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On a non-commutative analogue of the Hölder inequality

Abstract. Given three measure spaces (X_i, μ_i), $i = 1, 2, 3$ and let $L^p(X_i, \mu_i)$ be the usual Lebesgue spaces. Let P be a product operator (see [7] and Definition 2.1). In [7] (Theorem 3.5) R. O'Neil proved that

$$\|P(f, g)\|_1 \leqslant \|f\|_p \|g\|_q,$$

where $1/p+1/q = 1$, $f \in L^p(X_1, \mu_1)$, $g \in L^q(X_2, \mu_2)$.

In Theorem 2.2 we deduce an analogous inequality for the spaces $\mathfrak{L}_m^{\delta\sigma}(\mathfrak{A})$ associated with a semifinite von Neumann algebra $\mathfrak{A}$ equipped with a regular gage m.

1. Preliminaries. Throughout, $\mathfrak{A}_i$ is a von Neumann algebra on a Hilbert space H_i, m_i is a regular gage on $\mathfrak{A}_i$. The triple $(H_i, \mathfrak{A}_i, m_i)$ is termed a regular gage space (see [9]).

DEFINITION 1.1 (see [8], [11]). For any closed, densely defined operator $a \in \mathfrak{A}_i$, we determine:

$$\eta_{|a|}(\lambda) = m_i(e_\lambda^\perp), \qquad \lambda \geqslant 0$$

(the m_i-distribution of a) and

$$a_{m_i}(\alpha) = a(\alpha) = \inf\{\lambda \in [0, \infty]: m_i(e_\lambda^\perp) \leqslant \alpha, \alpha > 0\}$$

(the rearrangement of a) where $\sqrt{a^* a} = |a| = \int_0^\infty \lambda de_\lambda$ is the spectral decomposition of $|a|$.

DEFINITION 1.2 (cf. [6], [11]). The $*$-algebra $\mathfrak{L}(\mathfrak{A}_i, m_i)$ of m_i-measurable operators is defined by

$$\mathfrak{L}_{m_i}(\mathfrak{A}_i) = \mathfrak{L}_{m_i} = \{a\eta\mathfrak{A}_i: a(\alpha) < \infty, \alpha > 0\}.$$

DEFINITION 1.3 (cf. [10], [6], [11]). A sequence of m_i-measurable operators $\{a_n\}$ is said to be *m_i-convergent* (*convergent in measure m_i*) to a m_i-measurable operator a ($a_n \xrightarrow{m_i} a$) if and only if $(a-a_n)(\alpha) \to 0$ as $n \to \infty$ for any $\alpha > 0$.

The following proposition is of great practical importance.

PROPOSITION 1.1 (see [1]). *If* $a_n \xrightarrow{m_i} a$ $(a, a_n \in \mathfrak{L}_{m_i}(\mathfrak{A}_i),\ n = 1, 2, \ldots)$, *then* $a_n(\alpha) \xrightarrow[n]{} a(\alpha)$ *at each point of continuity of the function* $a(\alpha)$.

The non-commutative Lorentz space $\mathfrak{L}^{\delta\sigma}_{m_i}(\mathfrak{A}_i)$ is the collection of all m_i-measurable operators a such that $\|a\|_{\delta\sigma} < \infty$, where (cf. [4], [1])

$$\|a\|_{\delta\sigma} = \begin{cases} \left\{\frac{\sigma}{\delta}\int\limits_0^\infty (\alpha^{1/\delta} a_{m_i}(\alpha))^\sigma \frac{d\alpha}{\alpha}\right\}^{1/\sigma}, & 0 < \delta, \qquad \sigma < \infty, \\ \sup\{\alpha^{1/\delta} a_{m_i}(\alpha)\colon\ \alpha > 0\}, & 0 < \delta \leqslant \infty,\ \sigma = \infty. \end{cases}$$

$\mathfrak{L}^{\delta\sigma}_{m_i}(\mathfrak{A}_i)$ is seen to be a quasi-Banach space and in some cases a Banach space (cf. [4], [11], [2]). Note that $\mathfrak{L}^{\delta\delta}_{m_i}(\mathfrak{A}_i) = \mathfrak{L}^{\delta}_{m_i}(\mathfrak{A}_i)$ (the non-commutative Lebesgue space introduced in [9], [11])

$$\mathfrak{L}^{\delta}_{m_i}(\mathfrak{A}_i) = \{a \in \mathfrak{L}_{m_i}\colon\ \|a\|_\delta = \{\int\limits_0^\infty a^\delta_{m_i}(\alpha)\,d\alpha\}^{1/\delta} < \infty\}.$$

For $0 < \sigma_1 \leqslant \sigma_2 \leqslant \infty$ we have (cf. [4]) $\|a\|_{\delta\sigma_2} \leqslant \|a\|_{\delta\sigma_1}$. Hence, $\mathfrak{L}^{\delta\sigma_1} \subset \mathfrak{L}^{\delta\sigma_2}$ for $\sigma_1 \leqslant \sigma_2$.

We define an analogue of $a_{m_i}(\alpha)$, $a \in \mathfrak{L}_{m_i}$

$$\underset{\sim}{a}(\mu) = \underset{\sim}{a}_{m_i}(\mu) = \frac{1}{\mu}\int\limits_0^\mu a_{m_i}(\alpha)\,d\alpha, \qquad \mu > 0.$$

It is clear that $\underset{\sim}{a}_{m_i}(\mu) \geqslant a_{m_i}(\mu)$, $\mu > 0$ and $\underset{\sim}{a}_{m_i}(\mu) < \infty$ for $\mu > 0$ if and only if $a \in \mathfrak{A}_i + \mathfrak{L}^1_{m_i}(\mathfrak{A}_i)$.

