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Separability of Hardy–Orlicz space of analytic functions in the half-plane. II

Abstract. This paper is the second part of the paper under the same title which was also published in Commentationes Mathematicae [12]. The paper contains two sections III and IV. In Section III we present properties of K_N class. K_N is a closed linear subspace of the space $\langle H_N, \|\cdot\|_N\rangle$. The space $\langle K_N, \|\cdot\|_N\rangle$ is separable, in contradiction to the space $\langle H_N, \|\cdot\|_N\rangle$. Section IV contains a study of the separability of Hardy–Orlicz space of analytic functions in the half-plane.

The paper is a continuation of paper [12]. We adopt the notation and continue the section numbering of paper I. We cite the results of both parts, I and II, writing the number of the section and the number of the result in the section; within the same section, the section number is omitted.

III. Space K_N

1.1. By K_N we denote the class of functions $F \in H_N$ for which a boundary function $F(i\cdot)$ is continuous on the whole real axis and satisfies the condition: $F(it) \to 0$ as $|t| \to \infty$.

1.2. THEOREM. *K_N is a closed linear subspace of the space $\langle H_N, \|\cdot\|_N\rangle$.*

Proof. It is clear that K_N is a linear subspace of H_N. Let now $\{F_n\}$ be a sequence of elements of the subspace K_N which is convergent in norm $\|\cdot\|_N$ to the function $F \in H_N$; i.e., such that $\|F_n - F\|_N \to 0$, as $n \to \infty$. Then we have $\|F_n - F_m\|_N \to 0$ as $n, m \to \infty$. Since the functions F_n for $n = 1, 2, \ldots$ possess continuous boundary functions, so, in virtue of Lemma 1.2 of Section II, we have

$$\|F_n - F_m\|_N \geqslant \|F_n - F_m\|_\infty = \sup\{|F_n(w) - F_m(w)|: \operatorname{Re} w \geqslant 0\}.$$

Hence we deduce that the sequence $\{F_n\}$ is uniformly convergent in the half-plane $\Omega^* = \{w \in C: \operatorname{Re} w \geqslant 0\}$. Therefore the function F being the limit of the sequence $\{F_n\}$ possesses a continuous boundary function $F(i\cdot)$ on the whole number axis. Also, it is clear that the condition $F_n(it) \to 0$ as $|t| \to \infty$ which is true for $n = 1, 2, \ldots$, passes by the uniform convergence of the sequence $\{F_n(i\cdot)\}$ into the limit $F(i\cdot)$. Thus $F \in K_N$.

1.3. THEOREM. *$\langle K_N, \|\cdot\|_N\rangle$ is a separable space, and the set of functions of the form*

$$Q(w) = \frac{2}{(1+w)^2} P\left(\frac{w-1}{w+1}\right),$$

where P means a polynomial with complex rational coefficients is dense in $\langle K_N, \|\cdot\|_N\rangle$.

Proof. Let $F \in K_N$. Then, let us observe that the function

$$\tilde{G}(z) = \begin{cases} F((1+z)/(1-z)) & \text{for } z \in \bar{D} \text{ and } z \neq 1, \\ 0 & \text{for } z = 1, \end{cases}$$

is analytic in D and, in virtue of Lindelöf's Theorem, continuous in the closed disc $\bar{D}$. Let the function $\tilde{G}$ have the expansion in a power series

$$\tilde{G}(z) = \tilde{a}_0 + \tilde{a}_1 z + \ldots + \tilde{a}_n z^n + \ldots \quad \text{for } z \in D.$$

We write

$$\tilde{S}_m(z) = \tilde{a}_0 + \tilde{a}_1 z + \ldots + \tilde{a}_m z^m$$

and further

$$\tilde{\mathfrak{G}}_m(z) = \frac{1}{m+1}(\tilde{S}_0(z) + \tilde{S}_1(z) + \ldots + \tilde{S}_m(z)) \quad \text{for } m = 0, 1, 2, \ldots$$

