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Projections in locally bounded F-spaces with symmetric bases

Introduction. Let X be an F-space, i.e., complete metrizable topological linear space. A closed linear subspace Y of X is said to be *complemented* (in X) if there is a continuous projection from X onto Y, or equivalently, if there exists a closed linear subspace Z of X such that X is the direct sum of Y and Z, i.e., $X = Y \oplus Z$. X is locally bounded if it has a bounded neighbourhood of zero.

In this note we first show that the theorem of Casazza, Kottman and Lin [3] on operators in Banach spaces with symmetric bases holds in locally bounded spaces. Then we consider separable locally bounded Orlicz sequence spaces l_M with $\lim_{t\to 0}\frac{M(t)}{t} = \infty$.

In this case we prove that every infinite-dimensional complemented subspace X of l_M contains a complemented subspace Y which is isomorphic to l_M.

1. Projections in locally bounded spaces. Let X be an F-space with a basis (e_n). Recall that a (basic) sequence (x_n) in X is said to be a *block basic sequence* with respect to (e_n) if, for every n,

$$x_n = \sum_{i=p_{n-1}}^{p_n} a_i e_i,$$

where $0 = p_0 < p_1 < \ldots$ are integers and (a_i) is a sequence of scalars. A basic sequence (x_n) is complemented in X if its closed linear span $[x_n]$ is complemented.

Two basic sequences (x_n) and (y_n) of X are called *equivalent* provided the series $\sum_{n=1}^{\infty} a_n x_n$ converges if and only if the series $\sum_{n=1}^{\infty} a_n y_n$ converges.

A bounded set $(x_\alpha)_{\alpha\in A}$ of points in $(X, |\cdot|)$ is called an *ε-net* if $|x_\alpha - x_\beta| \geqslant \varepsilon$ for all $\alpha, \beta \in A$, $\alpha \neq \beta$.

PROPOSITION 1.1. *Let (x_n) be an ε-net in an F-space $(X, |\cdot|)$ with basis (e_n). Then there is a subsequence (x_{n_k}) of (x_n) such that:*

(i) $(y_k) = (x_{n_{2k+1}} - x_{n_{2k}})$ *is a basic sequence;*

(ii) (y_k) *is equivalent to some block basic sequence* (z_k) *of* (e_n);

(iii) $\sum_{k=1}^{\infty} |z_k - y_k| < \infty$.

Proof. See [7], Proposition II.5.7 or [4], Proposition 3.1.

Recall that if an F-space is locally bounded, then its topology may be given by a p-norm $\|\cdot\|$ for some $p \in (0, 1]$ (cf. [7], p. 61). $(X, \|\cdot\|)$ is then called a *p-Banach space.*

A bounded subset of an F-space $(X, |\cdot|)$ is precompact if it does not contain an infinite ε-net. An operator $K\colon X \to Y$ between two F-spaces is compact if it maps a neighbourhood of zero in X to a precompact set in Y.

The next Proposition is an easy generalization of a result of Bessaga and Pełczyński [1].

PROPOSITION 1.2. *Let* $(X, \|\cdot\|)$ *be a p-Banach space and let* (x_n) *be a complemented basic sequence in* X *with biorthogonal functionals* (x_n^*). *If* $(y_n) \subset X$ *and* $\sum_{n=1}^{\infty} \|x_n - y_n\| \|x_n^*\| < \infty$, *then there exists an integer* n_0 *such that* $(y_n)_{n \geq n_0}$ *is a complemented basic sequence equivalent to* $(x_n)_{n \geq n_0}$.

Proof. Let P be a continuous projection from X onto $[x_n]$. We define an operator $K\colon X \to X$ by

$$K(x) = \sum_{i=1}^{\infty} x_i(Px)(x_i - y_i),$$

and observe that K is compact and $(I-K)(x_n) = y_n$ for all n.

The desired result follows from the Riesz theory of compact operators (cf. [6], B. I. § 5).

A basis (e_n) of a locally bounded F-space is symmetric if it is equivalent to the basis $(x_{\pi(n)})$ for any permutation π of integers.