The following equality is obvious geometrically

$$\mu\underset{\sim}{a}(\mu) = \mu a(\mu) + \int\limits_{a(\mu)}^\infty \eta_{|a|}(\lambda)\,d\lambda.$$

Proposition 1.2 (cf. [4], [2]). *For any* $a \in \mathfrak{A}_i + \mathfrak{L}^1_{m_i}$ *and* $1 < \delta \leqslant \infty$, $1 \leqslant \sigma \leqslant \infty$

$$\|a\|_{\delta\sigma} \leqslant {}_\sim\|a\|_{\delta\sigma} \leqslant \frac{\delta}{\delta - 1}\|a\|_{\delta\sigma} \qquad \left(\frac{\infty}{\infty - 1} = 1\right),$$

where

$${}_\sim\|a\|_{\delta\sigma} = \begin{cases} \left\{\frac{\sigma}{\delta}\int\limits_0^\infty (\alpha^{1/\delta}\underset{\sim}{a}(\alpha))^\sigma \frac{d\alpha}{\alpha}\right\}^{1/\sigma}, & 0 < \delta < \infty,\ 0 < \sigma < \infty, \\ \sup\{\alpha^{1/\delta}\underset{\sim}{a}(\alpha)\colon\ \alpha > 0\}, & 0 < \delta \leqslant \infty,\ \sigma = \infty. \end{cases}$$

For any $a, b \in \mathfrak{A}_i + \mathfrak{L}^1_{m_i}$: $(\underset{\sim}{a + b})(\mu) \leqslant \underset{\sim}{a}(\mu) + \underset{\sim}{b}(\mu)$ (cf. [8], [12], [2]). In consequence, $\mathfrak{L}^{\delta\sigma}_{m_i}(\mathfrak{A}_i)$ are Banach spaces for $\delta > 1$, $\sigma \geqslant 1$ with norms ${}_\sim\|\cdot\|_{\delta\sigma}$.

Note that $\mathfrak{L}_{m_i}^{\delta\delta}(\mathfrak{A}_i) = \mathfrak{L}_{m_i}^{\delta}(\mathfrak{A}_i)$ are Banach spaces with norms $\|\cdot\|_\delta$ (see [9], [5], [11]) for $\delta \geqslant 1$.

The set of all orthogonal projections from $\mathfrak{A}_i$ is denoted by $\mathrm{Proj}(\mathfrak{A}_i)$.

2. Product operators.

DEFINITION 2.1 (cf. [7], Definition 3.2). Given three regular gage spaces $(H_i, \mathfrak{A}_i, m_i)$, $i = 1, 2, 3$. A bilinear operator P which maps measurable operators from $\mathfrak{L}_{m_1}(\mathfrak{A}_1)$ and $\mathfrak{L}_{m_2}(\mathfrak{A}_2)$ into measurable operators from $\mathfrak{L}_{m_3}(\mathfrak{A}_3)$ is called a *product operator* if

(i) $\|P(a, b)\| \leqslant \|a\| \, \|b\|$,
(ii) $\|P(a, b)\|_1 \leqslant \|a\|_1 \, \|b\|$,
(iii) $\|P(a, b)\|_1 \leqslant \|a\| \, \|b\|_1$.

LEMMA 2.1. *If P is a product operator and*

$$c = P(a, b), \qquad a \in \mathfrak{L}_{m_1}(\mathfrak{A}_1), \qquad b \in \mathfrak{L}_{m_2}(\mathfrak{A}_2),$$

where $ap^\perp = 0$, $m_1(p) = \delta$, $|a| \leqslant \beta 1$, then for $\alpha > 0$

(i) $\alpha \underline{c}(\alpha) \leqslant \beta\delta \underline{b}(\delta)$,
(ii) $\underline{c}(\alpha) \leqslant \beta \underline{b}(\alpha)$.

Proof. Let $\mu > 0$ and let $b_\mu = u(|b|\, e_\mu + \mu e_\mu^\perp)$, $b^\mu = b - b_\mu = u(|b|\, e_\mu^\perp - \mu e_\mu^\perp)$, where $b = u|b|$ (the polar decomposition of b), $|b| = \int\limits_0^\infty \lambda de_\lambda$ (the spectral decomposition of $|b|$),

$$c = P(a, b) = P(a, b_\mu) + P(a, b^\mu) = c_1 + c_2.$$

It is easy to see that $|b^\mu| = (|b| - \mu)\, e_\mu^\perp$ and

$$\eta_{|b^\mu|}(\lambda) \leqslant m_2(e_{\lambda+\mu}^\perp) = \eta_{|b|}(\lambda + \mu).$$

In consequence

$$\|c_1\| \leqslant \|a\| \, \|b_\mu\| \leqslant \beta\mu, \qquad \|c_1\|_1 \leqslant \|a\|_1 \, \|b_\mu\| \leqslant \beta\delta\mu,$$

$$\|c_2\|_1 \leqslant \|a\| \, \|b^\mu\|_1 \leqslant \beta \int\limits_\mu^\infty \eta_{|b|}(\lambda)\, d\lambda.$$

Let further $\mu = b(\delta)$. Then

$$\alpha \underline{c}(\alpha) \leqslant \alpha \underline{c}_1(\alpha) + \alpha \underline{c}_2(\alpha) = \int\limits_0^\alpha c_1(\gamma)\, d\gamma + \int\limits_0^\alpha c_2(\gamma)\, d\gamma \leqslant \|c_1\|_1 + \|c_2\|_1$$

$$\leqslant \beta\delta b(\delta) + \beta \int\limits_{b(\delta)}^\infty \eta_{|b|}(\lambda)\, d\lambda = \beta\delta \underline{b}(\delta).$$

To prove the second inequality of our lemma, we let $\mu = b(\alpha)$. We have

$$\alpha \underline{c}(\alpha) \leqslant \alpha \underline{c}_1(\alpha) + \alpha \underline{c}_2(\alpha) \leqslant \int_0^\alpha c_1(\gamma)\,d\gamma + \int_0^\alpha c_2(\gamma)\,d\gamma \leqslant \alpha \|c_1\| + \|c_2\|_1$$

$$\leqslant \alpha\beta b(\alpha) + \beta \int_{b(\alpha)}^{\infty} \eta_{|b|}(\lambda)\,d\lambda = \beta\alpha \underline{b}(\alpha)$$

and (ii) follows, by dividing by α.

