ANNALES SOCIETATIS MATHEMATICAE POLONAE
Series I: COMMENTATIONES MATHEMATICAE XXVI (1986)
ROCZNIKI POLSKIEGO TOWARZYSTWA MATEMATYCZNEGO
Seria I: PRACE MATEMATYCZNE XXVI (1986)

J. URBANOWICZ (Warszawa)

On a certain non-linear boundary value problem for the one-dimensional bicaloric equation

1. Introduction. The aim of this paper is to prove the existence of a solution of a non-linear boundary value problem for the one-dimensional bicaloric equation. We reduce the problem to a system of Volterra equations which contains both the first and the second kind of equations and then we apply Gevrey's method to reduce this system to a system containing only second kind of Volterra equations. The last system is solved by using Schauder's fixed point theorem. In order to apply the aforesaid procedure we need some integrals connected with the bicaloric equation, the properties of which are given in the first part of this paper. Let us note that up to now the boundary value problems for the bicaloric equation have been examined only either in the linear case (see Nicolescu [11], Borzymowski [2], Brzeziński [4], Musiałek [10], Milewski [9]), or in the non-linear case for special types of domains (see Barański, Musiałek [1], Brzeziński [3]).

2. Properties of integrals. Let T be a positive number, whereas $\chi_i(t)$ $(i = 1, 2)$ functions defined in the interval $\langle 0, T\rangle$ and fulfilling the condition:

(2.1) $$|\chi_i(t_2)-\chi_i(t_1)| \leqslant K_{\chi_i}|t_2-t_1|^{(1+\varkappa)/2} \quad (i = 1, 2),$$

where $\varkappa\in(0, 1\rangle$, $K_{\chi_i} > 0$.

We assume that the curves of equations $x = \chi_i(t)$ $(i = 1, 2)$ do not intersect for $t\in\langle 0, T\rangle$.

We introduce the following notation:

(2.2) $$U_\chi(x, t; \varphi) = \int_0^t \omega(x, t, \chi(\tau), \tau)\varphi(\tau)\,d\tau,$$

(2.3) $$J_\chi(x, t; \varphi) = \int_0^t (x-\chi(\tau))D_x\omega(x, t, \chi(\tau), \tau)\varphi(\tau)\,d\tau,$$

(2.4) $$H_{S_T}(x, t; f) = -\frac{1}{2\sqrt{\pi}}\iint_{S_t}(t-\tau)\omega(x, t, \xi, \tau)f(\xi, \tau)\,d\xi\,d\tau,$$

$$K_{ij}(t,\tau) = D_t\left(\int\limits_{\tau}^{t} \frac{\omega\big(\chi_i(z), z, \chi_j(\tau), \tau\big)}{\sqrt{t-z}}\,dz\right) \quad (i, j = 1, 2), \tag{2.5}$$

(2.6)

$$G_{ij}(t,\tau) = \int\limits_{\tau}^{t} \frac{1}{\sqrt{t-z}} \big(\chi_i(z) - \chi_j(\tau)\big) D_x \omega\big(\chi_i(z), z, \chi_j(\tau), \tau\big)\,dz \quad (i, j = 1, 2),$$

where $\omega(x, t, \xi, \tau) = \dfrac{1}{\sqrt{t-\tau}} \exp -\dfrac{(x-\xi)^2}{4(t-\tau)}$,

$$S_t = \{(\xi, t)\colon \chi_1(\tau) < \xi < \chi_2(\tau);\ 0 < \tau < t\},$$

$$\chi(t) = \chi_1(t) \quad \text{or} \quad \chi(t) = \chi_2(t).$$

We are going to formulate several theorems concerning the properties of the above integrals. They will be used in the next section in the examination of the boundary problem.

THEOREM 1 (see [7], p. 1473–1489). *If condition* (2.1) *is satisfied and if the function $\varphi(t)$ is defined in $\langle 0, T\rangle$ and fulfils the condition*

$$|\varphi(t_2) - \varphi(t_1)| \leqslant K_\varphi |t_2 - t_1|^{h_\varphi}; \quad \varphi(0) = 0,$$

then the inequality

$$\big|U_\chi(\chi(t_2), t_2; \varphi) - U_\chi(\chi(t_1), t_1; \varphi)\big| \leqslant \text{const}\ K_\varphi |t_2 - t_1|^{1/2 + \min(h_\varphi, \varkappa/2)}$$

holds for $t_1, t_2 \in \langle 0, T\rangle$, where $K_\varphi > 0$; $0 < h_\varphi < \frac{1}{2}$.

