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JAN SZAJKOWSKI (Zielona Góra)

Separability of Hardy–Orlicz spaces of analytic functions in the half-plane, I

Abstract. In this paper we present properties of H_N class ([11]), defined by the formula

$$H_N = \bigcap_{\psi} H^{\psi},$$

where the product extends over all N-functions ψ and H^{ψ} is the Hardy–Orlicz class of analytic functions in the half-plane considered in [8]. We need the H_N class to study of separability of Hardy–Orlicz space $H^{*\psi}$ of analytic functions in the half-plane.

This paper can be regarded as a continuation of papers [8] and [9] which contain the study of Hardy–Orlicz spaces of analytic functions in the half-plane. Some results of papers [8] and [9] and other papers will be needed here. We collect them in the first section.

I. Orlicz and Hardy–Orlicz spaces

1.1. An increasing and convex function $\psi(u)$ for $u \geqslant 0$ is called an *N-function*, if it satisfies the following conditions:

$$(0_1)\ \lim_{u\to 0^+} \frac{\psi(u)}{u} = 0 \quad \text{and} \quad (\infty_1)\ \lim_{u\to\infty} \frac{\psi(u)}{u} = \infty$$

([3], Chapter I).

1.2. Each N-function ψ can be written in the form

$$\psi(u) = \int_0^u p(t)\,dt \qquad (u \geqslant 0),$$

where p is a positive and non-decreasing function for $t > 0$ and such that

$$\lim_{t\to 0^+} p(t) = 0 \quad \text{and} \quad \lim_{t\to\infty} p(t) = \infty$$

([3], Chapter I).

1.3. An N-function ψ is said to *satisfy condition* (Δ_2), if for some constant $d > 1$ the inequality holds

$$\psi(2u) \leqslant d \cdot \psi(u) \quad \text{for } u \geqslant 0.$$

2.1. Let f be a complex-valued function, defined and measurable on the interval $(-\infty, \infty)$. We define

$$\varrho_\psi(f) = \int_{-\infty}^{\infty} \psi(|f(t)|)\,dt.$$

In the space of all complex-valued functions, defined and measurable on $(-\infty, \infty)$ the functional $\varrho_\psi(\cdot)$ is a convex modular in the sense of Musielak and Orlicz.

2.2. By L^ψ we denote the class of all complex-valued functions f, measurable on $(-\infty, \infty)$ for which $\varrho_\psi(f) < \infty$, by $L^{*\psi}$ the class of all functions f such that $kf \in L^\psi$ for a certain $k > 0$ (in general dependent on f) and by $L^{0\psi}$ the class of all functions f such that $kf \in L^\psi$ for every $k > 0$.

In the space of measurable on $(-\infty, \infty)$ complex-valued functions the class L^ψ is an absolutely convex set and the classes $L^{*\psi}$ and $L^{0\psi}$ are linear subspaces. The class L^ψ is called *Orlicz class*, $L^{*\psi}$ *Orlicz spaces* and $L^{0\psi}$ the *space of finite elements of* $L^{*\psi}$.

In the space $L^{*\psi}$ the formula

$$\|f\|_\psi = \inf\{\varepsilon > 0\colon\ \varrho_\psi(f/\varepsilon) \leqslant 1\} \qquad \text{for } f \in L^{*\psi}$$

defines the homogeneous norm in $L^{*\psi}$.

The space $L^{*\psi}$ is a Banach space with the norm $\|\cdot\|_\psi$ ([5], [6]).

3.1. We denote by N a set of functions G analytic in the disc $D = \{z \in \boldsymbol{C}\colon |z| < 1\}$, for which

$$\sup\Big\{\int_0^{2\pi} \log^+ |G(r\cdot e^{i\theta})|\,d\theta\colon\ 0 \leqslant r < 1\Big\} < \infty,$$

where $\log^+ u = \log\sup\{1, u\}$ for $u \geqslant 0$.

3.2. If $G \in N$, then for almost every θ there exists the limit

$$\lim G(z) = G(e^{i\theta}),$$

where z tends to $e^{i\theta}$ between two chords of the disc D starting at the point $e^{i\theta}$ ([12], Chapter VII).

3.3. Let ψ be an N-function and G an analytic function in D. We define

$$\varrho_\psi(G) = \sup\Big\{\int_0^{2\pi} \psi(|G(r\cdot e^{i\theta})|)\,d\theta\colon\ 0 \leqslant r < 1\Big\}.$$

By $H^\psi(D)$ we denote the class of all functions G analytic in D such that $\varrho_\psi(G) < \infty$ and by $H^{*\psi}(D)$ the class of all functions G such that $kG \in H^\psi(D)$ for a certain $k > 0$ (in general dependent on G).