We observe that functions $\tilde{\mathfrak{G}}_m(e^{i\theta})$ of the real variable θ are Cesàro means of a Fourier's series of the function $\tilde{G}(e^{i\theta})$ of the real variable (see [2], p. 16). Since the function $\tilde{G}(e^{i\theta})$ for reals θ is continuous and 2π-periodic, thus on the ground of Fejér's Theorem ([2], p. 23 and [13], Chapter III, (3.4)), the sequence $\{\tilde{\mathfrak{G}}_m(e^{i\theta})\}$ is uniformly convergent to the function $\tilde{G}(e^{i\theta})$. By means of the Maximum Principle, we have

$$\sup\{|\tilde{G}(z) - \tilde{\mathfrak{G}}_m(z)|:\ z \in \bar{D}\} = \sup\{|\tilde{G}(e^{i\theta}) - \tilde{\mathfrak{G}}_m(e^{i\theta})|:\ \theta \in \boldsymbol{R}\}.$$

Therefore the sequence $\{\tilde{\mathfrak{G}}_m(z)\}$ is uniformly convergent to the function $\tilde{G}(z)$ with respect to $z \in \bar{D}$. Further, let

$$G(z) = \frac{2}{(1-z)^2} F\left(\frac{1+z}{1-z}\right) = \frac{2}{(1-z)^2}\tilde{G}(z) \quad \text{for } z \in \bar{D} \text{ and } z \neq 1.$$

Since the function F belongs to H^1 in Ω, thus in view of Theorem 1.9 of Section II the function G belongs to H^1 in D. Let the function G have the expansion in a power series

$$G(z) = a_0 + a_1 z + \ldots + a_n z^n + \ldots \quad \text{for } z \in D.$$

We write

$$S_m(z) = a_0 + a_1 z + \ldots + a_m z^m$$

and further

$$\mathfrak{G}_m(z) = \frac{1}{m+1}(S_0(z) + S_1(z) + \ldots + S_m(z)) \quad \text{for } m = 0, 1, 2, \ldots$$

In range of the real variable θ the functions $\mathfrak{G}_m(e^{i\theta})$ are Cesàro means of a Fourier's series of the function $G(e^{i\theta})$. Since the function $G(e^{i\theta})$ is integrated in the interval $[0, 2\pi)$, thus by means of Fejér's type Theorem ([13], Chapter IV, (5.5)), we have

$$(*) \qquad \int_0^{2\pi} |G(e^{i\theta}) - \mathfrak{G}_m(e^{i\theta})|\, d\theta \to 0 \quad \text{as } m \to \infty.$$

In virtue of the connection $\tilde{G}(z) = \frac{1}{2}(1-z)^2 G(z)$ for $z \in \bar{D}$ we have the dependence

$$\tilde{a}_n = \tfrac{1}{2}(a_n - 2a_{n-1} + a_{n-2}) \quad \text{for } n = 0, 1, 2, \ldots$$

under the adoption that $a_{-1} = a_{-2} = 0$. We observe that

$$\tfrac{1}{2}(1-z)^2 S_m(z) = \tilde{S}_m(z) + \tfrac{1}{2}(-2a_m + a_{m-1}) z^{m+1} + \tfrac{1}{2}\cdot a_m z^{m+2}$$

for $m = 0, 1, 2, \ldots$ In that case for $z \in \bar{D}$ and $m = 0, 1, 2, \ldots$ we have

$$\begin{aligned} |\tfrac{1}{2}(1-z)^2 S_m(z) - \tilde{S}_m(z)| &\leqslant \tfrac{1}{2}(2|a_m| + |a_{m-1}|)|z|^{m+1} + \tfrac{1}{2}|a_m|\cdot|z|^{m+2} \\ &\leqslant \tfrac{1}{2}(2|a_m| + |a_{m-1}|) + \tfrac{1}{2}|a_m| \\ &= \tfrac{1}{2}(3|a_m| + |a_{m-1}|). \end{aligned}$$

Thus for $z \in \bar{D}$ we have

$$\begin{aligned} |\tfrac{1}{2}(1-z)^2 \mathfrak{G}_m(z) - \tilde{\mathfrak{G}}_m(z)| &= \left|\frac{1}{m+1}\sum_{n=0}^{m} \left(\tfrac{1}{2}(1-z)^2 S_n(z) - \tilde{S}_n(z)\right)\right| \\ &\leqslant \frac{1}{m+1}\sum_{n=0}^{m} |\tfrac{1}{2}(1-z)^2 S_n(z) - \tilde{S}_n(z)| \\ &\leqslant \frac{1}{m+1}\left(\tfrac{3}{2}\sum_{n=0}^{m}|a_n| + \tfrac{1}{2}\sum_{n=1}^{m}|a_{n-1}|\right) \\ &\leqslant \frac{2}{m+1}\sum_{m=0}^{m}|a_n|. \end{aligned}$$