THEOREM 2. *If condition* (2.1) *is fulfilled and in addition the function $\varphi(t)$ is bounded and integrable in $\langle 0, T\rangle$, $\big(\sup_{t\in\langle 0,T\rangle} |\varphi(t)| \leqslant \tilde{K}_\varphi\big)$, then the following inequalities*

$$|U_\chi(x_2, t; \varphi) - U_\chi(x_1, t; \varphi)| \leqslant \text{const}\ \tilde{K}_\varphi |x_2 - x_1| \quad \text{for } (x_2, t), (x_1, t) \in \bar{S}_T, \tag{2.7}$$

$$|U_\chi(x, t_2; \varphi) - U_\chi(x, t_1; \varphi)| \leqslant \text{const}\cdot \tilde{K}_\varphi |t_2 - t_1|^{\theta/2} \quad \text{for } (x, t_1), (x, t_2) \in \bar{S}_T, \tag{2.8}$$

$$|D_x^k U_\chi(x, t; \varphi)| \leqslant \frac{\text{const}\cdot \tilde{K}_\varphi}{|x - \chi(t)|^{k-1}} \quad \text{for } (x, t) \in S_T, \tag{2.9}$$

$$|D_x^k U_\chi(x, t_1; \varphi) - D_x^k U_\chi(x, t_2; \varphi)| \leqslant \frac{\text{const}\cdot \tilde{K}_\varphi |t_2 - t_1|^{\theta\beta}}{\min\big[|x - \chi(t_1)|^{k-1+2\beta}, |x - \chi(t_2)|^{k-1+2\beta}\big]} \quad \text{for } (x, t_2), (x, t_1) \in S_T, \tag{2.10}$$

(2.11) $$|D_x^k U_\chi(x_2, t; \varphi) - D_x^k U_\chi(x_1, t; \varphi)| \leqslant \frac{\text{const}\cdot \tilde{K}_\varphi |x_2 - x_1|^{2\theta\beta}}{\min[|x_2-\chi(t)|^{k-1+2\beta}|x_1-\chi(t)|^{k-1+2\beta}]}$$

for $(x_2, t), (x_1, t) \in S_T$; $0 < \theta < 1$, $0 < \beta < \frac{1}{2}\varkappa$, $k = 1, 2, 3$,

are satisfied.

Proof. Carried out with the use of classical methods of the potential theory.

Remark 1. Theorems 1 and 2 are true if U_χ is replaced by J_χ.

THEOREM 3. *If condition* (2.1) *is satisfied and the function* $\varphi(t)$ *is defined and continuous in the interval* $\langle 0, T\rangle$, *then*

(2.12) $$\lim_{x\to\chi(t)} D_x J_\chi(x, t; \varphi) = -\sqrt{\pi}\,\operatorname{sgn}(x-\chi(t))\,\varphi(t) + D_x J_\chi(\chi(t), t; \varphi); \quad t > 0$$

and

(2.13) $$L_0^2[J_\chi(x, t; \varphi)] = 0 \quad \textit{in } S_T, \textit{ where } L_0 = D_x^2 - D_t;\ L_0^2 \overset{\text{df}}{=} L_0[L_0].$$

Proof. Is similar to that in paper [11], p. 278, with necessary modifications connected with the weakening of the assumptions on $\varphi(t)$.

THEOREM 4. *If condition* (2.1) *is satisfied, then*

(2.14) $$|D_t G_{ij}(t, \tau)| \leqslant \frac{\text{const}}{|t-\tau|^{1-\varkappa/2}},$$

(2.15) $$|D_t G_{ij}(t_2, \tau) - D_t G_{ij}(t_1, \tau)| \leqslant \frac{\text{const}\cdot|t_2-t_1|^{\varkappa/2}}{|t_1-\tau|^{1-\varkappa/2}},$$

$$0 \leqslant \tau < t_1 \leqslant t_2 \leqslant T.$$

Proof. Carried out with the use of classical methods of the potential theory.

Remark 2. Theorem 4 remains true when we substitute $D_t G_{ij}(t, \tau)$ for $K_{ij}(t, \tau)$.

THEOREM 5. *If the function* $f(\xi, \tau)$ *is defined, continuous and integrable in* S_T *and fulfils Hölder's condition of the form*(¹)

(2.16) $$|f(\xi_1, \tau) - f(\xi_2, \tau)| \leqslant K(S_T^*)|\xi_1 - \xi_2|^{h_f} \quad \textit{in a domain } S_T^* \subset S_T,$$

then the following equality

(2.17) $$L_0^2[H_{S_T}(x, t; f)] = f(x, t) \quad \textit{is true in } S_T^*.$$

(¹) The coefficient $K(S_T^*)$ depends in general on S_T^* and $h_f \in (0, 1\rangle$.

Proof. The proof is similar to that of an analogous property for the potential of spatial charge related to the heat conduction equation (see [8]).

