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Ministerstwo Nauki i Szkolnictwa Wyższego

MAREK WILHELM (Wrocław)

Some negative examples concerning nearly continuity

Abstract. Examples of real surjections and bijections having many points at which they are not nearly continuous (nearly open) are given.

1. Introduction. Let f be a real function; the definitions of the sets $C_n(f)$ and $O_n(f)$ consisting of all points of nearly continuity and, respectively, nearly openness of f are recalled at the beginnings of Sections 2 and 3. (Here the terms *nearly continuity* and *nearly openness* are used in the sense of Pták [6] and Pettis [5].) The set $C_n(f)$ is always residual [1], while $O_n(f)$ is of power continuum provided $\operatorname{Rng} f$ contains a second category Borel set (Proposition 1). Nevertheless, $C_n(f)$ may happen to be co-dense, or even of measure zero, while $O_n(f)$ may happen to be nowhere dense and of measure zero, even for a continuous surjection (Cantor's function). In Theorem 1 we show an example of an open surjection f with $C_n(f)$ of measure zero, in Theorem 2 — a bijection f with $f^{-1} = f$ and $C_n(f) \cup \overline{O_n(f)}$ of measure zero. In Theorem 3 we construct a nearly continuous bijection f such that $O_n(f) \cup C_n(f^{-1})$ is co-dense. In the last section we consider compositions of nearly continuous (nearly open) mappings and give an example of a nearly continuous and nearly open bijection f such that $C_n(f^2) \cup O_n(f^2)$ is co-dense (Theorem 4).

2. An open surjection with small $C_n(f)$. Given a real function f ($\operatorname{Dom} f = R \supset \operatorname{Rng} f$), the set $C_n(f)$ of all points of nearly continuity of f is defined as follows: $x \in C_n(f)$ iff for every open set V containing $f(x)$, x belongs to $\operatorname{Int} \overline{f^{-1}(V)}$ (the property was considered e.g. in [1], [3], [5]–[7]). Obviously $C(f) \subset C_n(f)$, where $C(f)$ stands for the set of all points of continuity of f. In [1] it was essentially shown that the set $C_n(f)$ is residual, i.e., its complement is of first category in R (for a generalization with simple proof see [7], Theorem 1). It is an elementary fact that each open real injection is continuous (f is said to be *open* if all images of open sets are open). For open surjections, however, the situation is quite different:

THEOREM 1. *There exists an open surjection $f\colon R \to R$ such that the set $C_n(f)$ is of measure zero.*

To prove this we need the following auxiliary statement.

LEMMA 1. *Let $A \subset R$ be a Borel set which is uncountably dense in itself. There exists an open surjection $g\colon A \to R$.*

Proof. Step I. Assume $A = C$ the Cantor ternary set. Consider the following equivalence relation $\sim$ in C: $c \sim d$ iff $c - d \in Q$ (the rationals). Let h be any bijection of $C/\sim$ (which is of power continuum) onto R, and put $g(c) = h([c])$, where $[c]$ denotes the equivalence class of c in $C/\sim$. Since each class $[c]$ is dense in C, $g(U) = R$ for every non-empty open set U in C.

Step II. Fix an open base $\{U_i\colon i \in N\}$ for the subspace A, where $U_i \neq \emptyset$ for all i. Since U_1 is an uncountable Borel set in R, it contains a homeomorphic copy A_1 of C (the Alexandroff–Hausdorff theorem; cf. [4], 37.I). Let i_2 be the first index i for which $U_i \cap A_1 = \emptyset$ (if such index exists); choose a copy A_2 of C in U_{i_2}. Let i_3 be the first index i for which $U_i \cap (A_1 \cup A_2) = \emptyset$ (if exists); choose a copy A_3 of C in U_{i_3}. And so on (until possible). There exists a disjoint family $\{A_j\colon j \in N_0\}$ of homeomorphic copies of C in A, where $N_0 \subset N$, such that

$$U_i \cap \bigcup_{j \in N_0} A_j \neq \emptyset \quad \text{for all } i \in N.$$