The function $G(e^{i\theta})$ is integrable with respect $\theta \in [0, 2\pi)$, thus, on the ground of Riemann–Lebesgue Theorem ([8], Chapter II, (4.4)), the sequence of its Fourier coefficients $\{a_n\}$ is convergent to zero. Hence we obtain

$$\frac{2}{m+1}\sum_{n=0}^{m}|a_n| \to 0 \quad \text{as } m \to \infty.$$

In that case the sequence $\{\frac{1}{2}(1-z)^2 (\mathfrak{G}_m(z)-\tilde{\mathfrak{G}}_m(z)\}$ tends to zero uniformly in the closed disc $\bar{D}$. Hence and from this that the sequence $\{\tilde{\mathfrak{G}}_m(z)\}$ is convergent to $G(z)$ uniformly in $\bar{D}$, we get that the sequence $\{\frac{1}{2}(1-z)^2 \mathfrak{G}_m(z)\}$ is uniformly convergent to $\tilde{G}(z)$ in $\bar{D}$.

Now, let us take an arbitrary number $\varepsilon > 0$. Then from this what we show up to here, we deduce that there exists an index m such that

$$\sup\{|\tilde{G}(z)-\tfrac{1}{2}(1-z)^2 \mathfrak{G}_m(z)|:\ z\in\bar{D}\} \leqslant \tfrac{1}{2}\varepsilon$$

and on the ground of (*) such that also

$$\int_0^{2\pi} |G(e^{i\theta})-\mathfrak{G}_m(e^{i\theta})|\, d\theta \leqslant \tfrac{1}{2}\varepsilon.$$

We choose complex rational numbers $b_0, b_1, b_2, \ldots, b_m$ such that

$$\left|\left(1-\frac{k}{m+1}\right)a_k-b_k\right| \leqslant \frac{\varepsilon}{4\pi(m+1)} \quad \text{for } k=0,1,2,\ldots,m.$$

The polynomial $P_m(z)=b_0+b_1 z+\ldots+b_m z^m$ has complex rational coefficients and

$$\begin{aligned} |\mathfrak{G}_m(z)-P_m(z)| &= \left|\frac{1}{m+1}\sum_{n=0}^{m} S_n(z)-P_m(z)\right| \\ &= \left|\sum_{k=0}^{m}\left(1-\frac{k}{m+1}\right)a_k z^k-\sum_{k=0}^{m} b_k z^k\right| \\ &\leqslant \sum_{k=0}^{m}\left|\left(1-\frac{k}{m+1}\right)a_k-b_k\right|\cdot|z|^k \\ &\leqslant (m+1)\frac{\varepsilon}{4\pi(m+1)}=\frac{\varepsilon}{4\pi} \end{aligned}$$

for all $z\in\bar{D}$, and thus the inequality holds

$$\sup\{|\mathfrak{G}_m(z)-P_m(z)|:\ z\in\bar{D}\} \leqslant \frac{\varepsilon}{4\pi}.$$

In that case we have

$$\begin{aligned} &\sup\{|\tilde{G}(z)-\tfrac{1}{2}(1-z)^2 P_m(z)|:\ z\in\bar{D}\} \\ &\leqslant \sup\{|\tilde{G}(z)-\tfrac{1}{2}(1-z)^2 \mathfrak{G}_m(z)|:\ z\in\bar{D}\}+\sup\{\tfrac{1}{2}|1-z|^2|\mathfrak{G}_m(z)-P_m(z)|:\ z\in\bar{D}\} \\ &\leqslant \frac{\varepsilon}{2}+\tfrac{1}{2}\cdot 2^2\cdot\frac{\varepsilon}{4\pi} \leqslant \varepsilon \end{aligned}$$

and

$$\int_0^{2\pi} |G(e^{i\theta})-P_m(e^{i\theta})|\,d\theta \leqslant \int_0^{2\pi} |G(e^{i\theta})-\mathfrak{G}_m(e^{i\theta})|\,d\theta + \int_0^{2\pi} |\mathfrak{G}_m(e^{i\theta})-P_m(e^{i\theta})|\,d\theta$$

$$\leqslant \frac{\varepsilon}{2}+2\pi\cdot\frac{\varepsilon}{4\pi} = \varepsilon.$$