THEOREM 6. *If the function* $f(\xi, \tau)$ *is defined and integrable in* S_T *and if*

(2.18)
$$|f(\xi, \tau)| \leqslant \frac{m_f}{\min[|\xi-\chi_1(\tau)|^p, |\xi-\chi_2(\tau)|^p]}, \quad (\xi, t)\in S_T;\ 0<p<1,\ m_f>0,$$

then the following sequence of inequalities is valid

(2.19) $$|H_{S_T}(x, t; f)| \leqslant \text{const}\cdot m_f t,$$

(2.20) $$|D_x H_{S_T}(x, t; f)| \leqslant \text{const}\cdot m_f t,$$

(2.21) $$|D_x^2 H_{S_T}(x, t; f)| \leqslant \text{const}\cdot m_f t^{\frac{1}{2}+\frac{1-p}{2}\theta},$$

(2.22) $$|D_x^3 H_{S_T}(x, t; f)| \leqslant \text{const}\cdot m_f t^{\frac{1-p}{2}\theta},$$

(2.23) $$|H_{S_T}(x, t; f)-H_{S_T}(x_1, t_1; f)| \leqslant \text{const}\cdot m_f[|x-x_1|+|t-t_1|],$$

(2.24) $$|D_x H_{S_T}(x, t; f)-D_x H_{S_T}(x_1, t_1; f)| \leqslant \text{const}\cdot m_f[|x-x_1|+|t-t_1|],$$

(2.25)
$$|D_x^2 H_{S_T}(x, t; f)-D_x^2 H_{S_T}(x_1, t_1; f)| \leqslant \text{const}\cdot m_f[|x-x_1|+|t-t_1|^{\frac{1}{2}+\frac{1-p}{2}\theta}],$$

(2.26)
$$|D_x^3 H_{S_T}(x, t; f)-D_x^3 H_{S_T}(x_1, t_1; f) \leqslant \text{const}\cdot m_f[|x-x_1|^{(1-p)\theta}+|t-t_1|^{\frac{1-p}{2}\theta}].$$

Proof. The proof is similar to that of analogous properties of the spatial charge potential related to the heat conduction equation (see [8]).

3. The boundary value problem. Let us consider the following problem.

Find a function $u(x, t)\in C(\bar{S}_T)$ which satisfies the equation

(3.1) $$L_0^2[u(x, t)] = F(x, t, u(x, t), D_x u(x, t), D_x^2 u(x, t), D_x^3 u(x, t))$$

in S_T and fulfils the initial conditions:

(3.2) $$u(x, 0) = 0, \quad D_t u(x, 0) = 0, \quad \chi_1(0) < x < \chi_2(0)$$

and the boundary conditions

(3.3) $$u(\chi_i(t), t) = f_i(t, u(\chi_i(t), t)), \quad i = 1, 2;\ 0 < t \leqslant T,$$

(3.4) $$D_x u(\chi_i(t), t) = g_i(t, u(\chi_i(t), t)), \quad i = 1, 2;\ 0 < t \leqslant T.$$

We make the following assumptions concerning the functions appearing in the problem.

I. The functions $f_i(t, u)$ and $g_i(t, u)$ $(i = 1, 2)$ are defined for $t \in \langle 0, T\rangle$; $u \in (-\infty, \infty)$ and satisfy the conditions:

$$f_i(0, 0) = 0, \quad g_i(0, 0) = 0,$$
$$|f_i(t_1, u_1) - f_i(t_2, u_2)| \leqslant K_{f_i}|u_2 - u_1| + K|t_2 - t_1|^{h_{f_i}+1/2},$$
$$|g_i(t_1, u_1) - g_i(t_2, u_2)| \leqslant K_{g_i}|u_2 - u_1| + K|t_2 - t_1|^{h_{g_i}},$$

where $K, K_{g_i}, K_{f_i} > 0$; $0 < h_{f_i} \leqslant \frac{1}{2}$, $0 < h_{g_i} \leqslant 1$.

II. The function $F(x, t, u, u_1, u_2, u_3)$ is defined and continuous for $(x, t) \in S_T$, $u, u_1, u_2, u_3 \in (-\infty, \infty)$ and fulfils the conditions

$$|F(x, t, u, u_1, u_2, u_3)| \leqslant \frac{M_F}{\min[|x - \chi_1(t)|^{p_F}; |x - \chi_2(t)|^{p_F}]} + m_f(|u|^{r_F} + |u_1|^{r_F^1} + |u_2|^{r_F^2} + |u_3|^{r_F^3}),$$

$$|F(x_1, t, u, u_1, u_2, u_3) - F(x_2, t, \bar{u}, \bar{u}_1, \bar{u}_2, \bar{u}_3)|$$
$$\leqslant K_F \left\{ \frac{|x_2 - x_1|^{h_F}}{\min[|x_1 - \chi_1(t)|^{p_F}, |x_2 - \chi_1(t)|^{p_F}, |x_1 - \chi_2(t)|^{p_F}, |x_2 - \chi_2(t)|^{p_F}]} + |u - \bar{u}|^{h_F} + |u_1 - \bar{u}_1|^{h_F^1} + |u_2 - \bar{u}_2|^{h_F^2} + |u_3 - \bar{u}_3|^{h_F^3} \right\},$$

where $0 \leqslant p_F < 1$, $0 < r_F, r_F^1 \leqslant 1$, $0 < r_F^2 < 1$, $0 < r_F^3 < \frac{1}{2}$; $M_F, m_F, K_F > 0$ and the exponents h_F, h_F^1, h_F^2, h_F^3 belong to the interval $(0, 1\rangle$.