Moreover, by $H^{0\psi}(D)$ we denote the class of all functions G such that $kG \in H^\psi(D)$ for every $k > 0$.

The class $H^\psi(D)$ is an absolutely convex set in the space of all analytic functions in D and the classes $H^{*\psi}(D)$ and $H^{0\psi}(D)$ are linear subspaces. The class $H^{\psi}(D)$ is called *Hardy–Orlicz class* in D, $H^{*\psi}(D)$ – *Hardy–Orlicz space* in D and $H^{0\psi}(D)$ – the *space of finite elements* in $H^{*\psi}(D)$.

In the space $H^{*\psi}(D)$ the formula

$$\|G\|_{\psi} = \inf\{\varepsilon > 0: \ \varrho_{\psi}(G/\varepsilon) \leqslant 1\} \qquad (G \in H^{*\psi}(D)),$$

defines the homogeneous norm in $H^{*\psi}(D)$.

The space $H^{*\psi}(D)$ is complete with respect to the norm $\|\cdot\|_{\psi}$, [4].

3.4. By $H^1(D)$ we denote the class of all functions G analytic in D, for which

$$\|G\|_1 = \sup \int_0^{2\pi} |G(r\cdot e^{i\theta})|\, d\theta: \ 0 \leqslant r < 1\} < \infty.$$

In the space of all analytic functions in D the class $H^1(D)$ is a linear set. The functional $\|\cdot\|_1$ is the homogeneous norm in $H^1(D)$ and the space $H^1(D)$ is complete with respect to this norm.

By $H^{\infty}(D)$ we denote the class of all functions G analytic and bounded in D. In the space of all analytic functions in D the class $H^{\infty}(D)$ is a linear set. The space $H^{\infty}(D)$ is complete with respect to norm

$$\|G\|_{\infty} = \sup\{|G(z)|: \ z \in D\} \qquad (G \in H^{\infty}(D)),$$

([1], [2], [12], Chapter VII).

3.5. For an arbitrary N-function ψ, the inclusions hold

$$H^{\infty}(D) \subset H^{0\psi}(D) \subset H^{\psi}(D) \subset H^{*\psi}(D) \subset H^1(D) \subset N.$$

Then, on the ground of 3.2, we get that the function G, belonging to any one of these classes, possesses, precisely to a set of measure zero, a boundary function defined the formula

$$G(e^{i\theta}) = \lim_{r\to 1^-} G(r\cdot e^{i\theta}), \qquad [4].$$

3.6. If $G \in H^1(D)$, then the following Poisson's integral formula holds

$$G(r\cdot e^{i\theta}) = \frac{1}{2\pi} \int_0^{2\pi} G(e^{i\tau}) \frac{1-r^2}{1-2r\cdot\cos(\theta-\tau)+r^2}\, d\tau \qquad (0 \leqslant r < 1).$$

Hence we easily obtain that if $G \in H^{*\psi}(D)$, then

$$\|G\|_{\psi} = \inf\{\varepsilon > 0: \ \int_0^{2\pi} \psi\big(|G(e^{i\theta})|/\varepsilon\big)\, d\theta \leqslant 1\}.$$

Similarly, if $G \in H^1(D)$, then

$$\|G\|_1 = \int_0^{2\pi} |G(e^{i\theta})|\, d\theta,$$

and, if $G \in H^{\infty}(D)$, then

$$\|G\|_{\infty} = \operatorname{ess\,sup}\{|G(e^{i\theta})|: 0 \leqslant \theta < 2\pi\}$$

([1], [2], [4], [12]).

4.1. We denote by $A(\Omega)$ the space of analytic functions in the half-plane $\Omega = \{w \in C: \operatorname{Re} w > 0\}$. Let ψ be an N-function and F an analytic function in Ω. We define

$$\varrho_{\psi}(F) = \sup\{\int_{-\infty}^{\infty} \psi(|F(x+iy)|)dy: x > 0\}.$$

In the space $A(\Omega)$ the functional $\varrho_{\psi}(\cdot)$ is a convex modular in the sense of Musielak and Orlicz.