Let us denote

$$Q_m(w) = \frac{2}{(1+w)^2}P_m\left(\frac{w-1}{w+1}\right).$$

By the first from the obtained inequality we get

$$\|F-Q_m\|_\infty = \sup\{|F(w)-Q_m(w)|:\ w\in\Omega\}$$
$$= \sup\{|\tilde{G}(z)-\tfrac{1}{2}(1-z)^2P_m(z)|:\ z\in D\} \leqslant \varepsilon,$$

and from the second one, in view of Theorem 1.9 of Section II and in virtue of the fact that

$$G(z) = (TF)(z) \quad \text{and} \quad P_m(z) = (TQ_m)(z) \quad (z\in D),$$

we have

$$\|F-Q_m\|_1 = \|G-P_m\|_1 = \int_0^{2\pi} |G(e^{i\theta})-P_m(e^{i\theta})|\,d\theta \leqslant \varepsilon.$$

So we obtain

$$\|F-Q_m\|_N = \sup\{\|F-Q_m\|_\infty,\ \|F-Q_m\|_1\} \leqslant \varepsilon,$$

what concludes the proof.

1.4. LEMMA. *Functions*

$$U_n(w) = \frac{(w-1)^{n-1}}{(w+1)^{n+1}} \quad \textit{for } w\in\Omega \textit{ and } n = 1, 2, \ldots$$

belong to the class K_N.

Proof. We observe that these functions are analytic on the whole complex plane except fot the point $w = -1$, thus, in particular, they are analytic in the half-plane $\Omega^* = \{w\in C:\ \operatorname{Re} w \geqslant 0\}$. Since for $x \geqslant 0$

$$((x+1)^2+y^2)-((x-1)^2+y^2) = 4x \geqslant 0$$

and also

$$(x+1)^2+y^2 \geqslant 1+y^2 \geqslant 1,$$

therefore for $w = x+iy \in \Omega^*$ we have

$$(*)\quad \frac{|w-1|^{n-1}}{|w+1|^{n+1}} = \left|\frac{w-1}{w+1}\right|^{n-1} \cdot \frac{1}{|w+1|^2} = \left(\sqrt{\frac{(x-1)^2+y^2}{(x+1)^2+y^2}}\right)^{n-1} \cdot \frac{1}{(x+1)^2+y^2}$$

$$\leqslant \frac{1}{(x+1)^2+y^2} \leqslant \frac{1}{1+y^2} \leqslant 1.$$

From this inequality we deduce further that

$$\int_{-\infty}^{\infty} |U_n(x+iy)|\,dy = \int_{-\infty}^{\infty} \frac{|(x+iy)-1|^{n-1}}{|(x+iy)+1|^{n+1}}\,dy \leqslant \int_{-\infty}^{\infty} \frac{dy}{1+y^2} = \pi \quad \text{for } x \geqslant 0;$$

also

$$|U_n(w)| \leqslant 1 \quad \text{for } w \in \Omega.$$

This proves, in virtue of Theorem 1.4 of Section II, that $U_n \in H_N$.

Next, from the fact that the function U_n in Ω^* is analytic, we get that the boundary function $U_n(iy)$ is continuous on the whole real axis of variable y. By inequality $(*)$ we conclude that $U_n(iy) \to 0$ as $|y| \to \infty$. Hence and from above we have $U_n \in K_N$ for $n = 1, 2, \ldots$

IV. Separability of space $H^{*\psi}$

1.1. THEOREM. *If $F \in H^{\psi}$, then functions*

$$(A_n F)(w) = \left(1-\left(\frac{w}{w+1}\right)^n\right)^2 F\left(\frac{1}{n}+w\right), \quad \textit{where } w \in \Omega \textit{ and } n = 1, 2, \ldots,$$

are elements of space K_N, such that

$$\varrho_\psi\left(\tfrac{1}{16}(F-A_n F)\right) \to 0 \quad \textit{as } n \to \infty.$$