We seek a solution of the problem in the form

$$(3.5) \quad u(x, t) = U_{\chi_1}(x, t; \varphi_1) + U_{\chi_2}(x, t; \varphi_2) + J_{\chi_1}(x, t; \psi_1) + J_{\chi_2}(x, t; \psi_2) + H_{S_T}(x, t; F(x, t, u, D_x u, D_x^2 u, D_x^3 u)).$$

Taking into consideration the boundary conditions (3.3) and (3.4) as well as employing: Theorems 1, 2, 3, 6 and Remark 1 we obtain the following system of equations:

$$(3.6) \quad f_i(t, u(\chi_i(t), t)) = U_{\chi_1}(\chi_i(t), t; \varphi_1) + U_{\chi_2}(\chi_i(t), t; \varphi_2) + J_{\chi_1}(\chi_i(t), t; \psi_1) + J_{\chi_2}(\chi_i(t), t; \psi_2) + H_{S_T}(\chi_i(t), t; F(\ldots)) \quad (i = 1, 2),$$

$$(3.7) \quad g_i(t, u(\chi_i(t), t)) = \sqrt{\pi}(-1)^i(\varphi_i(t) + \psi_i(t)) + D_x U_{\chi_1}(\chi_i(t) t; \psi_1) + D_x U_{\chi_2}(\chi_i(t), t; \varphi_2) + D_x J_{\chi_1}(\chi_i(t), t; \psi_1) + D_x J_{\chi_2}(\chi_i(t), t; \psi_2) + D_x H_{S_T}(\chi_i(t), t; F(\ldots)) \quad (i = 1, 2),$$

where $\varphi_1, \varphi_2, \psi_1, \psi_2$ are unknown functions satisfying certain conditions which will be formulated in the sequel. Using the Gevrey method (see [5],

[6]), we reduce system (3.6), (3.7) to Volterra second kind system of the form

$$(3.8)\quad \varphi_i(t)+\sum_{j=1}^{2}\frac{1}{\pi}\int_0^t D_t\left[\int_\tau^t\frac{\omega(\chi_i(z), z, \chi_j(\tau), \tau)}{\sqrt{t-z}}\right]\varphi_j(\tau)\,d\tau+$$

$$+\sum_{j=1}^{2}\frac{1}{\pi}\int_0^t D_t\left[\int_\tau^t\frac{\chi_i(z)-\chi_j(\tau)}{\sqrt{t-z}}D_x\omega(\chi_i(z), z, \chi_j(\tau), \tau)\,dz\right]\psi_j(\tau)\,d\tau$$

$$=p_i\big(t, u(\chi_j(t), t)\big),$$

$$(3.9)\quad \psi_i(t)+\frac{(-1)^i}{\sqrt{\pi}}\sum_{j=1}^{2}\int_0^t D_x\omega(\chi_i(t), t, \chi_j(\tau), \tau)\varphi_j(\tau)\,d\tau+$$

$$+\frac{(-1)^i}{\sqrt{\pi}}\sum_{j=1}^{2}\int_0^t D_x[(x-\chi_j(\tau))D_x\omega(x, t, \chi_j(\tau), \tau)]\Big|_{x=\chi_j(t)}\psi_j(\tau)\,d\tau-$$

$$-\sum_{j=1}^{2}\frac{1}{\pi}\int_0^t D_t\left[\int_\tau^t\frac{\omega(\chi_i(z), z, \chi_j(\tau), \tau)}{\sqrt{t-z}}\right]\varphi_j(\tau)\,d\tau-$$

$$-\sum_{j=1}^{2}\frac{1}{\pi}\int_0^t D_t\left[\int_\tau^t\frac{\chi_i(z)-\chi_j(\tau)}{\sqrt{t-z}}D_x\omega(\chi_i(z), z, \chi_j(\tau), \tau)\right]\psi_j(\tau)\,d\tau$$

$$=\frac{(-1)^i}{\sqrt{\pi}}g_i\big(t, u(\chi_i(t), t)\big)-\frac{(-1)^i}{\sqrt{\pi}}D_xH_{S_T}(\chi_i(t), t; F(\ldots))-p_i\big(t, u(\chi_i(t), t)\big),$$

where

$$p_i\big(t, u(\chi_i(t), t)\big)=\frac{1}{\pi}\frac{W_i\big(t, u(\chi_i(t), t)\big)}{\sqrt{t}}-$$

$$(3.10)\qquad -\frac{1}{2\pi}\int_0^t\frac{W_i\big(z, u(\chi_i(z), z)\big)-W_i\big(t, u(\chi_i(t), t)\big)}{(t-z)^{3/2}}\,dz,$$

$$W_i\big(t, u(\chi_i(t), t)\big)=f_i\big(t, u(\chi_i(t), t)\big)-H_{S_T}(\chi_i(t), t, F(\ldots))\qquad (i=1, 2).$$

