let $E \subset [a, b] \setminus G$ and $mE = b - a$. Consider any subdivision D: $x_0 < x_1 < \ldots < x_m$ of $[a, b]$ with $x_i \in E$. Then

$$\sum_{i=1}^{m} |f(x_i) - f(x_{i-1})| = \sum_{i=1}^{m} |F(x_i) - F(x_{i-1})| \leqslant V[F; a, b].$$

This gives $V^*[f; a, b] \leqslant V[f; E] \leqslant V[F; a, b]$ and so $f \in BV^*[a, b]$.

Next, consider a subdivision $a \leqslant u_1 < u_2 < \ldots < u_{2p} < b$ of $[a, b]$. Then

$$|f(u_1)-f(a)| + \sum_{i=2}^{2p} |f(u_i)-f(u_{i-1})| + |f(b)-f(u_{2p})|$$

$$\geqslant \sum_{i=1}^{p} |f(u_{2i})-f(u_{2i-1})| = \sum_{i=1}^{p} |F(u_{2i})+v_i-F(u_{2i-1})|$$

$$\geqslant \sum_{i=1}^{p} v_i - \sum_{i=1}^{p} |F(u_{2i})-F(u_{2i-1})|$$

$$\geqslant \sum_{i=1}^{p} v_i - V[F; a, b].$$

Since $F \in BV[a, b]$ and $\sum v_n$ is divergent, it follows that $f \notin BV[a, b]$. The proof is now complete following the definition of $f(x)$ and Lemma 1.4.

Using Definition 1.1 and Lemma 1.1 we obtain

LEMMA 2.2. *If $\alpha(x)$ and $\beta(x)$ are of bounded essential variation in $[a, b]$, then $\alpha \pm \beta$ is also of bounded essential variation in $[a, b]$ and $V^*[\alpha \pm \beta; a, b] \leqslant V^*[\alpha; a, b] + V^*[\beta; a, b]$.*

COROLLARY 2.1. *If α, β are of bounded essential variation in $[a, b]$, then $V^*[\alpha \pm \beta; a, b] \geqslant |V^*[\alpha; a, b] - V^*[\beta; a, b]|$.*

THEOREM 2.1. *If $f(x)$ and $f_n(x)$, $n = 1, 2, \ldots$, are defined on $[a, b]$ and $\{f_n(x)\}$ converges to $f(x)$ almost everywhere in $[a, b]$, then*

$$\liminf_{n \to \infty} V^*[f_n; a, b] \geqslant V^*[f; a, b]. \tag{2.1}$$

Proof. Let G be a subset of $[a, b]$ such that $\{f_n(x)\}$ converges to $f(x)$ at each point of G. We first suppose that $V^*[f; a, b] < +\infty$. Consider a subset E of G with $mE = b-a$ and $V[f; E] < +\infty$. Let $\varepsilon > 0$ be arbitrary. There exists a subdivision D: $u_0 < u_1 < \ldots < u_m$ of $[a, b]$ with $u_i \in E$ such that:

$$\sum_{i=1}^{m} |f(u_i)-f(u_{i-1})| > V[f; E] - \varepsilon.$$

Since $f_n(u_i) \to f(u_i)$ as $n \to \infty$ for each i ($i = 0, 1, \ldots, m$), there is an n_0 such that for $n \geqslant n_0$ we have $|f_n(u_i)-f(u_i)| < \varepsilon/2m$, $i = 0, 1, \ldots, m$. For $n \geqslant n_0$, we have $|f_n(u_i)-f_n(u_{i-1})| + \varepsilon/m > |f(u_i)-f(u_{i-1})|$ for $i = 1, 2, \ldots, m$. Then for $n \geqslant n_0$

$$\sum_{i=1}^{m} \{|f_n(u_i)-f_n(u_{i-1})| + \varepsilon/m\} > \sum_{i=1}^{m} |f(u_i)-f(u_{i-1})| > V[f; E] - \varepsilon.$$