By Step I, there exists $g_j\colon A_j \to R$ with the property that $g_j(U) = R$ for every non-empty open set U in A_j ($j \in N_0$). Let $g\colon A \to R$ be any mapping extending all functions g_j. Given $i \in N$, $U_i \cap A_j \neq \emptyset$ for some $j \in N_0$, and so

$$g(U_i) \supset g(U_i \cap A_j) = R.$$

Proof of Theorem 1. Choose a disjoint sequence $\{A_i\colon i \in N\}$ of Cantor sets of positive measures in $R \setminus C$ so that

$$m\Big(R \setminus \bigcup_1^\infty A_i\Big) = 0.$$

(Here we use modified Cantor sets, having suitably large positive measures; see e.g. [2], 8.4.) Put $A_0 = 0$ and $A = R \setminus \bigcup_0^\infty A_i$. By Lemma 1 (all Cantor sets are homeomorphic [2], 8.23), there are open surjections $g_i\colon A_i \to (i-1, i+1)$ for $i = 0, 1, 2, \ldots$ and (A is a residual G_δ-set) there is an open surjection $g\colon A \to (-\infty, 0)$. The functions g_i and g build up a mapping $f\colon R \to R$, which is clearly an open surjection. For any $i \in N$ we have

$$f^{-1}\big((i-1, i+1)\big) \subset A_{i-1} \cup A_i \cup A_{i+1},$$

where the set on the right is nowhere dense. Thus $x \notin C_n(f)$ whenever $x \in \bigcup_1^\infty A_i$, which yields the assertion.

3. A bijection with small $C_n(f)$ and $O_n(f)$. We say that a mapping $f\colon R \to R$ is *nearly open at* x if for every open neighbourhood U of x, $f(x)$ is in

$\operatorname{Int}\overline{f(U)}$ (cf. [3], [5]–[7]). The set of all such points x will be denoted by $O_n(f)$; we have always $O(f) \subset O_n(f)$, where $O(f)$ is the set of all points of openness of f ($x \in U$ open $\Rightarrow$ $f(x) \in \operatorname{Int} f(U)$). When f is an injection with $\operatorname{Rng} f = C$ (Cantor's set), $O_n(f)$ is empty; one can also build an injection f with $O_n(f) = \emptyset$ and $\operatorname{Rng} f$ of full measure ($= \bigcup_1^\infty A_i$, where A_i as in the last proof).

PROPOSITION 1. *Let $f: R \to R$. If* $\operatorname{Rng} f$ *is of second category, the set $O_n(f)$ is uncountable. If* $\operatorname{Rng} f$ *contains a second category Borel set, $O_n(f)$ is of power continuum.*

Proof. Consider the multifunction $F = f^{-1}$; by Theorem 1 of [7], the set $L_n(F)$ of all points of nearly lower semicontinuity of F is residual in R. (In fact, only a countable base for the range space of F is needed here.) It follows from the definitions that $O_n(f)$ contains $f^{-1}(L_n(F))$. Since $L_n(F) \cap \operatorname{Rng} f$ is of second category, it is uncountable, and so is $f^{-1}(L_n(F))$. Under the second assumption, $L_n(F) \cap \operatorname{Rng} f$ contains a second category Borel set, which is of power continuum (by the Alexandroff–Hausdorff theorem).

In particular, $O_n(f)$ is of power continuum provided f is surjective. But even then $O_n(f)$ may be "small" in both category and measure theoretic senses of the word. An example is supplied by the Cantor function: a non-decreasing continuous mapping of $[0, 1]$ onto itself, which is constant on each of the intervals constituting $[0, 1] \setminus C$ (see e.g. [2], 8.15). For this function g, the set $O_n(g)$ is contained in C, and g may be easily extended to a non-decreasing continuous surjection $f: R \to R$ for which $O_n(f)$ is nowhere dense and of measure zero. On the other hand, it may happen that all the four sets $C_n(f)$, $O_n(f)$, $C_n(f^{-1})$ and $O_n(f^{-1})$ are simultaneously "small" (when f^{-1} exists).

THEOREM 2. *There exists a bijection $f: R \to R$ such that $f^{-1} = f$ and the set $\overline{C_n(f) \cup O_n(f)}$ is of measure zero.*

Proof. Let A, A_0, A_1, ... be as in the proof of Theorem 1. Choose a disjoint sequence $\{I_i: i = 0, 1, 2, \ldots\}$ of open intervals so that each intersection $A_0 \cap I_i$ be of power continuum and

$$A_0 \setminus \bigcup_0^\infty I_i = \{1\}.$$

Fix a point $a \in A$. We can now define a bijection f of A_0 onto $R \setminus A_0$ so that

$$\begin{aligned} f(A_0 \cap I_0) &= A \setminus \{a\}, \\ f(A_0 \cap I_i) &= A_i \quad \text{for } i \in N, \\ f(1) &= a, \end{aligned}$$

and extend it to a bijection $f\colon R \to R$ defining

$$f(x) = f^{-1}(x) \quad \text{for } x \in R \setminus A_0.$$

For any point $x \in A_i$, where $i \geqslant 1$, we have $f(x) \in I_i$ and

$$f^{-1}(I_i) = f^{-1}(I_i \cap A_0) \cup f^{-1}(I_i \setminus A_0) \subset A_i \cup A_0,$$

which is a nowhere dense set. Thus $C_n(f)$ is contained in $R \setminus \bigcup_1^\infty A_i$, which is of measure zero.