In order to simplify certain formulas we introduce the following symbols

$$\varphi_{i+2}(t)=\psi_i(t)\quad (i=1, 2),\qquad \chi_{i+2}(t)=\chi_i(t)\quad (i=1, 2),$$

$$(3.11)\quad \frac{(-1)^i}{\sqrt{\pi}}g_i\big(t, u(\chi_i(t), t)\big)-\frac{(-1)^i}{\sqrt{\pi}}D_xH_{S_T}(\chi_i(t), t; F(\ldots))-$$

$$-p_i\big(t, u(\chi_i(t), t)\big)=p_{i+2}\big(t, u(\chi_{i+2}(t), t)\big).$$

In the aforementioned notation system (3.8), (3.9) takes the form

$$\varphi_i(t)+\sum_{j=1}^{4} M_{ij}(t,\tau)\varphi_j(\tau)\,d\tau = p_i\big(t, u(\chi_i(t),t)\big) \tag{3.12}$$

$$(i = 1, 2, 3, 4),$$

where $M_{ij}(t,\tau)$ are linear combinations of the expressions

$$D_t\left[\int_\tau^t \frac{\omega(\chi_i(z), z, \chi_j(\tau), \tau)}{\sqrt{t-z}}\,dz\right];\quad D_t\left[\int_\tau^t \frac{\chi_i(z)-\chi_j(\tau)}{\sqrt{t-z}} D_x\omega(\chi_i(z), z, \chi_j(\tau), \tau)\,dz\right];$$

$$D_x\omega(\chi_i(t), t, \chi_j(\tau), \tau);\quad D_x\big[(x-\chi_j(\tau))D_x\omega(x, t, \chi_j(\tau), \tau)\big]\big|_{x=\chi_i(t)}.$$

Thus, in order to examine the boundary problem in question it is enough to solve the following non-linear system of integro-differential equations:

$$u(x,t) = \sum_{j=1}^{2} \omega(x, t, \chi_j(\tau), \tau)\varphi_j(\tau)\,d\tau + \tag{3.13}$$

$$+\sum_{j=1}^{2}\int_0^t (x-\chi_j(\tau))D_x\omega(x, t, \chi_j(\tau), \tau)\psi_j(\tau)\,d\tau + H_{S_T}(x, t; F(x, t, v_1, v_2, v_3)),$$

$$\begin{aligned} v_i(x,t) = {} & \sum_{j=1}^{2}\int_0^t D_x^i\omega(x, t, \chi_j(\tau), \tau)\varphi_j(\tau)\,d\tau + \\ & + \sum_{j=1}^{2}\int_0^t D_x^i\big[(x-\chi_j(\tau))D_x\omega(x, t, \chi_j(\tau), \tau)\big]+\psi_j(\tau)\,d\tau + \\ & + D_x^i H_{S_T}(x, t; F(x, t, u, v_1, v_2, v_3)) \quad (i = 1, 2, 3), \end{aligned} \tag{3.14}$$

$$\varphi_i(t)+\sum_{j=1}^{4}\int_0^t M_{ij}(t,\tau)\varphi_j(\tau)\,d\tau = p_i\big(t, u(\chi_i(t),t)\big) \quad (i = 1, 2, 3, 4). \tag{3.15}$$

We will prove the existence of a solution to this system using Schauder's fixed point theorem.

Let Λ be the space of the systems $\Phi = [u(x,t), v_1(x,t), v_2(x,t), v_3(x,t), \varphi_1(t), \varphi_2(t), \psi_1(t), \psi_2(t)]$ of real functions where $u(x,t)$, $\varphi_i(t)$ and $\psi_i(t)$ are defined and continuous on the sets $\bar{S}_T$ and $\langle 0, T\rangle$ respectively and $v_i(x,t)$ $(i = 1, 2, 3)$ are defined and continuous in S_T and satisfy the condition:

$$\sup_{(x,t)\in S_T}\left\{\min\left(|x-\chi_1(t)|^{i-1+2\gamma}, |x-\chi_2(t)|^{i-1+2\gamma}\right)|v_i(x,t)|\right\} < \infty$$

$$(i = 1, 2, 3);\ 0 < \gamma < \tfrac{1}{2}.$$

Linear operations in the space Λ are defined in the ordinary manner, the norm is given by the formula

$$\|\Phi\| = \sup_{(x,t)\in S_T} |u(x,t)| + \sum_{i=1}^{3} \sup_{(x,t)\in S_T} \min\left(|x-\chi_1(t)|^{i-1+2\gamma},\right.$$

$$\left. |x-\chi_2(t)|^{i-1+2\gamma} |v_i(x,t)|\right) + \sum_{i=1}^{2} \left(\sup_{t\in\langle 0,T\rangle} |\varphi_i(t)| + \sup_{t\in\langle 0,T\rangle} |\psi_i(t)|\right)$$

and the distance between two points Φ_1 and Φ_2 of the space is understood as $|\Phi_1-\Phi_2|$.