So, $V[f_n; E] + \varepsilon > V[f; E] - \varepsilon \geqslant V^*[f; a, b] - \varepsilon$. Since $mE = b-a$, $E \subset G$ is arbitrary and addition of more points on E does not decrease the sum, it follows that

$$V^*[f_n; a, b] + \varepsilon \geqslant V^*[f; a, b] - \varepsilon. \tag{2.2}$$

Inequality (2.2) is true for each $n \geqslant n_0$ and hence

$$\liminf_{n\to\infty} V^*[f_n; a, b] \geqslant V^*[f; a, b].$$

If $V^*[f; a, b] = +\infty$, then $V[f; E] = +\infty$ for $E \subset [a, b]$ with $mE = b-a$. Let E_1 be a fixed subset of $[a, b]$ with $mE_1 = b-a$ and $E_1 \subset G$. Then for $\varepsilon > 0$ arbitrary, there exists a subdivision D_1: $u_0 < u_1 < \ldots < u_p$ of $[a, b]$ with $u_i \in E_1$ such that

$$\sum_{i=1}^{p} |f(u_i) - f(u_{i-1})| > 1/\varepsilon.$$

The procedure as above shows that $V^*[f_n; a, b] + \varepsilon > 1/\varepsilon$ for $n \geqslant n_1(\varepsilon)$. Hence $\liminf_{n\to\infty} V^*[f_n; a, b] = +\infty$. This completes the proof.

Note 2.1. It may be noted that everywhere convergence of the sequence $\{f_n(x)\}$ to $f(x)$ or even the uniform convergence is not sufficient to ensure the equality in (2.1). For example, consider $f_n(x) = \frac{1}{n}\sin nx$, $0 \leqslant x \leqslant \pi$. Then $\{f_n(x)\}$ converges uniformly to $f(x) \equiv 0$ in $[0, \pi]$. By Lemma 1.5, $V^*[f_n; 0, \pi] = V[f_n; 0, \pi] = 2$ for each n and $V^*[f; 0, \pi] = 0$. So,

$$\lim_{n\to\infty} V^*[f_n; 0, \pi] = 2 \neq V^*[f; 0, \pi].$$

The equality in (2.1) will be established in the next section under certain additional hypotheses.

3. Convergence in bounded essential variation. Let $g_0(x)$ and $g_n(x)$, $n = 1, 2, \ldots$ be defined on $[a, b]$ and let $\{g_n(x)\}$ converge to $g_0(x)$ at each point of a subset G of $[a, b]$ with $mG = b-a$. By Lemma 1.1, there exist $A_n \subset G$ with $mA_n = b-a$ such that $V[g_n; A_n] = V^*[g_n; a, b]$, $n = 0, 1, \ldots$ Let $A^0 = \bigcap_{n=0}^{\infty} A_n$ so that $mA^0 = b-a$ and $V[g_n; A^0] = V^*[g_n; a, b]$ for each $n \geqslant 0$.

PROPERTY A*. The sequence $\{g_n(x)\}$ is said *to possess Property* A* on $[a, b]$ if a subdivision D_0: $\xi_0 < \xi_1 < \ldots < \xi_\mu$ of $[a, b]$ with $\xi_i \in A^0$ and a positive integer m exists such that $|g_n(x') - g_n(x'')| \geqslant |g_m(x') - g_m(x'')|$ whenever $n \geqslant m$ and x', x'' belong to the same subinterval $[\xi_{i-1}, \xi_i]$, $0 \leqslant i \leqslant \mu+1$, $\xi_{-1} = a$, $\xi_{\mu+1} = b$.

Let Δ be the collection of all subdivisions D: $x_0 < x_1 < \ldots < x_p$ of $[a, b]$ with $x_i \in A^0$. If $\varphi(x)$ be any function defined on $[a, b]$, we write

$$(\varphi, D) = \sum_{i=1}^{p} |\varphi(x_i) - \varphi(x_{i-1})|.$$

We present two lemmas without proof.

LEMMA 3.1. $\lim_{n\to\infty} (g_n, D) = (g_0, D)$ *for every* $D \in \Delta$.