The set $R \setminus A_0$ is open and $f(R \setminus A_0) = A_0$ is nowhere dense; this implies that $O_n(f)$ has no point in $R \setminus A_0$. Thus $O_n(f)$ is contained in the Cantor set A_0.

4. Inverse of a nearly continuous bijection. Each continuous real injection is open; in particular, the inverse of a continuous bijection is itself continuous. Do these statements remain true when the words *nearly* are added? The answer is negative:

THEOREM 3. *There exists a nearly continuous bijection $f\colon R \to R$ such that the set $O_n(f) \cup C_n(f^{-1})$ is co-dense. (The inverse f^{-1} is nearly open but not nearly continuous.)*

LEMMA 2. *Let S and T be different elements of R/Q, $s \in S$ (fixed), $A = S \cup T$, $U = (s-1, s+1)$. There exists a bijection $f\colon A \to A$ such that $f(s) = s$, $f(a) = a$ for all $a \in A \setminus U$, $f|S$ and $f|T$ are continuous, f is not nearly open at s.*

Proof. Fix some $x \in (s-1, s) \setminus A$ and $y \in (s, s+1) \setminus A$. Let f_1 be a decreasing homeomorphism of $S \cap (x, s)$ onto $T \cap (s, y)$; let f_2 be an increasing homeomorphism of $S \cap (s-1, x)$ onto $S \cap (s-1, s)$; let f_3 be an increasing homeomorphism of $T \cap (s-1, y)$ onto $T \cap (s-1, s)$. Define

$$f(a) = f_i(a) \quad \text{whenever} \quad a \in \operatorname{Dom} f_i \quad (i = 1, 2, 3)$$

and

$$f(a) = a \quad \text{for all the remaining } a \in A.$$

Since the critical points x and y do not belong to A, both $f|S$ and $f|T$ are continuous. Put $V = A \cap (x, z)$, where $z = \frac{1}{2}(s+y)$, and take any point $a \in V$. If $a \in S$, then $f(a) \geqslant s$; if $a \in T$, then $f(a) < f(z) < s$. Thus $f(V)$ is disjoint with the interval $(f(z), s)$, and so

$$f(s) = s \notin \operatorname{Int} \overline{f(V)}.$$

LEMMA 3. *There exists a sequence $\{s_i\colon i \in N\}$ dense in R, $s_i - s_j \notin Q$ whenever $i \neq j$, and a permutation $p\colon N \to N$ such that*

$$s_i + 7i - 1 < s_{p(i)} < s_i + 7i + 1, \quad i \in N.$$

Proof. Let $N = \bigcup_1^\infty N_k$, where the sets N_k are pairwise disjoint and infinite, and let N_0 be a selector from the sets N_k. Choose a set $\{s_i\colon i \in N_0\}$ dense in R so that $s_i - s_j \notin Q$ whenever $i \neq j$. Let p be any permutation of N such that $p(N_k) = N_k$ and $p|N_k$ has no proper invariant set ($k \in N$). It is now easy to choose numbers s_i for all $i \in N_1 \setminus N_0$ so that the desired double inequality be satisfied for all $i \in N_1$ and $s_i - s_j \notin Q$ whenever $i \neq j$, $i, j \in N_0 \cup N_1$. Similarly we can choose s_i for all $i \in N_2 \setminus N_0$ so that the double inequality be satisfied for all $i \in N_2$ and $s_i - s_j \notin Q$ whenever $i \neq j$, $i, j \in N_0 \cup N_1 \cup N_2$. And so on.

Proof of Theorem 3. Let s_i and p be as in Lemma 3. Fix a number $t \in R$ in such a way that

$$s_i + t - s_j \notin Q, \qquad i, j \in N.$$

Given $i \in N$, apply Lemma 2 to $S_i = s_i + Q$, $T_i = s_i + t + Q$, $A_i = S_i \cup T_i$, $U_i = (s_i - 1, s_i + 1)$; there exists a bijection $f_i\colon A_i \to A_i$ such that

$$f_i(s_i) = s_i,$$
$$f_i(a) = a \quad \text{for all } a \in A_i \setminus U_i,$$
$$f_i|S_i \quad \text{and} \quad f_i|T_i \quad \text{are continuous},$$
$$f_i \quad \text{is not nearly open at } s_i.$$