Proof. Before we pass to the basic proof of the theorem, we shall consider the following functions

$$V_n(w) = \left(1-\left(\frac{w}{w+1}\right)^n\right)^2, \quad \text{where } n = 1, 2, \ldots$$

These functions are analytic on the whole complex plane except at the point $w = -1$, thus, in particular, are analytic in the half-plane $\Omega^* = \{w \in C\colon \operatorname{Re} w \geqslant 0\}$. Let $w = x+iy \in \Omega^*$. Since

$$\left(((1+x)^2+y^2)-(x^2+y^2)\right) = 1+2x > 0,$$

therefore we have

$$(1)\quad \left|\frac{w}{w+1}\right| = \sqrt{\frac{x^2+y^2}{(1+x)^2+y^2}} < 1.$$

From this inequality we deduce further that

$$V_n(w) \to 1 \quad \text{as } n \to \infty \tag{2}$$

also

$$|V_n(w)| \leqslant \left(1 + \left|\frac{w}{w+1}\right|^n\right)^2 \leqslant 4 \tag{3}$$

and

$$|1 - V_n(w)| = \left|2\left(\frac{w}{w+1}\right)^n - \left(\frac{w}{w+1}\right)^{2n}\right| \leqslant 2\left|\frac{w}{w+1}\right|^n + \left|\frac{w}{w+1}\right|^{2n} \leqslant 3. \tag{4}$$

We observe that

$$1 - \left(\frac{w}{w+1}\right)^n = \frac{1}{(w+1)^n} \sum_{k=1}^{n} \binom{n}{k} w^{n-k}.$$

Hence, on the ground of inequality (1) and inequality

$$|w+1| = \sqrt{(1+x)^2 + y^2} \geqslant \sqrt{1+y^2} \geqslant 1,$$

we get

$$\left|1 - \left(\frac{w}{w+1}\right)^n\right| \leqslant \frac{1}{|w+1|^n} \sum_{k=1}^{n} \binom{n}{k} |w|^{n-k}$$

$$= \frac{1}{|w+1|} \sum_{k=1}^{n} \binom{n}{k} \left|\frac{w}{w+1}\right|^{n-k} \cdot \frac{1}{|w+1|^{k-1}}$$

$$\leqslant \frac{1}{|w+1|} \sum_{k=1}^{n} \binom{n}{k} \leqslant \frac{2^n}{|w+1|} \leqslant \frac{2^n}{\sqrt{1+y^2}}$$

and further

$$|V_n(w)| \leqslant \frac{4^n}{1+y^2}. \tag{5}$$

Now, let $F \in H^{\psi}$. From the fact that the function F in Ω is analytic we get that the function $F(1/n+w)$ in the half-plane $\operatorname{Re} w > -1/n$ is also analytic. In particular, the function $F(1/n+w)$ is analytic in Ω^*. On the ground of 4.6 of Section I we have the estimation

$$|F(w)| \leqslant \psi^{-1}\left(\frac{\varrho_\psi(F)}{\pi \cdot \operatorname{Re} w}\right) \quad \text{for } w \in \Omega.$$

Thus

$$|F(1/n+w)| \leqslant \psi^{-1}\left(\frac{\varrho_\psi(F)}{\pi \cdot \operatorname{Re}(1/n+w)}\right) \leqslant \psi^{-1}\left(\frac{n}{\pi}\varrho_\psi(F)\right) \quad \text{for } w \in \Omega^*.$$

Taking $M_n = \psi^{-1}\left(\frac{n}{\pi}\varrho_\psi(F)\right)$, we write down the above inequality in the form

$$|F(1/n+w)| \leqslant M_n \quad \text{for } w \in \Omega^*. \tag{6}$$

By inequalities (3) and (6), we get

$$|(A_n F)(w)| = |V_n(w)| \cdot |F(1/n+w)| \leqslant 4M_n \quad \text{for } w \in \Omega^*,$$

in particular for $w \in \Omega$, what proves that $A_n F \in H^\infty$.Whereas by inequalities (5) and (6) we have

$$\int_{-\infty}^{\infty} |(A_n F)(x+iy)|\, dy = \int_{-\infty}^{\infty} |V_n(x+iy)| \cdot |F(1/n+x+iy)|\, dy$$