LEMMA 3.2. *If* $V^*[g_n; a, b] \leqslant K$ *for all n, where K is a finite number, then* $V^*[g_0; a, b] \leqslant K$.

LEMMA 3.3. *If the sequence* $\{g_n(x)\}$ *possesses Property* A* *on* $[a, b]$ *and if* $V^*[g_n; a, b] > K$ *for all n, K being fixed, then a* $D \in \Delta$ *exists such that* $(g_n, D) > K$ *for all n.*

Proof. Let $D_1 \in \Delta$ where $D_1 \supset D_0$, D_0 being the subdivision in relation to Property A*. It is easily seen that $(g_n, D_1) \geqslant (g_m, D_1)$ when $n \geqslant m$. Since $V^*[g_i; a, b] > K$ for each i, $1 \leqslant i \leqslant m$, an element D_2 of Δ exists such that $(g_i; D_2) > K$ for each i, $1 \leqslant i \leqslant m$.

If $D = D_1 \cup D_2$, then $D \in \Delta$ and $(g_n, D) > K$ for all n. This proves the lemma.

LEMMA 3.4. *If* $\{g_n(x)\}$ *and all its subsequences possess Property* A* *on* $[a, b]$ *and if* $V^*[g_0; a, b] < K$, *K being fixed, then* $V^*[g_n; a, b] \leqslant K$ *for all n except possibly for a finite number.*

Proof. If possible, suppose that the lemma is false. Then there exists a sequence of positive integers $\{n_i\}$ with $n_i \to \infty$ such that $V^*[g_{n_i}; a, b] > K$. Using Lemma 3.3 and then Lemma 3.1 we obtain $(g_0, D) \geqslant K$ for some $D \in \Delta$. So $V^*[g_0; a, b] \geqslant K$. The contradiction proves the lemma.

THEOREM 3.1. *If* $\{g_n(x)\}$ *and all its subsequences possess Property* A* *on* $[a, b]$ *and* $V^*[g_n; a, b]$ *is finite for each n, then*

$$\lim_{n\to\infty} V^*[g_n; a, b] = V^*[g_0; a, b].$$

Proof. We first suppose that $V^*[g_0; a, b] < +\infty$. Then there exists a positive number K such that $V^*[g_0; a, b] < K$. By Lemma 3.4, there exists an integer n_0 such that $V^*[g_n; a, b] \leqslant K$ for $n \geqslant n_0$.

Let $L = \limsup V^*[g_n; a, b]$ and $l = \liminf V^*[g_n; a, b]$. There exists a sequence $\{n_i\}$ of positive integers such that $V^*[g_{n_i}; a, b] \to L$, as $i \to \infty$.

If $\varepsilon > 0$ is arbitrary, an integer i_0 exists such that

$$(3.1) \qquad L - \varepsilon < V^*[g_{n_i}; a, b] < L + \varepsilon \quad \text{when } i \geqslant i_0.$$

So, by Lemma 3.2,

$$(3.2) \qquad V^*[g_0; a, b] \leqslant L + \varepsilon.$$

Utilizing Lemma 3.3 and then Lemma 3.2, we obtain, from the first inequality in (3.1), $(g_0, D) \geq L-\varepsilon$ for some $D \in \Delta$. This gives

$$V^*[g_0; a, b] = V[g_0; A^0] \geq L-\varepsilon. \tag{3.3}$$

As $\varepsilon > 0$ is arbitrary, combining (3.2) and (3.3) it follows that $V^*[g_0; a, b] = L$. Similarly, we obtain $V^*[g_0; a, b] = l$. Hence $\lim_{n\to\infty} V^*[g_n; a, b] = V^*[g_0; a, b]$.

If $V^*[g_0; a, b] = +\infty$, then there cannot exist any subsequence $\{n_i\}$ of positive integers for which the sequence $\{V^*[g_{n_i}; a, b]\}$ is bounded, because in that case $V^*[g_0; a, b]$ would be bounded, by Lemma 3.2, and this would imply that $V^*[g_0; a, b]$ is finite. Hence $\lim_{n\to\infty} V^*[g_n; a, b] = +\infty$. This completes the proof.