It is not difficult to show that Λ is a Banach space. In the aforementioned space, we will consider the set E of all points whose coordinates satisfy the conditions:

$$\sup_{(x,t)\in S_T} |u(x,t)| \leqslant R_u;$$

$$\sup_{(x,t)\in S_T} \left\{\min\left(|x-\chi_1(t)|^{i-1}, |x-\chi_2(t)|^{i-1}\right)|v_i(x,t)|\right\} \leqslant R_{v_i},$$

$$\sup_{t\in\langle 0,T\rangle} |\varphi_i(t)| \leqslant K_\varphi T^{h_\varphi}, \qquad \varphi(0)=0, \qquad |\varphi_i(t_2)-\varphi_i(t_1)| \leqslant K_\varphi |t_2-t_1|^{h_\varphi}$$

$(i = 1, 2)$, $\sup_{t\in\langle 0,T\rangle} |\psi_i(t)| \leqslant K_\psi$ $(i = 1, 2)$, where

$$h_\varphi \leqslant \min\left[h_{f_1}, h_{f_2}, \tfrac{1}{2}\varkappa, h_{g_1}, h_{g_2}, \frac{h_{g_1}}{2(1-h_{g_1})}, \frac{h_{g_2}}{2(1-h_{g_2})}, \frac{1-\max(p_F, r_F^2, 2r_F^3)}{2}\theta\right].$$

R_u, R_{v_i}, K_φ, and K_ψ above, are parameters that will be appropriately chosen in the sequel.

It is evident that the set E is closed and convex.

We are going to transform this set by the operation

$$(3.16) \qquad \bar{u}(x,t) = \sum_{j=1}^{2} \int_0^t \omega\left(x,t,\chi_j(\tau),\tau\right)\varphi_j(\tau)\,d\tau +$$

$$+ \sum_{j=1}^{2} \int_0^t \left(x-\chi_j(\tau)\right) D_x\left(x,t,\chi_j(\tau),\tau\right)\psi_j(\tau)\,d\tau + H_{S_T}\left(x,t; F(x,t,u,v_1,v_2,v_3)\right),$$

$$(3.17) \qquad \bar{v}_i(x,t) = \sum_{j=1}^{2} \int_0^t D_x^i \omega\left(x,t,\chi_j(\tau),\tau\right)\varphi_j(\tau)\,d\tau +$$

$$+ \sum_{j=1}^{2} \int_0^t D_x^i\left[\left(x-\chi_j(\tau)\right) D_x \omega\left(x,t,\chi_j(\tau),\tau\right)\right]\psi_j(\tau)\,d\tau +$$

$$+ D_x^i H_{S_T}\left(x,t; F(x,t,u,v_1,v_2,v_3)\right),$$

$$(3.18)\quad \bar{\varphi}_i(t)+\sum_{j=1}^{4}\int_0^t M_{ij}(t,\tau)\bar{\varphi}_j(\tau)d\tau = p_i\big(t,\bar{u}(\chi_i(t),t)\big)\ (^2)\qquad (i=1,2,3,4).$$

The image of a point $\Phi=[u, v_1, v_2, v_3, \varphi_1, \varphi_2, \varphi_3, \varphi_4]$ will be denoted by $\bar{\Phi}=[\bar{u}, \bar{v}_1, \bar{v}_2, \bar{v}_3, \bar{\varphi}_1, \bar{\varphi}_2, \bar{\varphi}_3, \bar{\varphi}_4]$ and the image of an entire set E by E'. We shall find sufficient conditions for the inclusions $E'\subset E$.

From (3.16) and (3.17) we obtain:

$$(3.19)\quad |\bar{u}(x,t)|\leqslant \text{const}\cdot[(K_\varphi+K_\varphi)T^{h_\varphi}+ + M_F+m_F(R_u^{r_F}+R_{v_1}^{r_F^1}+R_{v_2}^{r_F^2}+R_{v_3}^{r_F^3})\,T^{(1-\max(p_F,r_F^2,2r_F^3)/2)\theta}],$$

$$(3.20)\quad |\bar{v}_i(x,t)|\leqslant \text{const}\cdot\left[\frac{K_\varphi T^{h_\varphi}+K_\psi}{\min(|x-\chi_1(t)|^{i-1},|x-\chi_2(t)|^{i-1})}+M_F+ +m_F(R_u^{r_F}+R_{v_1}^{r_F^1}+R_{v_2}^{r_F^2}+R_{v_3}^{r_F^3})\right]T^{(1-\max(p_F,r_F^2,r_F^3)/2)\theta}.$$