If X is a locally bounded F-space with symmetric basis (e_n), then its topology may be defined by p-norm $\|\cdot\|$ (cf. [5] p. 113) such that

$(*)$ $\|x\| = \sup \{ \| \sum_{i \in F} \alpha_i a_i e_i \| \colon |\alpha_i| \leq 1, \quad F \subset N \}$, where $x = \sum_{i=1}^{\infty} a_i e_i$;

$(**)$ $\|e_n\| = 1$ for $n \in N$.

Thus $J\colon X \to c_0$ defined by $J(\sum_{i=1}^{\infty} a_i e_i) = (a_i)$ is continuous and $\|Jx\|_{c_0} \leq \|x\|$ for all $x \in X$.

LEMMA 1.3. *Let* X *be a locally bounded F-space with symmetric basis* (e_n) *and let* Q *be a continuous operator on* X. *If* $JQ(e_n)$ *is a non-precompact set in* c_0, *then* $Q(X)$ *contains a subspace* Y *which is isomorphic to* X *and complemented in* X.

Proof. Since X is locally bounded we may assume that the topology on X is given by p-norm $\|\cdot\|$ satisfying (*) and (**). The assumption of the lemma implies that there exists an infinite set $P \subset N$ such that $\{Q(e_i)\colon i \in P\}$ and $\{JQ(e_i)\colon i \in P\}$ are ε-nets in X and c_0, respectively.

Applying Proposition 1.1 we can construct two basic sequences (y_k) and (z_k) with the following properties:

(a) $y_k = Q(e_{i_{2k+1}} - e_{i_{2k}})$, $i_k \in P$;

(b) $z_k = \sum_{i \in A_k} \alpha_i e_i$ is a block basic sequence;

(c) $\sum_{n=1}^{\infty} \|z_n - y_n\| < \infty$.

Since $\|Jy_k\|_{c_0} \geq \varepsilon$ and $\|z_k - y_k\| \geq \|J(z_k - y_k)\|_{c_0}$, it follows from (c) that

(d) $\|Jz_k\|_{c_0} = \max\{|\alpha_i|\colon i \in A_k\} \geq \eta$, where $k \in N$ and $\eta > 0$.

Since for every strictly increasing sequence of integers (n_k) the basic sequence (e_{n_k}) is equivalent to (e_n) (cf. [5], Proposition 3.a.3), by (c) it follows that (z_n) is equivalent to e_{k_n}. Moreover, as in [2] we may define a continuous projection $R\colon X \to [z_n]$ by

$$R(x) = \sum_{n=1}^{\infty} (b_{i_n}/a_{i_n}) z_n \quad \text{if } x = \sum_{n=1}^{\infty} b_n e_n,$$

where $|a_{i_n}| = \max\{|\alpha_i|\colon i \in A_n\}$, $\forall n$.

Since $\sup\|z_n^*\| < \infty$, it follows from Proposition 1.2 that, for some $n_0 \in N$, $(y_n)_{n \geq n_0}$ is a complemented basic sequence equivalent to $(z_n)_{n \geq n_0}$. Thus $[y_n]_{n \geq n_0}$ is isomorphic to X and complemented in X.

The next theorem is a generalization of a result of Casazza, Kottman and Lin [3].

THEOREM 1.4. *Let X be a locally bounded F-space with symmetric basis (e_n). Then for every continuous operator Q on X either $Q(X)$ or $(I-Q)(X)$ contains a subspace Y which is isomorphic to X and complemented in X.*

Proof. Since either $\{JQ(e_n)\}_{n \in N}$ or $\{(I-JQ)(e_n)\}_{n \in N}$ is non-precompact in c_0 our result follows from the preceding lemma.

2. Projections in Orlicz sequence spaces. An Orlicz function M is a continuous non-decreasing map from R_+ to R_+ such that $M(0) = 0$ and $\lim_{t \to 0} M(t) = \infty$. The Orlicz sequence space l_M is the vector space of all scalar sequences (x_n) such that $\sum_{n=1}^{\infty} M(|\varepsilon x_n|) < \infty$ for some $\varepsilon > 0$. We define

$$B_M(\varepsilon) = \{x\colon \sum_{n=1}^{\infty} M(|x_n|) \leq \varepsilon\}$$

and then $\{rB_M(\varepsilon): r > 0, \varepsilon > 0\}$ is a base of neighbourhoods of 0 for an F-space topology on l_M. It is well known that in every separable Orlicz sequence space the unit vectors (e_n) form a symmetric basis.