LEMMA 2.2 (basic lemma on product operators). *If P is a product operator and $c = P(a, b)$, where either $b \notin \mathfrak{A}_2 + \mathfrak{L}^1_{m_2}$ or $b \in \mathfrak{A}_2 + \mathfrak{L}^1_{m_2}$, $a \in \mathfrak{A}_1$ or $b \in \mathfrak{A}_2$, $a \in \mathfrak{L}^1_{m_1}$, then for any $\mu > 0$*

$$\mu \underline{c}(\mu) \leqslant \int_{a(\mu)}^{\infty} m_1(e_\lambda^\perp)\, \underline{b}(m_1(e_\lambda^\perp))\,d\lambda + \mu \underline{b}(\mu)\, a(\mu),$$

where $|a| = \int_0^\infty \lambda de_\lambda$.

Proof. Fix $\mu > 0$. Consider a doubly infinite sequence $\{\lambda_n\}_{n=-\infty}^{\infty}$ such that

$$\lambda_0 = a(\mu), \quad \lambda_n \leqslant \lambda_{n+1}, \quad \lim \lambda_n = \infty \quad \text{as } n \to \infty,$$
$$\lim \lambda_n = 0 \quad \text{as } n \to -\infty.$$

Let $a_n = u[(\lambda_n - \lambda_{n-1}) e^\perp_{\lambda_n} + (|a| - \lambda_{n-1}) e_{(\lambda_{n-1}, \lambda_n\rangle}]$, where $a = u|a|$, $|a| = \int_0^\infty \lambda de_\lambda$ and $c^{(n)} = P(a_n, b)$. It is clear that

$$|a_n| \leqslant (\lambda_n - \lambda_{n-1}) 1, \quad a_n e_{\lambda_{n-1}} = 0.$$

Fromm Lemma 2.1 (i) we obtain for $n = 1, 2, \ldots$

$$\mu \underline{c}^{(n)}(\mu) \leqslant (\lambda_n - \lambda_{n-1})\, m_1(e^\perp_{\lambda_{n-1}})\, \underline{b}(m_1(e^\perp_{\lambda_{n-1}}))$$

and by the second inequality of Lemma 2.1 we obtain for $n = 0, -1, -2, -3, \ldots$

$$\mu \underline{c}^{(n)}(\mu) \leqslant (\lambda_n - \lambda_{n-1})\, \underline{b}(\mu) \cdot \mu.$$

Furthermore,

$$c = P(a, b) = P(u(|a| - \lambda_0) e^\perp_{\lambda_0}, b) + P(u(|a| e_{\lambda_0} + \lambda_0 e^\perp_{\lambda_0}), b) = c_1 + c_2,$$

$$\mu \underline{c}(\mu) \leqslant \mu \underline{c}_1(\mu) + \mu \underline{c}_2(\mu).$$

Let $a^{(l)} = \sum_{n=1}^{l} a_n$ and $a^{(-l)} = \sum_{n=-l}^{0} a_n$, $l = 0, 1, 2, \ldots$ It is easy to see that

$$a^{(l)} e_{\lambda_l} = a^{(l)} e_{(\lambda_0, \lambda_l\rangle} = u(|a| - \lambda_0) e^\perp_{\lambda_0} e_{(\lambda_0, \lambda_l\rangle} = u(|a| - \lambda_0) e^\perp_{\lambda_0} e_{\lambda_l},$$

and so

$$u(|a|-\lambda_0)e_{\lambda_0}^{\perp}-a^{(l)}=\big(u(|a|-\lambda_0)e_{\lambda_0}^{\perp}-a^{(l)}\big)e_{\lambda_l}^{\perp}=u(|a|-\lambda_1)e_{\lambda_l}^{\perp}.$$

In consequence

$$c_1=P\big(u(|a|-\lambda_0)e_{\lambda_0}^{\perp},\,b\big)=P(a^{(l)},\,b)+P\big(u(|a|-\lambda_l)e_{\lambda_l}^{\perp},\,b\big)=c_3^{(l)}+c_4^{(l)},$$

$$\mu_{\mathfrak{c}_1}(\mu)\leqslant\mu_{\mathfrak{c}_3^{(l)}}(\mu)+\mu_{\mathfrak{c}_4^{(l)}}(\mu)$$

$$\leqslant\sum_{n=1}^{l}(\lambda_n-\lambda_{n-1})m_1(e_{\lambda_{n-1}}^{\perp})\mathfrak{b}\big(m_1(e_{\lambda_{n-1}}^{\perp})\big)+\mu_{\mathfrak{c}_4^{(l)}}(\mu).$$

If $a\in\mathfrak{L}_{m_1}^{1}$, $b\in\mathfrak{A}_2$, then $u(|a|-\lambda_l)e_{\lambda_l}^{\perp}\to 0$ in $\mathfrak{L}_{m_1}^{1}$ and

$$\big\|P\big(u(|a|-\lambda_l)e_{\lambda_l}^{\perp},\,b\big)\big\|_1\leqslant\|u(|a|-\lambda_l)e_{\lambda_l}^{\perp}\|_1\cdot\|b\|\to 0\qquad\text{as } l\to\infty.$$

Moreover,

$$\mu_{\mathfrak{c}_4^{(l)}}(\mu)=\int_0^{\mu}c_4^{(l)}(\alpha)\,d\alpha\leqslant\|c_4^{(l)}\|_1\to 0\qquad\text{as } l\to\infty.$$

