ANNALES SOCIETATIS MATHEMATICAE POLONAE
Series I: COMMENTATIONES MATHEMATICAE XXVI (1986)
ROCZNIKI POLSKIEGO TOWARZYSTWA MATEMATYCZNEGO
Seria I: PRACE MATEMATYCZNE XXVI (1986)

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A density result for abstract equations

1. Many authors have studied generic properties of differential and functional equations in Banach spaces (for extended literature see [6], [4], [3]). Results of De Blasi–Myjak [2] and Pianigiani [7] are complement to those findings for ordinary differential equations. Pianigiani, generalizing the result of De Blasi and Myjak, has proved that in an infinite-dimensional Banach space the set of all functions f for which the Cauchy problem $x'(t) = f(t, x(t))$, $x(t_0) = x_0$ has no solutions on any interval containing t_0, is dense in the space of all continuous functions with the topology of uniform convergence. Till now, there was no analogous theorem for the other problems. We obtain a density result for the implicit differential equation $x'(t) = f(t, x(t), x'(t))$. We prove a general theorem and obtain some conclusions for the mentioned implicite differential equation and others. But, unfortunately, the result of Pianigiani does not follow from the theorem of this paper. This will be discussed on the end of the paper.

2. Denote by
D – a topological space,
E – an infinite-dimensional normed space with a norm $|\cdot|$,
C – the space of all continuous functions $F\colon D \to E$ with the metric d:

$$d(F, G) = \sup\{|F(x)-G(x)|\,(1+|F(x)-G(x)|)^{-1}\colon x \in D\},$$

K – a finite subset of E,
$B(x, r) = \{y\colon y \in E, |x-y| < r\}$,
$\bar{B}(x, r)$ – the closure of $B(x, r)$ in E,
$S(x, r) = \bar{B}(x, r) \setminus B(x, r)$,
where $x \in E$, $r > 0$.

For $F \in C$ consider the inclusion

$$F(x) \in K. \tag{1}$$

Denote by M the set of all $F \in C$ for which inclusion (1) has no solutions.

Our main result is

THEOREM. *The interior of M is dense in C.*

Proof. Suppose that $F \in C$, $\varepsilon > 0$, is arbitrary, and

$$C_0 = \{F \in C\colon \inf\{|F(x)-a|\colon x \in D,\ a \in K\} > 0\}.$$

Let $0 < r < \frac{1}{2}\varepsilon$ be such that $B(a, r) \cap B(b, r) = \emptyset$ for all $a, b \in K$, $a \neq b$. There exists a retract function $r_a\colon \bar{B}(a, r) \to S(a, r)$ for all $a \in K$ (see Collorary 5.1, [1], p. 109, and note that $S(a, r)$ is the retract of $E \setminus \{a\}$).

Let $S = \bigcup_{a \in K} B(a, r)$ and $R\colon E \to E$ be the function defined in the following way:

$$R(x) = \begin{cases} x & \text{for } x \notin S, \\ r_a(x) & \text{for } x \in B(a, r),\ a \in K, \end{cases}$$

$x \in E$. It is obvious that $\inf\{|R(x)-a|\colon x \in E, a \in K\} = r$ and R is a continuous function.

Suppose that $G = RF$, then

$$\inf\{|G(x)-a|\colon x \in D,\ a \in K\} \geqslant \inf\{|R(y)-a|\colon y \in E,\ a \in K\} = r > 0,$$

so $G \in C_0$. We get

$$d(F, G) \leqslant \sup\{|F(x)-G(x)|\colon x \in D\} \leqslant \sup\{|y-R(y)|\colon y \in E\} \leqslant 2r < \varepsilon.$$

We have obtained that C_0 is a dense subset of C. Since C_0 is open and $C_0 \subset M$; the proof is completed.

Remark 1. The infinite dimension of E is essential in the proof of the theorem, because a finite-dimensional space Y is not homeomorphic to $Y \setminus \{0\}$.

For instance, if $D = E = R^n$, $K = \{0\}$ and $F(x) = x$ for $x \in D$, then it is impossible to find a function $G \in C_0$ such that $d(G, F) < 1$, so C_0 is not a dense subset of C in this case.

Remark 2. We can change an assumption, that E is a infinite-dimensional normed space by the assumption that E is a metric space and

$$\forall_{\eta > 0} \ \forall_{a \in K} \ \exists_{0 < \varepsilon < \eta} \ \exists_{r_a} \big(r_a \in C[\bar{B}(a, \varepsilon); S(a, \varepsilon)] \wedge \forall_{x \in S(a,\varepsilon)} \ r_a(x) = x\big).$$

It means that for any $\eta > 0$ and $a \in K$ there exists a sphere $S(a, \varepsilon)$ with radius smaller than η which is a retract of the ball $\bar{B}(a, \varepsilon)$.

3. We consider some applications of the theorem to the equation theory.

Example 1. Suppose that $D = D_1 \times D_2$, where D_1, D_2 are topological spaces. From the theorem we get that the set of all functions $F \in C$ for which the inclusion

$$F(x, y) \in K$$

cannot be solved with respect to x or y, has the dense interior in C.

EXAMPLE 2. Assume that $D = E$. It is easy to prove that the set of all continuous functions $f\colon E \to E$, which have no fixed point ($x \in E$ is a fixed point of f if $x = f(x)$), has the dense interior in C.

EXAMPLE 3. Suppose that $D = (a, b) \times E \times E$, a, b are real numbers, $a < b$, $K = \{0\}$. For $f \in C$ consider the implicit differential equation of the type

$$g\big(t, x(t), x'(t)\big) = f\big(t, x(t), x'(t)\big), \tag{2}$$

where $g \in C$ is fixed. Let $t_0 \in (c, d) \subset (a, b)$. We say that a function $u\colon (c, d) \to E$ is a solution of (2) at t_0 if u possesses a weak derivative u' at t_0 and the equality

$$g\big(t_0, u(t_0), u'(t_0)\big) = f\big(t_0, u(t_0), u'(t_0)\big)$$

is true.

Assume that N is the set of all functions $f \in C$ for which equation (2) has no solutions at any point $t_0 \in (a, b)$.

COROLLARY. *N has the dense interior in C.*

Proof. Let C_0 means the same as in the proof of the theorem and

$$g + C_0 = \{g + F\colon F \in C_0\}.$$

If $f = g + F$ and $F \in C_0$, then $f(t, x, y) \neq g(t, x, y)$ for all $(t, x, y) \in D$, so equation (2) has no solutions. Consequently, $g + C_0 \subset N$. Of course, $g + C_0$ is open and dense in C, so the proof is completed.

Remark 3. We can replace x' and x in equation (2) by any operators on x and the assertion of the corollary will be the same.

Moreover, we can extend Examples 1–3 to the spaces E fulfilling the condition from Remark 2 (E must be a linear metric space in Examples 2 and 3).

We can formulate an analogous conclusion for partial differential equations, integral equations and others.

Remark 4. Till now we said that Pianigiani has proved that the set P of all continuous functions $h\colon R \times E \to E$ (where E is a Banach space), for which the Cauchy problem

$$x'(t) = h\big(t, x(t)\big), \qquad x(t_0) = x_0,$$

has no solutions, is dense in the space of all continuous functions from $R \times E$ into E. But the complement of P is residual (see [5]), therefore it is dense, too. Consequently P has no dense interior and the result of Pianigiani cannot be obtained from the theorem of this paper.

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