4. The space (X, d)**.** Let X denote the set of all functions $x(t)$ such that $x \in BV^*[0, 1]$. To each pair of functions x, y in X we associate the real number $d(x, y)$ defined by

$$d(x, y) = \int_0^1 |x(t)-y(t)|\, dt + |V^*(x) - V^*(y)|, \tag{4.1}$$

where the integral is taken in the Lebesgue sense and $V^*(x)$ stands for $V^*[x; 0, 1]$. The existence of the integral on the right of (4.1), is assured by the fact that corresponding to each $x \in X$ there exists an $x^* \in BV[0, 1]$ such that x^* equals x almost everywhere in $[0, 1]$ and $V[x^*; 0, 1] = V^*[x; 0, 1]$ (see Lemmas 1.3 and 1.4). If $x(t) = y(t)$ almost everywhere in $[0, 1]$ and $x, y \in X$, then the integral part as well as the variation part in (4.1) vanish separately and consequently $d(x, y) = 0$. But if $d(x, y) = 0$, then $V^*(x) = V^*(y)$ and $x(t) = y(t)$ almost everywhere in $[0, 1]$. It follows therefore that d is a pseudometric for X and therefore (X, d) is a pseudometric space.

In the following results we shall use the terms closed, sphere, compact, separability, etc., with reference to the pseudometric d for X.

THEOREM 4.1. *The space* (X, d) *is separable.*

To prove the theorem we require the following definition and results.

DEFINITION 4.1 (cf. [2]; § 1). Let $f(x)$ be defined on $[a, b]$ and $E \subset [a, b]$ with $mE = b-a$ and $V[f; E] = V^*[f; a, b]$. Let D: $x_0 < x_1 < \ldots < x_m$ be any subdivision of $[a, b]$ with $x_i \in E$. We denote by $B(x) = B(x; f, D)$ the function whose graph is the polygonal line joining the points $(x_i, f(x_i))$, $0 \leq i \leq m$, $B(x) = B(x_0)$ for $a \leq x \leq x_0$, $B(x) = B(x_m)$ for $x_m \leq x \leq b$. $B(x)$ is said to be a *polygonal function associated with* $f(x)$ *relative to the subdivision* D.

It is clear that

$$\sum_{i=1}^{m} |f(x_i)-f(x_{i-1})| = \sum_{i=1}^{m} |B(x_i)-B(x_{i-1})|$$
$$= V[B; a, b] = V^*[B; a, b].$$

The last equality is obtained owing to the continuity of the polygonal function $B(x)$ (see Lemma 1.5). It then follows that

$$V^*[f; a, b] = V[f; E] \geqslant V^*[B; a, b]. \tag{4.2}$$

THEOREM 4.2 (cf. [2]; § 2). *If $f \in BV^*[a, b]$, then it is possible to choose a sequence $\{B_n(x)\}$ of polygonal functions such that $B_n(x)$ converges to $f(x)$ almost everywhere in $[a, b]$ and*

$$\lim_{n\to\infty} V^*[B_n; a, b] = V^*[f; a, b].$$

The following lemma is needed to prove Theorem 4.2.

LEMMA 4.1. *If $f \in BV^*[a, b]$ and $a \leqslant x_1 < b$, then for $\varepsilon > 0$ arbitrary there exists $\delta > 0$ such that $V^*[f; x_1, x] < \varepsilon$ for $x_1 < x < x_1+\delta$.*

Proof. Let f_1^* be the reduced function of f in $[x_1, b]$. The right-hand continuity of f_1^* at x_1 shows that to each $\varepsilon > 0$ there exists a $\delta > 0$ such that $V[f_1^*; x_1, x] < \varepsilon$ whenever $x_1 < x < x_1+\delta$. This implies that $V^*[f_1^*; x_1, x] < \varepsilon$ whenever $x_1 < x < x_1+\delta$. Since $f_1^* = f$ almost everywhere in $[x_1, x_1+\delta]$, this yields $V^*[f; x_1, x] < \varepsilon$ whenever $x_1 < x < x_1+\delta$.