Assume that $a\in\mathfrak{A}_1$. It is clear that $a_n=0$ for n sufficiently large and $a^{(l)}=u(|a|-\lambda_0)e_{\lambda_0}^{\perp}$ for l sufficiently large. Finally

$$\mu_{\mathfrak{c}_1}(\mu)\leqslant\sum_{n=1}^{\infty}(\lambda_n-\lambda_{n-1})m_1(e_{\lambda_{n-1}}^{\perp})\mathfrak{b}\big(m_1(e_{\lambda_{n-1}}^{\perp})\big)$$

for $a\in\mathfrak{L}_{m_1}^{1}$, $b\in\mathfrak{A}_2$ or $a\in\mathfrak{A}_1$, $b\in\mathfrak{L}_{m_2}(\mathfrak{A}_2)$.

The series on the right is an infinite Riemann sum tending (with proper choice of λ_n) to the integral

$$\int_{a(\mu)}^{\infty}m_1(e_{\lambda}^{\perp})\mathfrak{b}\big(m_1(e_{\lambda}^{\perp})\big)\,d\lambda.$$

Therefore,

$$\mu_{\mathfrak{c}_1}(\mu)\leqslant\int_{a(\mu)}^{\infty}m_1(e_{\lambda}^{\perp})\mathfrak{b}\big(m_1(e_{\lambda}^{\perp})\big)\,d\lambda.$$

We use the second inequality of Lemma 2.1 to evaluate $\mathfrak{c}_2(\mu)$. For any $\xi\in e_{\lambda_0}^{\perp}(H)$: $a^{(-l)}\xi=(\lambda_0-\lambda_{-l-1})\xi$ and for any $\xi\in e_{(\lambda_{-k-1},\lambda_{-k}\rangle}(H)$, $0\leqslant k\leqslant l$: $a^{(-l)}\xi=u(|a|\xi-\lambda_{-l-1}\xi)$, and for any $\xi\in e_{\lambda_{-l-1}}(H)$: $a^{(-l)}\xi=0$. Hence

$$c_2=P\big(u(|a|e_{\lambda_0}+\lambda_0 e_{\lambda_0}^{\perp}),\,b\big)$$

$$=P(a^{(-l)},\,b)+P\big(u(\lambda_{-l-1}e_{\lambda_{-l-1}}+|a|\cdot e_{\lambda_{-l-1}}),\,b\big)=c_5^{(-l)}+c_6^{(-l)}$$

and from Lemma 2.1 (ii)

$$\mu \underline{c}_5^{(-l)} \leqslant \mu \sum_{n=-l}^{0} (\lambda_n - \lambda_{n-1}) \underline{b}(\mu) = \mu \underline{b}(\mu)(\lambda_0 - \lambda_{-l-1}).$$

Let further $b \in \mathfrak{A}_2(\mathfrak{L}_{m_2}^1)$. Then, $\|c_6^{(-l)}\| (\|c_6^{(-l)}\|_1) \to 0$ as $l \to \infty$. Therefore, for any $\mu > 0$, $a \in \mathfrak{L}_{m_1}$, $b \in \mathfrak{L}_{m_2}$

$$\mu \underline{c}_2(\mu) \leqslant \mu \underline{b}(\mu) \lambda_0 = \mu a(\mu) \underline{b}(\mu).$$

Finally

$$\mu \underline{c}(\mu) \leqslant \mu \underline{c}_1(\mu) + \mu \underline{c}_2(\mu) \leqslant \int_{a(\mu)}^{\infty} m_1(e_\lambda^\perp) \underline{b}(m_1(e_\lambda^\perp)) \, d\lambda + \mu a(\mu) \underline{b}(\mu)$$

for either $a \in \mathfrak{L}_{m_1}^1$, $b \in \mathfrak{A}_2$ or $a \in \mathfrak{A}_1$, $b \in \mathfrak{A}_2 + \mathfrak{L}_{m_2}^1$ or $b \notin \mathfrak{A}_2 + \mathfrak{L}_{m_2}^1$.

COROLLARY 2.1 (cf. [11], Theorem 3.3). *If P is a product operator, $c = P(a, b)$, where either $a \in \mathfrak{A}_1$, $b \in \mathfrak{A}_2 + \mathfrak{L}_{m_2}^1$ or $a \in \mathfrak{L}_{m_1}^1$, $b \in \mathfrak{A}_2$ or $a \notin \mathfrak{A}_1 + \mathfrak{L}_{m_1}^1$ or $b \notin \mathfrak{A}_2 + \mathfrak{L}_{m_2}^1$, then*

$$\mu \underline{c}(\mu) \leqslant \int_0^\mu a(\alpha) b(\alpha) \, d\alpha.$$

Proof. If $0 \neq a \notin \mathfrak{A}_1 + \mathfrak{L}_{m_1}^1$ or $0 \neq b \notin \mathfrak{A}_2 + \mathfrak{L}_{m_2}^1$, then

$$\int_0^\mu a(\alpha) b(\alpha) \, d\alpha = \infty, \qquad \mu > 0.$$

If $a = 0$ or $b = 0$, then $\underline{c}(\mu) = 0$ and $\int_0^\mu a(\alpha) b(\alpha) \, d\alpha = 0$, $\mu > 0$. Assume that $a \in \mathfrak{A}_1$, $b \in \mathfrak{A}_2 + \mathfrak{L}_{m_2}^1$ or $a \in \mathfrak{L}_{m_1}^1$, $b \in \mathfrak{A}_2$. We have

$$\int_{a(\mu)}^{\infty} m_1(e_\lambda^\perp) \underline{b}(m_1(e_\lambda^\perp)) \, d\lambda = (\lambda = a(\alpha)) = -\int_0^\mu \alpha \underline{b}(\alpha) \, da(\alpha)$$

$$= -\alpha b(\alpha) a(\alpha)|_0^\mu + \int_0^\mu a(\alpha) b(\alpha) \, d\alpha$$

$$= -\mu \underline{b}(\mu) a(\mu) + \int_0^\mu a(\alpha) b(\alpha) \, d\alpha.$$