Next, using Theorems 1, 2, 4, 6 and Remarks 1 and 2, we have

$$|\bar{u}(\chi_i(t),t)|\leqslant \text{const}\cdot[K_\varphi+K_\psi+M_F+m_F(R_u^{r_F}+R_{v_1}^{r_F^1}+R_{v_2}^{r_F^2}+R_{v_3}^{r_F^3})]\,t^{1/2+h_\varphi},$$

$$|\bar{u}(\chi_i(t_1),t_1)-\bar{u}(\chi_i(t_2),t_2)| \leqslant \text{const}\cdot[K_\varphi+K_\psi+M_F+m_F(R_u^{r_F}+R_{v_1}^{r_F^1}+R_{v_2}^{r_F^2}+R_{v_3}^{r_F^3})]\,|t_2-t_1|^{1/2+h_\varphi}.$$

In view of the definition of functions W_i (p. 172) and the above estimates we have

$$(3.21)\quad \big|W_i\big(t,\bar{u}(\chi_i(t),t)\big)\big| \leqslant \text{const}\cdot\{K_{f_i}[K_\varphi+K_\psi+M_F+m_F(R_u^{r_F}+R_{v_1}^{r_F^1}+R_{v_2}^{r_F^2}+R_{v_3}^{r_F^3})]+ +K+M_F+m_F(R_u^{r_F}+R_{v_1}^{r_F^1}+R_{v_2}^{r_F^2}+R_{v_3}^{r_F^3})\}\,t^{1/2+h_\varphi}=\text{const}\cdot\tilde{K}_{f_i}t^{1/2+h_\varphi},$$

$$(3.22)\quad \big|W_i\big(t_2,\bar{u}(\chi_i(t_2),t_2)\big)-W_i\big(t_1,\bar{u}(\chi_i(t_1),t_1)\big)\big| \leqslant \text{const}\cdot\{K_{f_i}[K_\varphi+K_\psi+M_F+m_F(R_u^{r_F}+R_{v_1}^{r_F^1}+R_{v_2}^{r_F^2}+R_{v_3}^{r_F^3})]+ +K+M_F+m_F(R_u^{r_F}+R_{v_1}^{r_F^1}+R_{v_2}^{r_F^2}+R_{v_3}^{r_F^3})\}\,|t_2-t_1|^{1/2+h_\varphi} \leqslant \text{const}\cdot\tilde{K}_{f_i}\cdot|t_2-t_1|^{1/2+h_\varphi}.$$

Finally, from the definition of functions p_i, inequalities (3.21), (3.22) and the lemma in paper [6], p. 1091, we obtain the following estimates:

$$(3.23)\quad \big|p_i\big(t,\bar{u}(\chi_i(t),t)\big)\big|\leqslant \text{const}\cdot\tilde{K}_{f_i}t^{h_\varphi}\qquad (i=1,2),$$

$$(3.24)\quad \big|p_i\big(t_2,\bar{u}(\chi_i(t_2),t_2)\big)-p_i\big(t_1,\bar{u}(\chi_i(t_1),t_1)\big)\big|\leqslant \text{const}\cdot\tilde{K}_{f_i}|t_2-t_1|^{h_\varphi}\qquad (i=1,2),$$

(2) In accordance with the notation on p. 172 (formula (3.10) and (3.11)).

(3.25) $|p_i(t, \bar{u}(\chi_i(t), t))|$

$$\leqslant \text{const}\cdot\{(K_{f_i}+K_{g_i})[K_\varphi+K_\psi+M_F+m_F(R_u^{r^F}+R_{v_1}^{r_F^1}+R_{v_2}^{r_F^2}+R_{v_3}^{r_F^3})]+M_F+ +m_F(R_{v_1}^{r^F}+R_{v_1}^{r^F}+R_{v_2}^{r_F^2}+R_{v_3}^{r_F^3})\}\, t^{h_\varphi} = \text{const}\cdot\tilde{K}_{f_ig_i} t^{h_\varphi} \quad (i = 3, 4),$$

(3.26) $$|p_i(t_2, \bar{u}(\chi_i(t_2), t_2))-p_i(t_1, \bar{u}(\chi_i(t_1); t_1))| \leqslant \text{const}\cdot\tilde{K}_{f_ig_i}|t_2-t_1|^{h_\varphi} \quad (i = 3, 4).$$

Using the formulas obtain above and basing on the fact that the solution of system (3.18) is of the form (see [6], p. 1089)

$$\bar{\varphi}_i(t) = p_i(t)+\int_0^t \sum_{j=1}^2 \mathscr{K}_{ij}(t, \tau)p_j(\tau)d\tau \quad (i = 1, 2, 3, 4),$$

where $\mathscr{K}_{ij}(t, \tau)$ is the appropriate resolvent kernel, we get the following inequalities (see Theorems 4 and 6)

$$\begin{aligned} |\bar{\varphi}_i(t)| &\leqslant \text{const}\cdot\tilde{K}_{f_i} t^{h_\varphi} && (i = 1, 2), \\ |\bar{\varphi}_i(t)| &\leqslant \text{const}\cdot\tilde{K}_{f_ig_i} t^{h_\varphi} && (i = 3, 4), \\ |\bar{\varphi}_i(t_2)-\bar{\varphi}_i(t_1)| &\leqslant \text{const}\cdot\tilde{K}_{f_i}|t_2-t_1|^{h_\varphi} && (i = 1, 2), \\ |\bar{\varphi}_i(t_2)-\bar{\varphi}_i(t_1)| &\leqslant \text{const}\cdot\tilde{K}_{f_ig_i}|t_2-t_1|^{h_\varphi} && (i = 3, 4). \end{aligned} \qquad t, t_1, t_2 \in \langle 0, T\rangle,$$