Proof of Theorem 4.2. Let $E \subset [a, b]$, $mE = b-a$ and $V[f; E] = V^*[f; a, b]$. We note first that the set of points of discontinuity of f in E with respect to E is countable (see Lemma 1.2). Let $\{D_n\}$ be a sequence of subdivisions D_n: $x_{n,0} < x_{n,1} < \ldots < x_{n,r_n}$ of $[a, b]$ with $x_{n,i} \in E$, $0 \leqslant i \leqslant r_n$, such that $D_n \subset D_{n+1}$, $\bar{D} = \cup D_n$ is everywhere dense in $[a, b]$ and $\bar{D}$ contains all the points of discontinuity of f in E with respect to E. Consider the sequence $\{B_n(x)\}$ with $B_n(x) = B(x; f, D_n)$ for each n.

We show that $B_n(x) \to f(x)$ at each point of $E\setminus\{b\}$. At each point of $\bar{D}$ the convergence is evident. If $\xi \in E\setminus(\{b\} \cup \bar{D})$, then for arbitrary $\varepsilon > 0$ we choose ξ' ($> \xi$) in $\bar{D}$ such that $|f(\xi)-f(\xi')| < \frac{1}{2}\varepsilon$ and $V^*[f; \xi, \xi'] < \frac{1}{2}\varepsilon$. The first inequality is obtained owing to the continuity of f at ξ with respect to E and the second inequality is obtained from Lemma 4.1. There exists a positive integer N such that $\xi' \in D_n$ for all $n \geqslant N$. Then for all $n \geqslant N$

$$\begin{aligned}
|B_n(\xi)-f(\xi)| &\leqslant |B_n(\xi)-B_n(\xi')|+|f(\xi')-f(\xi)| \\
&\leqslant V[B_n; \xi, \xi']+\tfrac{1}{2}\varepsilon \\
&= V^*[B_n; \xi, \xi']+\tfrac{1}{2}\varepsilon \qquad \text{(see Lemma 1.5)} \\
&\leqslant V^*[f; \xi, \xi']+\tfrac{1}{2}\varepsilon \qquad \text{(using (4.2))} \\
&< \varepsilon.
\end{aligned}$$

As $mE = b-a$, it follows that $B_n(x) \to f(x)$ almost everywhere in $[a, b]$.

From Theorem 2.1 we get $\liminf_{n\to\infty} V^*[B_n; a, b] \geqslant V^*[f; a, b]$. Using (4.2) we obtain $\limsup_{n\to\infty} V^*[B_n; a, b] \leqslant V^*[f; a, b]$. This completes the proof of Theorem 4.2.

Proof of Theorem 4.1. Let Q denote the set of all polygonal functions in X with rational corners. Clearly Q is countable.

Let $x(t) \in X$. By Theorem 4.2, it is possible to choose a sequence of polygonal functions $\{B_n(t)\}$ in X such that $B_n(t) \to x(t)$ almost everywhere in $[0, 1]$ and $V^*(B_n) \to V^*(x)$. For each $B_n(t)$ we can choose a polygonal function $P_n(t)$ in Q such that $|B_n(t) - P_n(t)| < 1/n$ everywhere in $[0, 1]$ and $|V^*(B_n) - V^*(P_n)| < 1/n$. So, the sequence $\{P_n(t)\}$ converges to $x(t)$ almost everywhere in $[0, 1]$ and $V^*(P_n) \to V^*(x)$. Therefore $d(P_n, x) \to 0$ as $n \to \infty$. This shows that x is an accumulation point of Q and hence Q is dense in X. This completes the proof.

Note 4.1. Whenever Theorem 4.2 is obtained, the proof of Theorem 4.1 is verbatim with the proof of Theorem 4.1 of [1]. However, for the sake of completeness, we give the proof.