It is now clear that

$$\mu \underline{c}(\mu) \leqslant \int_0^\mu a(\alpha) b(\alpha) \, d\alpha.$$

Remark 2.1. Assume that $a_n \xrightarrow{m_1} a$ implies $c_n = P(a_n, b) \xrightarrow{m_3} P(a, b)$ and let further $a_n = ae_n \in \mathfrak{A}_1$, $|a| = \int_0^\infty \lambda de_\lambda$. From Proposition 1.1 we obtain

$$\mu \underset{\sim}{c}(\mu) = \int_0^\mu c(\alpha)\,d\alpha \leqslant \liminf_n \int_0^\mu c_n(\alpha)\,d\alpha = \liminf_n \mu \underset{\sim}{c}_n(\mu)$$

$$\leqslant \liminf_n \int_0^\mu a_n(\alpha)\,b(\alpha)\,d\alpha = \int_0^\mu a(\alpha)\,b(\alpha)\,d\alpha$$

(since $a_n(\alpha)\uparrow a(\alpha)$, $\alpha > 0$) for any $a \in \mathfrak{L}_{m_1}$, $b \in \mathfrak{L}_{m_2}$.

THEOREM 2.1 (cf. [7], Theorem 3.3). *A bilinear operator P is a product operator if and only if for $c = P(a, b)$, where either $a \notin \mathfrak{A}_1 + \mathfrak{L}^1_{m_1}$ or $b \notin \mathfrak{A}_2 + \mathfrak{L}^1_{m_2}$ or $a \in \mathfrak{A}_1$, $b \in \mathfrak{A}_2 + \mathfrak{L}^1_{m_2}$ or $a \in \mathfrak{L}^1_{m_1}$, $b \in \mathfrak{A}_2$*

$$\mu \underset{\sim}{c}(\mu) \leqslant \int_0^\mu a(\alpha)\,b(\alpha)\,d\alpha.$$

Proof. The necessity of the condition follows from Corollary 2.1. To prove the sufficiency we may assume, without loss of generality, that $a \in \mathfrak{A}_1$, $b \in \mathfrak{A}_2 + \mathfrak{L}^1_{m_2}$ or $a \in \mathfrak{L}^1_{m_1}$, $b \in \mathfrak{A}_2$. We have

$$\|c\| = \lim_{\mu\to 0} \underset{\sim}{c}(\mu) \leqslant \limsup_{\mu\to 0} \frac{1}{\mu}\int_0^\mu a(\alpha)\,b(\alpha)\,d\alpha \leqslant a(0+)\,b(0+) = \|a\|\,\|b\|,$$

$$\|c\|_1 = \lim_{\mu\to\infty} \mu \underset{\sim}{c}(\mu) \leqslant \lim_{\mu\to\infty} \int_0^\mu a(\alpha)\,b(\alpha)\,d\alpha \leqslant \|a\|\,\|b\|_1\;(\|a\|_1\,\|b\|).$$

COROLLARY 2.2 *A bilinear operator P is a product operator if and only if*

(i) $\|P(a, b)\| \leqslant \|a\|\,\|b\|$,

(ii) $\|P(a, b)\|_1 \leqslant \int_0^\infty a(\alpha)\,b(\alpha)\,d\alpha$,

where either $a \notin \mathfrak{A}_1 + \mathfrak{L}^1_{m_1}$ or $b \notin \mathfrak{A}_2 + \mathfrak{L}^1_{m_2}$ or $a \in \mathfrak{A}_1$, $b \in \mathfrak{A}_2 + \mathfrak{L}^1_{m_2}$ or $a \in \mathfrak{L}^1_{m_1}$, $b \in \mathfrak{A}_2$.

THEOREM 2.2 (Hölder's inequality, cf. [7], Theorem 3.5, [3], [5], [11]). *If P is a product operator: $c = P(a, b)$, where either $a \in \mathfrak{A}_1$, $b \in \mathfrak{A}_2 + \mathfrak{L}^1_{m_2}$ or $a \in \mathfrak{L}^1_{m_1}$, $b \in \mathfrak{A}_2$, and if $a \in \mathfrak{L}^{\delta_1\sigma_1}_{m_1}$, $b \in \mathfrak{L}^{\delta_2\sigma_2}_{m_2}$, where $1/\delta_1 + 1/\delta_2 = 1$, $1/\sigma_1 + 1/\sigma_2 \geqslant 1$, then $c \in \mathfrak{L}^1_{m_3}$ and*

(i) $\|c\|_1 \leqslant \max\{\delta_1/\sigma_1, \delta_2/\sigma_2\}\,\|a\|_{\delta_1\sigma_1}\,\|b\|_{\delta_2\sigma_2}$, $\delta_i, \sigma_i \neq \infty$, $i = 1, 2$. *In the particular case: $\delta_1 = \sigma_1$, $\delta_2 = \sigma_2$*

$$\|c\|_1 \leqslant \|a\|_{\delta_1}\,\|b\|_{\delta_2}.$$

(ii) $\|c\|_1 \leqslant \|a\|_{\delta_1\infty}\,\|b\|_{\delta_2\sigma_2}$, $\sigma_1 = \infty$,

(iii) $\|c\|_1 \leqslant \|a\|_{\delta_1\sigma_1}\,\|b\|_{\delta_2\infty}$, $\sigma_2 = \infty$.