$$\leqslant 4^n M_n \int_{-\infty}^{\infty} \frac{dy}{1+y^2} = \pi \cdot 4^n \cdot M_n$$

for $x \geqslant 0$, what proves that also $A_n F \in H^1$. Thus $A_n F \in H^1 \cap H^\infty = H_N$. The function $A_n F$ as the product of functions V_n and $F(1/n+\cdot)$ analytic in Ω^* is analytic in this half-plane, also. Hence we get that the boundary function $(A_n F)(it)$ is continuous on the whole real axis of variable t. In virtue of inequalities (5) and (6) we have the inequality

$$|(A_n F)(it)| = |V_n(it)| \cdot |F(1/n+it)| \leqslant 4^n M_n/(1+t^2),$$

from which it follows that

$$(A_n F)(it) \to 0 \quad \text{as } |t| \to \infty.$$

In this way we have shown that $A_n F \in K_N$ for an arbitrary positive integer n.

Further, from inequality (4), for an arbitrary real number t and positive integer n we have

$$\psi\left(\tfrac{1}{8}|F(it) - V_n(it)F(it)|\right) = \psi\left(\tfrac{1}{8}|1 - V_n(it)| \cdot |F(it)|\right) \leqslant \psi\left(\tfrac{3}{8}|F(it)|\right) \leqslant \psi\left(|F(it)|\right),$$

and on the ground of (2) for almost all real number t

$$\psi\left(\tfrac{1}{8}|F(it) - V_n(it)F(it)|\right) \to 0 \quad \text{as } n \to \infty.$$

Since the function $\psi(|F(i\cdot)|)$ is integrable in the interval $(-\infty, \infty)$, thus by Lebesgue Theorem we get

$$\varrho_\psi\left(\tfrac{1}{8}(F - V_n F)\right) = \int_{-\infty}^{\infty} \psi\left(\tfrac{1}{8}|F(it) - V_n(it)F(it)|\right) dt \to 0 \quad \text{as } n \to \infty.$$

On the ground of inequality (3) we have

$$\begin{aligned}\varrho_\psi\big(\tfrac{1}{8}(V_n F - A_n F)\big) &= \int_{-\infty}^{\infty} \psi\big(\tfrac{1}{8}|V_n(it)F(it) - V_n(it)F(1/n+it)|\big)\,dt \\ &= \int_{-\infty}^{\infty} \psi\big(\tfrac{1}{8}|V_n(it)|\cdot|F(it) - F(1/n+it)|\big)\,dt \\ &\leqslant \int_{-\infty}^{\infty} \psi\big(\tfrac{1}{2}|F(it) - F(1/n+it)|\big)\,dt \\ &= \varrho_\psi\big(\tfrac{1}{2}(F(i\cdot) - F(1/n+i\cdot))\big).\end{aligned}$$

As $n \to \infty$, the right-hand side of this inequality tends to zero, in virtue of 4.7 of Section I. Thus

$$\varrho_\psi\big(\tfrac{1}{8}(V_n F - A_n F)\big) \to 0 \quad \text{as } n \to \infty.$$

Now, by the inequality

$$\begin{aligned}\varrho_\psi\big(\tfrac{1}{16}(F - A_n F)\big) &= \varrho_\psi\big(\tfrac{1}{2}\cdot\tfrac{1}{8}(F - V_n F) + \tfrac{1}{2}\cdot\tfrac{1}{8}(V_n F - A_n F)\big) \\ &\leqslant \tfrac{1}{2}\varrho_\psi\big(\tfrac{1}{8}(F - V_n F)\big) + \tfrac{1}{2}\varrho_\psi\big(\tfrac{1}{8}(V_n F - A_n F)\big)\end{aligned}$$

we obtain

$$\varrho_\psi\big(\tfrac{1}{16}(F - A_n F)\big) \to 0 \quad \text{as } n \to \infty.$$

1.2. THEOREM. *If $F \in H^{0\psi}$, then $\{A_n F\}$ is a sequence of elements of space K_N such that*

$$\|F - A_n F\|_\psi \to 0 \quad \textit{as } n \to \infty.$$

Hence it follows that K_N is a norm dense set in the space $H^{0\psi}$.