Thus, a sufficient condition for the inclusion $E' \subset E$ is the following system of inequalities

$$\text{const}\cdot[(K_\varphi+K_\psi)T^h+M_F+m_F(R_u^{r^F}+R_{v_1}^{r_F^1}+R_{v_2}^{r_F^2}+R_{v_3}^{r_F^3})T^{\frac{1-\max(p_F, r_F^2, 2r_F^3)}{2}\theta}] \leqslant R_u,$$

$$\text{const}\cdot[K_\varphi T^{h_\varphi}+K_\psi+M_F+m_F(R_u^{r^F}+R_{v_1}^{r_F^1}+R_{v_2}^{r_F^2}+R_{v_3}^{r_F^3})T^{\frac{1-\max(p_F, r_F^2, 2r_F^3)}{2}\theta}] \leqslant R_{v_i}$$

$$(i = 1, 2, 3),$$

(3.27) $$\text{const}\cdot\{K_{f_i}[K_\varphi+K_\psi+M_F+m_F(R_u^{r^F}+R_{v_1}^{r_F^1}+R_{v_2}^{r_F^2}+R_{v_3}^{r_F^3})]+ +K+M_F+m_F(R_u^{r^F}+R_{v_1}^{r_F^1}+R_{v_2}^{r_F^2}+R_{v_3}^{r_F^3})\} \leqslant K_\varphi,$$

$$\text{const}\,\{(K_{f_i}+K_{g_i})[K_\varphi+K_\psi+M_F+m_F(R_u^{r^F}+R_{v_1}^{r_F^1}+R_{v_2}^{r_F^2}+R_{v_3}^{r_F^3})]+ +K+M_F+m_F(R_u^{r^F}+R_{v_1}^{r_F^1}+R_{v_2}^{r_F^2}+R_{v_3}^{r_F^3})\} \leqslant K_\psi \quad (i = 1, 2).$$

It is easily seen that (3.27) holds true if we assume that m_F, K_{f_i}, K_{g_i} are sufficiently small whereas the parameters K_φ, K_ψ, R_{v_i}, R_u are suitably chosen. Now we proceed to proving the compactness of the set E'. Using Theorems 2 and 6, Remark 1, the definition of the set E as well as equality (3.17), we obtain the inequalities

(3.28) $|\bar{v}_i(x, t_2)-\bar{v}_i(x, t_1)| \leqslant$

$$\frac{\operatorname{const}\cdot\left(|t_2-t_1|+|t_2-t_1|^{\frac{1-\max(p_F, r_F^2, 2r_F^3)}{2}\theta}\right)}{\min[|x-\chi_1(t_2)|^{i-1+2\beta}, |x-\chi_2(t_1)|^{i-1+2\beta}, |x-\chi_1(t_1)|^{i-1+2\beta}, |x-\chi_2(t_2)|^{i-1+2\beta}]},$$

$(x, t_2), (x, t_1) \in S_T;\ \beta \in (0, \frac{1}{2}\varkappa)$,

(3.29) $|\bar{v}_i(x_2, t)-\bar{v}_i(x_1, t)| \leqslant$

$$\frac{\operatorname{const}\cdot\left(|x_2-x_1|+|x_2-x_1|^{1-\max(p_F, r_F^2, 2r_F^3)\theta}\right)}{\min[|x_1-\chi_1(t)|^{i-1+2\beta}, |x_2-\chi_1(t)|^{i-1+2\beta}, |x_1-\chi_2(t)|^{i-1+2\beta}, |x_2-\chi_2(t)|^{i-1+2\beta}]},$$

$(x_2, t), (x_1, t) \in S_T;\ 0<\theta<1;\ \beta \in (0, \frac{1}{2}\varkappa)$,

which guarantee the compactness of E' (see [13]).

The continuity of operation (3.16)–(3.18) can be proved on the basis of Theorems 2 and 6 and Remark 1, by an argument similar to that used in deriving inequalities (3.27).

Thus assumptions of Schauder's fixed point theorem are satisfied and we can conclude the existence of a fixed point of operation (3.16)–(3.18) and hence, due to Theorems 2, 5 and Remark 1, the existence of a solution $u(x, t)$ of the considered problem.

As a result we can formulate the following final theorem.

THEOREM. *If assumptions* I *and* II *are satisfied and if the system of inequalities* (3.27) *is true, then the considered problem has at least one solution.*

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