Proof. (i) We have

$$\|c\|_1 = \lim_{\mu\to\infty} \int_0^{\mu} c(\alpha)\,d\alpha \leqslant \int_0^{\infty} a(\alpha)b(\alpha)\,d\alpha = \int_0^{\infty} (\alpha^{1/\delta_1}/\alpha^{1/\gamma_1})\,a(\alpha)(\alpha^{1/\delta_2}/\alpha^{1/\gamma_2})\,b(\alpha)\,d\alpha$$

$$\leqslant \Big\{\int_0^{\infty} \alpha^{\gamma_1/\delta_1} a^{\gamma_1}(\alpha)(1/\alpha)\,d\alpha\Big\}^{1/\gamma_1} \Big\{\int_0^{\infty} \alpha^{\gamma_2/\delta_2} b^{\gamma_2}(\alpha)(1/\alpha)\,d\alpha\Big\}^{1/\gamma_2}$$

$$= (\delta_1/\gamma_1)^{1/\gamma_1}(\delta_2/\gamma_2)^{1/\gamma_2}\|a\|_{\delta_1\gamma_1}\|b\|_{\delta_2\gamma_2}$$

$$\leqslant \max\{\delta_1/\sigma_1,\ \delta_2/\sigma_2\}\,\|a\|_{\delta_1\sigma_1}\|b\|_{\delta_2\sigma_2}$$

where $1/\gamma_1+1/\gamma_2 = 1$, $0 \leqslant 1/\gamma_i \leqslant 1/\sigma_i$, $i = 1, 2$.

(ii) If $\sigma_1 = \infty$, then

$$\|c\|_1 \leqslant \int_0^{\infty} \alpha^{1/\delta_1} a(\alpha)(\alpha^{1/\delta_2}/\alpha)\,b(\alpha)\,d\alpha \leqslant \|a\|_{\delta_1\infty}\|b\|_{\delta_2 1} \leqslant \|a\|_{\delta_1\infty}\|b\|_{\delta_2\sigma_2}.$$

The case $\sigma_2 = \infty$ follows analogously.

THEOREM 2.3 (cf. [7], Theorem 3.4). *If P is a product operator, $c = P(a, b)$, where either $a \in \mathfrak{A}_1$, $b \in \mathfrak{A}_2 + \mathfrak{L}^1_{m_2}$ or $a \in \mathfrak{L}^1_{m_1}$, $b \in \mathfrak{A}_2$, and if $a \in \mathfrak{L}^{\delta_1\sigma_1}_{m_1}$, $b \in \mathfrak{L}^{\delta_2\sigma_2}_{m_2}$, where $1/\delta_1+1/\delta_2 = 1/\delta < 1$ and $\sigma \geqslant 1$ is any number such that $1/\sigma_1+1/\sigma_2 \geqslant 1/\sigma$, then $c \in \mathfrak{L}^{\delta\sigma}_{m_3}$ and*

$$\|c\|_{\delta\sigma} \leqslant \theta(\delta, \sigma, \delta_1, \sigma_1, \delta_2, \sigma_2)\|a\|_{\delta_1\sigma_1}\|b\|_{\delta_2\sigma_2}.$$

Moreover,

$\theta(\delta, \sigma, \delta_1, \sigma_1, \delta_2, \sigma_2) =$

(i) $\delta/(\delta-1)(\sigma/\delta)^{1/\sigma}\max\{(\delta_1/\sigma_1)^{1/\sigma}, (\delta_2/\sigma_2)^{1/\sigma}\}$; $\sigma, \sigma_1, \sigma_2 \neq \infty$,

(ii) $\delta/(\delta-1)(\delta_1/\delta)^{1/\sigma}$; $\sigma, \sigma_1 \neq \infty$, $\sigma_2 = \infty$,

(iii) $\delta/(\delta-1)(\delta_2/\delta)^{1/\sigma}$; $\sigma, \sigma_2 \neq \infty$, $\sigma_1 = \infty$,

(iv) $(1/\sigma'_2)^{1/\sigma'_2}(\delta/(\delta-1))^{1/\sigma'_2}(\delta_2/\sigma_2)^{1/\sigma_2}$; $\sigma, \sigma_1 = \infty$, $1 < \sigma_2 < \infty$, $1/\sigma_2+1/\sigma'_2 = 1$,

(v) $(1/\sigma'_1)^{1/\sigma'_1}(\delta/(\delta-1))^{1/\sigma'_1}(\delta_1/\sigma_1)^{1/\sigma_1}$; $\sigma, \sigma_2 = \infty$, $1 < \sigma_1 < \infty$, $1/\sigma_1+1/\sigma'_1 = 1$,

(vi) δ_2; $\sigma, \sigma_1 = \infty$, $\sigma_2 = 1$,

(vii) δ_1; $\sigma, \sigma_2 = \infty$, $\sigma_1 = 1$,

(viii) $\delta/(\delta-1)$; $\sigma, \sigma_1, \sigma_2 = \infty$.

Proof. (i) Let us suppose that $1/\gamma_1+1/\gamma_2 = 1$, $1/\gamma_i \leqslant \sigma/\sigma_i$, $i = 1, 2$. We have

$$\|c\|_{\delta\sigma} \leqslant \left\{\frac{\sigma}{\delta}\int_0^\infty \mu^{\sigma/\delta-1}\,\underline{c}^{\sigma}(\mu)\,d\mu\right\}^{1/\sigma} = \left\{\frac{\sigma}{\delta}\int_0^\infty (\mu\underline{c}(\mu))^{\sigma}\,\mu^{\sigma/\delta-\sigma-1}\,d\mu\right\}^{1/\sigma}$$