Proof. Let $F \in H^{0\psi}$. We take an arbitrary number $k > 0$. Since $16kF \in H^\psi$ and the operators A_n are linear, therefore on the ground of Theorem 1.1 we have

$$\varrho_\psi\big(k(F - A_n F)\big) = \varrho_\psi\big(\tfrac{1}{16}(16kF - A_n[16kF])\big) \to 0 \quad \text{as } n \to \infty.$$

Hence we obtain, in virtue of 4.9 of Section I, that $\|F - A_n F\|_\psi \to 0$ as $n \to \infty$. The fact that $\{A_n F\}$ is a sequence of elements of space K_N follows from Theorem 1.1.

1.3. THEOREM. *K_N is a modular dense set in the space $H^{*\psi}$.*

Proof. Let $F \in H^{*\psi}$. Then there exists the constant $k > 0$ such that $kF \in H^\psi$. By means of Theorem 1.1 and from the fact that the operators are linear, we get

$$\varrho_\psi\big(\tfrac{1}{16}k(F - A_n F)\big) = \varrho_\psi\big(\tfrac{1}{16}(kF - A_n[kF])\big) \to 0 \quad \text{as } n \to \infty,$$

what proves that the sequence $\{A_n F\}$ is modular convergent to F in $H^{*\psi}$.

The fact that $\{A_n F\}$ is a sequence of elements of K_N space follows from the linearity of operators A_n and K_N space and also from Theorem 1.1.

Since $H^{*\psi} \supset H^{0\psi} \supset H_N \supset K_N$, so by Theorem 1.3 we obtain as the suggestion

1.4. THEOREM. *$H^{0\psi}$ is a modular dense set in the space $H^{*\psi}$.*

1.5. THEOREM. *$\langle H^{0\psi}, \|\cdot\|_\psi\rangle$ is a separable space, and the set of functions of the form*

$$Q(w) = \frac{2}{(1+w)^2} P\left(\frac{w-1}{w+1}\right),$$

where P means a polynomial with complex rational coefficients, is dense in $\langle H^{0\psi}, \|\cdot\|_\psi\rangle$.

The theorem follows immediately from Theorems 1.2, 1.3 of Section III and 1.8 of Section II.

1.6. THEOREM. *$H^{*\psi}$ is modular separable, and the set of functions of the form*

$$Q(w) = \frac{2}{(1+w)^2} P\left(\frac{w-1}{w+1}\right),$$

where P means a polynomial with complex rational coefficients, is modular dense in $H^{*\psi}$.

The theorem follows clearly from Theorems 1.4 and 1.5.

1.7. THEOREM. *$\langle H^{*\psi}, \|\cdot\|_\psi\rangle$ is separable if and only if ψ satisfies the condition (Δ_2).*

Proof. If ψ satisfies the condition (Δ_2), then $H^{*\psi} = H^{0\psi}$ and from Theorem 1.5 we deduce that $\langle H^{*\psi}, \|\cdot\|_\psi\rangle$ is separable.

Next, let us assume that ψ does not satisfy the condition (Δ_2). Then there exists a sequence of positive numbers $\{u_n\}$ such that

$$\psi(2u_n) > 2^n \psi(u_n).$$

In the interval $(-\infty, \infty)$ we distinguish a sequence of pairwise disjoint sets $\{E_n\}$ such that

$$\text{mes } E_n = 1/2^n \psi(u_n).$$

We define a sequence $\{g_n\}$ of real functions:

$$g_n(t) = \begin{cases} u_n & \text{for } t \in E_n, \\ 0 & \text{elsewhere in } (-\infty, \infty). \end{cases}$$

We observe that

$$\varrho_\psi(g_n) = \int_{-\infty}^{\infty} \psi(|g_n(t)|)\,dt = \psi(u_n) \text{ mes } E_n = \psi(u_n)\frac{1}{2^n \psi(u_n)} = \frac{1}{2^n}.$$

Subsequently, we define the family of real functions

$$g_\eta(t) = \sum_{n=1}^{\infty} \eta_n g_n(t) \quad \text{for } t \in (-\infty, \infty),$$

where $\eta = \{\eta_n\}$ is an arbitrary sequence of terms 0 and 1. Let us observe that

$$0 \leqslant g_\eta(t) \leqslant \sum_{n=1}^{\infty} g_n(t) \quad \text{for } t \in (-\infty, \infty).$$