LEMMA 2.1. *Let M be an Orlicz function satisfying $\underline{\lim}_{t\to 0} M(t)/t > 0$, and let (e_n) be the unit vectors in l_M. An operator A: $l_M \to l_1$ is compact if and only if the set $\{A(e_n): n \in N\}$ is precompact.*

Proof. Suppose the set $\{A(e_n): n \in N\}$ is precompact. Since $\underline{\lim}_{t\to 0} M(t)/t > 0$, we may assume $M(t) \geqslant t$, $t \geqslant 0$. Thus

$$U = \{x = (x_n) \in l_M: \sum_{n=1}^{\infty} M(|x_n|) \leqslant 1\} \subset V,$$

where

$$V = \{x = (x_n) \in l_M: \sum_{n=1}^{\infty} |x_n| \leqslant 1\}.$$

Hence V is a neighbourhood of zero in l_M. Moreover, $A(V)$ is contained in the closed absolutely convex hull of $\{A(e_n): n \in N\}$.

Let $(X, |\cdot|)$ be an F-space with a separating dual, and let τ_1 be the Mackey topology of X, i.e., the finest locally convex topology weaker than the original topology (cf. [8]). We denote the completion of (X, τ_1) by $\hat{X}$.

LEMMA 2.2. *Let X be an F-space with separating dual X^* and let Y be an infinite-dimensional subspace of X. If the operator* id: $Y \hookrightarrow \hat{X}$ *is compact, then Y does not contain an inifinite-dimensional complemented subspace of X.*

Proof. If $Z \subset Y$ is a complemented subspace of X, then $X = Z \oplus W$ for some subspace W of X. From the assumption it follows that the operator id_Z: $Z \hookrightarrow \hat{Z}$ is compact. Hence there is a neighbourhood of zero U such that $\mathrm{id}_Z(U)$ is precompact. Since a closed convex hull of $\mathrm{id}_Z(U)$ is a neighbourhood of zero in Z (cf. [8], Proposition 3), Z is finite-dimensional.

THEOREM 2.3. *Let l_M be a separable locally bounded Orlicz sequence space and let X be an infinite-dimensional complemented subspace of l_M. If $\lim_{t\to 0} M(t)/t = \infty$, then X contains a subspace Y which is isomorphic to l_M and complemented in l_M.*

Proof. Let P: $l_M \to l_M$ be a continuous projection on X and let e_n^* be the biorthogonal functionals to the unit vectors e_n in l_M. Let us observe that $\hat{l}_M = l_1$ (cf. [4] Theorem 3.3). By Lemmas 2.1 and 2.2, the set $A = \{P(e_i)\}_{i \in N}$ is non-precompact in l_1. Now, we show that A is non-precompact in c_0. Suppose that our statement is false. Then there exist $\varepsilon > 0$ and a sequence $v_k = P(e_{i_{2k+1}} - e_{i_{2k}})$ such that

(i) $\|v_k\|_{l_1} = \sum_{i=1}^{\infty} |e_i^*(v_k)| \geqslant \varepsilon$;

(ii) $\|v_k\|_{c_0} = \sup_i |e_i^*(v_k)| \xrightarrow[k \to \infty]{} 0.$

Since the sequence (v_k) is bounded in l_M, we have

(iii) $\sum_{i=1}^{\infty} M(|e_i^*(v_k)|) \leqslant \delta$, where $\delta > 0$ and $k \in N$.

By (i) and (iii), for each $k \in N$ there exists $n(k)$ such that

$$\frac{M(|e_{n(k)}^*(v_k)|)}{|e_{n(k)}^*(v_k)|} \leqslant \delta/\varepsilon.$$

Since $\lim_{t \to 0} M(t)/t = \infty$ we get $\inf_k |e_{n(k)}^*(v_k)| > 0$, contradiction with (ii). Hence A is non-precompact in c_0 and our result follows from Lemma 1.3.

Remark. Theorem 2.3 does not hold in all Orlicz spaces (cf. [4], p. 276, or [5], Theorem 4.b.12).

As an easy consequence of Theorem 2.3 and Pełczyński's decomposition technique [5], we have the following result of Stiles [9].

COROLLARY 2.4. *Every complemented infinite-dimensional subspace of l_p $(0 < p < 1)$ is isomorphic to l_p.*

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