(by Hardy's inequality, see e.g. [4])

$$\leqslant \delta/(\delta-1)\left\{\frac{\sigma}{\delta}\int_0^\infty \alpha^{\sigma/\delta-1}\,a^{\sigma}(\alpha)\,b^{\sigma}(\alpha)\,b^{\sigma}(\alpha)\,d\alpha\right\}^{1/\sigma}$$

$$= \delta/(\delta-1)\left\{\frac{\sigma}{\delta}\int_0^\infty (\alpha^{\sigma/\delta_1}/\alpha^{1/\gamma_1})\,a^{\sigma}(\alpha)\,(\alpha^{\sigma/\delta_2}/\alpha^{1/\gamma_2})\,b^{\sigma}(\alpha)\,d\alpha\right\}^{1/\sigma}$$

(by Hölder's inequality)

$$\leqslant \delta/(\delta-1)(\sigma/\delta)^{1/\sigma}\left\{\int_0^\infty (\alpha^{1/\delta_1}\,a(\alpha))^{\sigma\gamma_1}\frac{d\alpha}{\alpha}\right\}^{1/\sigma\gamma_1}\cdot\left\{\int_0^\infty (\alpha^{1/\delta_2}\,b(\alpha))^{\sigma\gamma_2}\frac{d\alpha}{\alpha}\right\}^{1/\sigma\gamma_2}$$

$$= \delta/(\delta-1)(\sigma/\delta)^{1/\sigma}(\delta_1/\gamma_1\sigma)^{1/\gamma_1\sigma}(\delta_2/\gamma_2\sigma)^{1/\gamma_2\sigma}\cdot\|a\|_{\delta_1(\gamma_1\sigma)}\,\|b\|_{\delta_2(\gamma_2\sigma)}$$

$$\leqslant \delta/(\delta-1)(\sigma/\delta)^{1/\sigma}\max\{(\delta_1/\sigma_1)^{1/\sigma},(\delta_2/\sigma_2)^{1/\sigma}\}\cdot\|a\|_{\delta_1\sigma_1}\,\|b\|_{\delta_2\sigma_2}.$$

(ii) $$\|c\|_{\delta\sigma} \leqslant \left\{\frac{\sigma}{\delta}\int_0^\infty \mu^{\sigma/\delta-1}\,\underline{c}(\mu)\,d\mu\right\}^{1/\sigma} = \left\{\frac{\sigma}{\delta}\int_0^\infty (\mu\underline{c}(\mu))^{\sigma}\,\mu^{\sigma/\delta-\sigma-1}\,d\mu\right\}^{1/\sigma}$$

$$\leqslant \delta/(\delta-1)\left\{\frac{\sigma}{\delta}\int_0^\infty \alpha^{\sigma/\delta-1}\,a^{\sigma}(\alpha)\,b^{\sigma}(\alpha)\,d\alpha\right\}^{1/\sigma}$$ (by Hardy's inequality)

$$= \delta/(\delta-1)\left\{\frac{\sigma}{\delta}\int_0^\infty (\alpha^{1/\delta_1}\,a(\alpha))^{\sigma}(\alpha^{1/\delta_2}\,b(\alpha))^{\sigma}\frac{d\alpha}{\alpha}\right\}^{1/\sigma}$$

$$\leqslant \delta/(\delta-1)(\sigma/\delta)^{1/\sigma}(\delta_1/\sigma)^{1/\sigma}\cdot\|a\|_{\delta_1\sigma}\,\|b\|_{\delta_2\infty}$$

$$\leqslant \delta/(\delta-1)(\delta_1/\delta)^{1/\sigma}\,\|a\|_{\delta_1\sigma_1}\,\|b\|_{\delta_2\infty}.$$

(iii) The proof of case (iii) is identical with that of case (ii).

(iv) $$\|c\|_{\delta\infty} \leqslant \sup\{\mu^{1/\delta} \underline{c}(\mu)\colon\ \mu>0\} \leqslant \sup\{\mu^{1/\delta-1}\int_0^\mu a(\alpha)b(\alpha)\,d\alpha\colon\ \mu>0\}$$

$$\leqslant \|a\|_{\delta_1\infty}\sup\{\mu^{1/\delta-1}\int_0^\mu \alpha^{-1/\delta_1} b(\alpha)\,d\alpha\colon\ \mu>0\}$$

$$= \|a\|_{\delta_1\infty}\sup\{\mu^{1/\delta-1}\cdot\int_0^\mu \alpha^{-1/\delta+1/\sigma_2}(\alpha^{1/\delta_2}/\alpha^{1/\sigma_2})\,b(\alpha)\,d\alpha\}$$

$$\leqslant \|a\|_{\delta_1\infty}\sup\left\{\mu^{1/\delta-1}\left\{\int_0^\mu \alpha^{-\sigma_2'/\delta+\sigma_2'/\sigma_2}\,d\alpha\right\}^{1/\sigma_2'}\cdot\left\{\int_0^\mu \alpha^{\sigma_2/\delta_2} b^{\sigma_2}(\alpha)\frac{d\alpha}{\alpha}\right\}^{1/\sigma_2}\right.$$

$$\leqslant (\delta/(\delta-1))^{1/\sigma_2'}(1/\sigma_2')^{1/\sigma_2'}(\delta_2/\sigma_2)^{1/\sigma_2}\|a\|_{\delta_1\infty}\|b\|_{\delta_2\sigma_2}.$$

(v) The proof of case (v) is the same as that of case (iv).

(vi) $$\|c\|_{\delta\infty} \leqslant \sup\{\mu^{1/\delta}\underline{c}(\mu)\colon\ \mu>0\} \leqslant \sup\left\{\mu^{1/\delta-1}\int_0^\mu a(\alpha)b(\alpha)\,d\alpha\colon\ \mu>0\right\}$$

$$\leqslant \|a\|_{\delta_1\infty}\sup\left\{\mu^{1/\delta-1}\int_0^\mu \alpha^{1-1/\delta}\alpha^{1/\delta_2} b(\alpha)\frac{d\alpha}{\alpha}\right\} \leqslant \delta_2\|a\|_{\delta_1\infty}\|b\|_{\delta_2 1}.$$

(vii) The proof of case (vii) is the same as that of case (vi).