Thus, for functions of this family we have

$$g_\psi(g_\eta) \leqslant \varrho_\psi\Big(\sum_{n=1}^{\infty} g_n(t)\Big) \leqslant \sum_{n=1}^{\infty} \psi(u_n)\frac{1}{2^n \psi(u_n)} = \sum_{n=1}^{\infty} \frac{1}{2^n} < \infty.$$

Now, let us take a sequence $\{f_\eta\}$ of real functions

$$f_\eta(t) = \tfrac{1}{2}\left(g_\eta(t) + \frac{1}{1+t^2}\right).$$

In virtue of the convexity of the N-function ψ and the fact that for $0 \leqslant u \leqslant 1$ the inequality $\psi(u) \leqslant \psi(1)u$ holds, we get

$$\varrho_\psi(f_\eta) = \int_{-\infty}^{\infty} \psi(f_\eta(t))\,dt = \int_{-\infty}^{\infty} \psi\left(\tfrac{1}{2}\cdot g_\eta(t) + \tfrac{1}{2}\cdot\frac{1}{1+t^2}\right)dt$$

$$\leqslant \tfrac{1}{2}\int_{-\infty}^{\infty} \psi(g_\eta(t))\,dt + \tfrac{1}{2}\int_{-\infty}^{\infty} \psi\left(\frac{1}{1+t^2}\right)dt$$

$$\leqslant \tfrac{1}{2}\varrho_\psi(g_\eta) + \frac{\psi(1)}{2}\int_{-\infty}^{\infty} \frac{dt}{1+t^2}$$

$$= \tfrac{1}{2}\varrho_\psi(g_\eta) + \tfrac{1}{2}\pi\cdot\psi(1) < \infty.$$

Next, similarly as in the proof of Theorem 1.7 of Section II, we state that there exists the integral

$$\int_{-\infty}^{\infty} \frac{|\ln f_\eta(t)|}{1+t^2}\,dt$$

and further that the functions defined by the formula

$$F_\eta(w) = \exp\left(\frac{1}{\pi}\int_{-\infty}^{\infty} \frac{tw+i}{t+iw}\cdot\frac{\ln f_\eta(t)}{1+t^2}\,dt\right)$$

are analytic in Ω and such that $|F_\eta(it)| = f_\eta(t)$ for almost all t from the interval $(-\infty, \infty)$. Moreover, in view of 4.8 of Section I, we state that the identity holds

$$\varrho_\psi(F_\eta) = \varrho_\psi(f_\eta).$$

Hence we deduce that $F_\eta \in H^\psi \subset H^{*\psi}$ for each η. Now, let us take two different sequences $\eta' = \{\eta'_k\}$ and $\eta'' = \{\eta''_k\}$. Then there exists an index n such that $\eta'_n \neq \eta''_n$. We state, taking into account 4.5 of Section I, that

$$\begin{aligned}
\varrho_\psi(4(F_{\eta'} - F_{\eta''})) &= \int_{-\infty}^{\infty} \psi(4|F_{\eta'}(it) - F_{\eta''}(it)|)\,dt \\
&\geqslant \int_{-\infty}^{\infty} \psi(4\,||F_{\eta'}(it)| - |F_{\eta''}(it)||)\,dt \\
&= \int_{-\infty}^{\infty} \psi(4|f_{\eta'}(t) - f_{\eta''}(t)|)\,dt \\
&= \int_{-\infty}^{\infty} \psi(2|g_{\eta'}(t) - g_{\eta''}(t)|)\,dt \geqslant \int_{-\infty}^{\infty} \psi(2g_n(t)) \\
&\geqslant \psi(2u_n)\,\mathrm{mes}\,E_n > 2^n \psi(u_n) \frac{1}{2^n \psi(u_n)} = 1.
\end{aligned}$$

This proves that $\|F_{\eta'} - F_{\eta''}\|_\psi > \frac{1}{4}$. Thus there exists in $H^{*\psi}$ continuum elements whose distances are $> \frac{1}{4}$. Hence the space $\langle H^{*\psi}, \|\cdot\|_\psi \rangle$ is not separable.

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