Define

$$f(x) = f_i(x) + s_{p(i)} - s_i \qquad \text{whenever } x \in A_i \ (i \in N)$$

and

$$f(x) = x \qquad \text{for all } x \in B := R \setminus \bigcup_1^\infty A_i.$$

We have

$$\begin{aligned} &f(s_i) = s_{p(i)} \in A_{p(i)}, \\ &f(x) = x + s_{p(i)} - s_i \in A_{p(i)} \setminus U_{p(i)} \qquad &&\text{for } x \in A_i \setminus U_i, \\ &f(x) \in A_{p(i)} \cap U_{p(i)} &&\text{for } x \in A_i \cap U_i. \end{aligned}$$

Since $f|A_i$ is a bijection of A_i onto $A_{p(i)}$ and p is a permutation, f is a bijection as well. All restrictions $f|S_i$, $f|T_i$ and $f|B$ are continuous, the sets being dense in R, and so f is nearly continuous. To complete the proof, we will show that each point s_i is not in $O_n(f)$ (whence $s_{p(i)}$ is not in $C_n(f^{-1})$). Let $x \in U_i$. If $x \in A_j$ for some $j \neq i$, then

$$\begin{aligned} f(x) &\in (x + s_{p(j)} - s_j - 2,\ x + s_{p(j)} - s_j + 2) \\ &\subset \big((s_i - 1) + (7j - 1) - 2,\ (s_i + 1) + (7j + 1) + 2\big) = (s_i + 7j - 4,\ s_i + 7j + 4), \end{aligned}$$

and so $f(x)$ is not in (s_i+7i-2, s_i+7i+2), which contains $U_{p(i)}$. If $x \in B$, then $f(x) = x$ also does not belong to $U_{p(i)}$. Thus $f(x)$ is not in $U_{p(i)}$ unless $x \in A_i$, and so f is not nearly open at s_i, because f_i is not.

5. Compositions. It is quite obvious that the composition gf is nearly continuous provided g is continuous and f nearly continuous, and that gf is nearly open provided f is open and g nearly open. We have also

PROPOSITION 2. (i) *If g is nearly continuous and f is continuous and open, then gf is nearly continuous.* (ii) *If f is nearly open and g is open and continuous, then gf is nearly open.*

Proof. Part (i) follows from the equality

$$f^{-1}(\operatorname{Int} \bar{B}) = \operatorname{Int} \overline{f^{-1}(B)}, \qquad B \subset R,$$

part (ii) from the inclusion

$$g(\operatorname{Int} \bar{A}) \subset \operatorname{Int} \overline{g(A)}, \qquad A \subset R.$$

(The above four statements about gf possess pointwise counterparts and remain true in arbitrary topological spaces.)

EXAMPLE 1. Put $f(x) = x$ for $x \in R \setminus Q$ and $f(x) = x+1$ for $x \in Q$; f is a nearly open and nearly continuous bijection. Let h_i be a bijection of R/Q onto $(i-1, i)$. Define $g(i) = i$ and $g(x) = h_i([x])$ for $x \in (i-1, i)$ $(i \in Z$, integers); g is an open surjection. No point of $Q \setminus Z$ belongs to $O_n(gf)$, which is thus co-dense.

EXAMPLE 2. Let $f\colon R \to R$ be the (non-decreasing continuous) Cantor function mentioned in the passage preceding Theorem 2 ($f(x+i) = g(x)+i$ for $x \in [0, 1)$ and $i \in Z$, where g is the Cantor function on $[0, 1]$). Let D denote the collection of all binary rational points in R. Put $g(x) = x$ for $x \in R \setminus D$ and $g(x) = -x$ for $x \in D$; g is a nearly continuous and nearly open bijection. No point of $\bigcup_{i \in Z} (C+i) \setminus D$ (C = Cantor's set in $[0, 1]$) belongs to $C_n(gf)$.

Using Cantor sets of positive measures in Example 2, we may obtain $C_n(gf)$ of arbitrarily small positive measure. But f and g cannot be chosen so that $C_n(gf)$ be of measure zero:

PROPOSITION 3. *Suppose $f\colon R \to R$ is continuous and $g\colon R \to R$ is nearly continuous. Then the set $R \setminus C_n(gf)$ is nowhere dense.*

Proof. Put $h = gf$ and assume $x \in R \setminus C_n(h)$. There is an open set W with $h(x) \in W$ and $x \notin \operatorname{Int} \overline{h^{-1}(W)}$. Since $f(x) \in C_n(g)$,

$$f(x) \in V := \operatorname{Int} \overline{g^{-1}(W)}.$$

Let U be any open neighbourhood of x contained in the open set $f^{-1}(V)$. There exists a non-empty open interval $I \subset U$ such that $I \cap h^{-1}(W) = \emptyset$.