(viii) $$\|c\|_{\delta\infty} \leqslant \sup\{\mu^{1/\delta}\underline{c}(\mu)\colon\ \mu>0\} \leqslant \sup\{\mu^{1/\delta-1}\int_0^\mu a(\alpha)b(\alpha)\,d\alpha\colon\ \mu>0\}$$

$$= \sup\{\mu^{1/\delta-1}\int_0^\mu (1/\alpha^{1/\delta})\alpha^{1/\delta_1}a(\alpha)\alpha^{1/\delta_2}b(\alpha)\,d\alpha\colon\ \mu>0\}$$

$$\leqslant \delta/(\delta-1)\|a\|_{\delta_1\infty}\|b\|_{\delta_2\infty}$$

for $\delta<\infty$ and $\delta=\infty$,

$$\|c\|_{\infty\infty} = \|c\| = \sup\{\underline{c}(\mu)\colon\ \mu>0\} \leqslant \sup\{1/\mu\int_0^\mu a(\alpha)b(\alpha)\,d\alpha\} \leqslant \|a\|\,\|b\|.$$

Remark 2.2. Assume that $a_n \xrightarrow{m_1} a$ implies $c_n = P(a_n, b) \xrightarrow{m_3} P(a, b)$. It follows from Remark 2.1 that $a \in \mathfrak{L}_{m_1}^{\delta_1\sigma_1}$, $b \in \mathfrak{L}_{m_2}^{\delta_2\sigma_2}$ may be assumed in place of either $a \in \mathfrak{A}_1$, $b \in \mathfrak{A}_2 + \mathfrak{L}_{m_2}^1$ or $a \in \mathfrak{L}_{m_1}^1$, $b \in \mathfrak{A}_2$ in Theorems 2.2 and 2.3.

Making use of Theorem 3 in [5] and Theorem 2.3 we easily obtain the following theorem:

THEOREM 2.4. *Let* $1 < \delta_1, \delta_2, \sigma_1, \sigma_2, \gamma_1, \gamma_2 \leqslant \infty$, $0 < \theta < 1$ *and let*

$$1/\gamma_1 = 1/\delta_1 + 1/\sigma_1 < 1, \quad 1/\delta = (1-\theta)/\delta_1 + \theta/\delta_2,$$
$$1/\gamma_2 = 1/\delta_2 + 1/\sigma_2 < 1, \quad 1/\sigma = (1-\theta)/\sigma_1 + \theta/\sigma_2,$$
$$1/\gamma = (1-\theta)/\gamma_1 + \theta/\gamma_2.$$

Then

$$\|P(a, b)\|_\gamma \leqslant (\gamma_1/(\gamma_1 - 1))^{1-\theta} (\gamma_2/(\gamma_2 - 1))^\theta \|a\|_\delta \|b\|_\sigma.$$

3. Concrete examples. Let A_i, $i = 1, 2$, be any two linear operators from $\mathfrak{L}_{m_i}(\mathfrak{A}_i)$ into $\mathfrak{L}_{m_3}(\mathfrak{A}_3)$. Let further

$$\|A_i(a)\| \leqslant \|a\|, \quad a \in \mathfrak{A}_i, \quad i = 1, 2,$$
$$\|A_i(a)\|_1 \leqslant \|a\|_1, \quad a \in \mathfrak{L}^1_{m_i}, \; i = 1, 2.$$

EXAMPLE 3.1. $H_1 = H_2 = H_3$, $m_1 = m_2 = m_3$, $\mathfrak{A}_1 = \mathfrak{A}_2 = \mathfrak{A}_3$.

(i) $P(a, b) = A_1(a) A_2(b)$ or

(ii) $P(a, b) = \frac{1}{2}[A_1(a) A_2(b) + A_2(b) A_1(a)]$.

EXAMPLE 3.2. $H_1 = H_2$, $\mathfrak{A}_1 = \mathfrak{A}_2$, $m_1 = m_2$. Let A_1 be a gage-preserving $*$-isomorphism from $\mathfrak{L}_{m_1}(\mathfrak{A}_1)$ into $\mathfrak{L}_{m_3}(\mathfrak{A}_3)$. We define

(i) $P(a, b) = A_1(ab)$ or

(ii) $P(a, b) = \frac{1}{2}[A_1(ab) + A_1(ba)]$.

EXAMPLE 3.3. Suppose that $m_i(I_i) = 1$, $i = 1, 2$, and let $(H_3, \mathfrak{A}_3, m_3)$ be the product gage space (see [9], [10]) of $(H_1, \mathfrak{A}_1, m_1)$ and $(H_2, \mathfrak{A}_2, m_2)$. We define

(i) $P(a, b) = a \otimes b$ (the tensor product) or

(ii) $P(a, b) = A_1(a) \otimes A_2(b)$,

where A_i are any two linear operators from $\mathfrak{L}_{m_i}(\mathfrak{A}_i)$ into $\mathfrak{L}_{m_i}(\mathfrak{A}_i)$, $i = 1, 2$, and

$$\|A_i(a)\| \leqslant \|a\|, \quad a \in \mathfrak{A}_i, \quad i = 1, 2,$$
$$\|A_i(a)\|_1 \leqslant \|a\|_1, \quad a \in \mathfrak{L}^1_{m_i}, \; i = 1, 2.$$

It is clear that

$$\|P(a, b)\| = \|a\| \, \|b\|.$$

From Corollary 8.3 in [10] it follows that

$$\|P(a, b)\|_1 = \|a \otimes b\|_1 = \|a\|_1 \|b\|_1 \leqslant \|a\| \, \|b\|_1 (\|a\|_1 \|b\|).$$

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