THEOREM 4.3. *No sphere in (X, d) is compact.*

Proof. Let y be a sphere in (X, d) with centre $\alpha(t) \in X$ and radius r ($\neq 0$). Consider the sequence $\{x_n(t)\}$ on $[0, 1]$ as follows:

$$x_n(t) = \alpha(t) + \alpha_n(t),$$

where $\alpha_n(t) = \frac{K}{n} \sin n\pi t$, $0 \leqslant t \leqslant 1$, $|K| \leqslant r/3$, $n = 1, 2, \ldots$ Clearly $\{x_n(t)\}$ converges uniformly to $\alpha(t)$ in $[0, 1]$. Further, by Lemma 1.5, $V^*(\alpha_n) = V(\alpha_n) = 2|K|$ for each n so that $\alpha_n \in X$. That $x_n \in X$ follows from the inequality $V^*(x_n) \leqslant V^*(\alpha) + V^*(\alpha_n)$ (see Lemma 2.2). Also

$$\begin{aligned} d(x_n, \alpha) &= \int_0^1 |x_n(t) - \alpha(t)|\, dt + |V^*(x_n) - V^*(\alpha)| \\ &\leqslant \int_0^1 |\alpha_n(t)|\, dt + V^*(\alpha_n) \\ &\leqslant |K|/n + 2|K| \leqslant 3|K|. \end{aligned}$$

Since $|K| \leqslant r/3$, $d(x_n, \alpha) \leqslant r$ for each n so that each $x_n \in y$.

If possible, let y be compact. Then there is a subsequence $\{x_{n_i}\}$ of $\{x_n\}$ which converges to an element x of y. Now $d(x_{n_i}, x) \to 0$ as $i \to \infty$ implies each of $\int_0^1 |x_{n_i}(t) - x(t)|\, dt$ and $|V^*(x_{n_i}) - V^*(x)|$ tending to 0 as $i \to \infty$. Uniform convergence of $\{x_{n_i}(t)\}$ to $\alpha(t)$ implies $\lim_{n\to\infty} \int_0^1 |x_{n_i}(t) - x(t)|\, dt = \int_0^1 |\alpha(t) - x(t)|\, dt$

and so $x(t) = \alpha(t)$ almost everywhere in [0, 1]. By Lemma 1.4, $V^*(x) = V^*(\alpha)$ and by Corollary 2.1, $|V^*(x_{n_i}) - V^*(x)| = |V^*(\alpha+\alpha_{n_i}) - V^*(\alpha)| \geqslant V^*(\alpha_{n_i}) = 2|K|$ for each i. The contradiction proves the theorem.

THEOREM 4.4. *The space (X, d) is not complete.*

Proof. Consider the sequence $\{\beta_n(t)\}$ on [0, 1] defined by $\beta_n(t) = \frac{1}{n}\sin n\pi t$, $n = 1, 2, \ldots$ By Lemma 1.5, $V^*(\beta_n) = 2$ for each n and so $\beta_n \in X$. For any two positive integers m and n

$$d(\beta_m, \beta_n) = \int_0^1 |\beta_m(t) - \beta_n(t)|\, dt + |V^*(\beta_m) - V^*(\beta_n)|$$
$$= \int_0^1 \left|\frac{\sin m\pi t}{m} - \frac{\sin n\pi t}{n}\right| dt$$
$$\leqslant \frac{1}{m} + \frac{1}{n}.$$

So, $d(\beta_m, \beta_n) \to 0$ as $m, n \to \infty$ and consequently $\{\beta_n\}$ is a Cauchy sequence in (X, d). If $\{\beta_n\}$ converges to a limit β in (X, d), we should have

$$(4.3) \qquad \lim_{n\to\infty} \int_0^1 |\beta_n(t) - \beta(t)|\, dt = 0 \quad \text{and} \quad \lim_{n\to\infty} |V^*(\beta_n) - V^*(\beta)| = 0.$$

The first shows that $\beta(t) = 0$ almost everywhere in [0, 1]. If $\beta \in X$, then $V^*(\beta) = 0$ and so $|V^*(\beta_n) - V^*(\beta)| = 2$. This contradicts the second relation of (4.3).

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