Since $f(I)\cap g^{-1}(W)=\emptyset$, $f(I)\subset V$, $g^{-1}(W)$ is dense in V and $f(I)$ is connected, f must be constant on I. Hence $I\subset C(h)\subset C_n(h)$.

In Section 4 we gave an example of a nearly continuous bijection f such that f^{-1} does not inherit this property. A related question is whether the composition gf of two nearly continuous bijections must inherit this property. The answer is again negative.

THEOREM 4. *There exists a nearly continuous and nearly open bijection* $f\colon R\to R$ *such that for the composition* $g=f^2$ *the set* $C_n(g)\cup O_n(g)\cup\cup C_n(g^{-1})\cup O_n(g^{-1})$ *is co-dense.*

Using a method similar as in Lemma 3 one can prove

LEMMA 4. *Let* $Q=\{r_i\colon i\in N\}$. *There exists a permutation* $p\colon N\to N$ *such that*

$$r_i+2^i-\tfrac{1}{5}<r_{p(i)}<r_i+2^i+\tfrac{1}{5},\qquad i\in N.$$

Proof of Theorem 4. Let r_i and p be as in Lemma 4. Choose a sequence $\{S_i\colon i\in N\}$ of different elements of $R/Q\setminus\{Q\}$. Define

$$f(r_i)=r_{p(i)}\qquad \text{for } i\in N,$$
$$f(x)=x+r_{p(i)}-r_i\qquad \text{for } x\in S_i$$

and

$$f(x)=x\qquad \text{for } x\in A:=(R\setminus Q)\setminus\bigcup_1^\infty S_i.$$

Evidently f is a bijection and Q, S_i, A are invariant sets. We have $R\setminus Q\subset C_n(f)$, because $f|A$ and $f|S_i$ are continuous and $\bar{A}=R=\bar{S}_i$. Similarly $R\setminus Q\subset O_n(f)$. Suppose U is an open neighbourhood of r_i ($i\in N$ fixed). Then

$$f(U)\supset f(U\cap S_i)=U\cap S_i+r_{p(i)}-r_i$$

and

$$\overline{f(U)}\supset U+r_{p(i)}-r_i\ni r_{p(i)}.$$

Thus $r_i\in O_n(f)$, and so $Q\subset O_n(f)$. Similarly $Q\subset C_n(f)$.

Now consider the composition $g=f^2$. Put $U=(r_i-\frac{1}{5},r_i+\frac{1}{5})$ ($i\in N$ fixed). We have

$$g(r_i)=r_{p^2(i)}\in V:=(r_i+2^i+2^{p(i)}-\tfrac{2}{5},\ r_i+2^i+2^{p(i)}+\tfrac{2}{5}).$$

Since $g|A$ is identity, $g(U\cap A)\cap V=\emptyset$. Since $2^k+2^{p(k)}\neq 2^i+2^{p(i)}$ unless $k=i$ (even the case $k=p(i)$ and $p(k)=i$ is impossible), $g(U\cap Q)\cap V=\{g(r_i)\}$. For every $k\in N$ we have $2^k+2^k\neq 2^i+2^{p(i)}$, which implies $g(U\cap S_k)\cap V=\emptyset$. Summing this up, $g(U)\cap V=\{g(r_i)\}$, and so $g(r_i)\notin \operatorname{Int}\overline{g(U)}$. Hence $Q\cap O_n(g)=\emptyset$. For analogous reasons Q has no point in $C_n(g)\cup C_n(g^{-1})\cup O_n(g^{-1})$.

References

[1] H. Blumberg, *New properties of all real functions*, Trans. Amer. Math. Soc. 24 (1922), 113–128.

[2] B. R. Gelbaum and J. M. H. Olmsted, *Counterexamples in Analysis*, San Francisco 1964.

[3] J. L. Kelley, *General Topology*, New York 1955.

[4] K. Kuratowski, *Topology*, Vol. I, New York 1966.

[5] B. J. Pettis, *Closed graph and open mapping theorems in certain topologically complete spaces*, Bull. London Math. Soc. 6 (1974), 37–41.

[6] V. Pták, *Completeness and the open mapping theorem*, Bull. Soc. Math. France 86 (1958), 41–74.

[7] M. Wilhelm, *Nearly lower semicontinuity and its applications*, in: *General Topology and its Relations to Modern Analysis and Algebra* V, Proc. Fifth Prague Topol. Symp. 1981, Berlin 1983, 692–698.

INSTYTUT MATEMATYCZNY POLITECHNIKI WROCŁAWSKIEJ
WROCŁAW, POLAND