By H^{ψ} we denote the class of all functions $F \in A(\Omega)$ such that $\varrho_{\psi}(F) < \infty$ and by $H^{*\psi}$ the class of all functions F such that $kF \in H^{\psi}$ for a certain $k > 0$ (in general, dependent on F). Moreover, by $H^{0\psi}$ we denote the class of all functions F such that $kF \in H^{\psi}$ for every $k > 0$.

The class H^{ψ} is an absolutely convex set in $A(\Omega)$ and the classes $H^{*\psi}$ and $H^{0\psi}$ are linear subspaces. Obviously, the inclusions hold

$$H^{0\psi} \subset H^{\psi} \subset H^{*\psi} \subset A(\Omega).$$

The class H^{ψ} we call *Hardy–Orlicz class* in Ω, $H^{*\psi}$ – *Hardy–Orlicz space* in Ω and $H^{0\psi}$ – the *space of finite elements* in $H^{*\psi}$.

If $\psi(u) = u^p$, where $p > 1$, then the space $H^{*\psi}$ is the known Hardy space in Ω for the power p.

In the space $H^{*\psi}$ the formula

$$\|F\|_{\psi} = \inf\{\varepsilon > 0: \varrho_{\psi}(F/\varepsilon) \leqslant 1\} \quad (F \in H^{*\psi}),$$

defines the homogeneous norm in $H^{*\psi}$.

The space $H^{*\psi}$ is a Banach space with respect to the norm $\|\cdot\|_{\psi}$, [8].

4.2. If $F \in H^{*\psi}$, then F has non-tangential limits in almost every point of the imaginary axis and a boundary function $F(i\cdot)$ belongs to the space $L^{*\psi}$, [8].

4.3. If $F \in H^{*\psi}$, then the following Poisson's integral formula holds

$$F(\xi+i\eta) = \frac{1}{\pi}\int_{-\infty}^{\infty} F(x+it)\frac{\xi-x}{(\xi-x)^2+(\eta-t)^2}dt, \quad \text{where } 0 < x < \xi.$$

For $F \in H^{*\psi}$ the formula holds as well

$$F(\xi+i\eta) = \frac{1}{\pi}\int_{-\infty}^{\infty} F(it)\frac{\xi}{\xi^2+(\eta-t)^2}dt, \quad \text{where } \xi > 0, \quad [9].$$

4.4. On the ground of 4.3 we easily get that if $F \in H^{*\psi}$, then $F(w)$ tends uniformly to zero, as $w \to \infty$, in the interior of every arbitrary half-plane $\{w \in C\colon \operatorname{Re} w \geqslant \delta\}$, where $\delta > 0$, [9].

4.5. Taking into account 4.3, also we easily obtain that for $F \in H^{*\psi}$ the identity holds

$$\varrho_\psi(F) = \varrho_\psi(F(i\cdot)),$$

and

$$\|F\|_\psi = \|F(i\cdot)\|_\psi, \quad [9].$$

4.6. If $F \in H^\psi$, then

$$|F(w)| \leqslant \psi^{-1}\left(\frac{\varrho_\psi(F)}{\pi\cdot\operatorname{Re} w}\right) \quad \text{for } \operatorname{Re} w > 0, \quad [9].$$

4.7. If $F \in H^\psi$, then

$$\lim_{x\to 0^+} \varrho_\psi\big(\tfrac{1}{2}(F(i\cdot) - F(x+i\cdot))\big) = 0, \quad [9].$$

4.8. If f is a non-negative function on the interval $(-\infty, \infty)$ and there exists the integral

$$\int_{-\infty}^{\infty} \frac{|\ln f(\tau)|}{1+\tau^2}\, d\tau,$$

then the formula

$$F(w) = \exp\left(\frac{1}{\pi}\int_{-\infty}^{\infty} \frac{itw-1}{it-w}\cdot\frac{\ln f(t)}{1+t^2}\, dt\right)$$

defines an analytic function in Ω such that $|F(it)| = f(t)$ for almost all t from the interval $(-\infty, \infty)$ and

$$\varrho_\psi(F) = \varrho_\psi(f), \quad [10].$$

4.9. Let $\{F_k\}$ be a sequence of functions and F the element from $H^{*\psi}$. Then $\|F_k - F\|_\psi \to 0$ as $k \to \infty$ if and only if $\varrho_\psi(\lambda(F_k - F)) \to 0$ as $k \to \infty$ for every number $\lambda > 0$, [11].

4.10. By H^1 we denote the class of all functions F analytic in Ω, for which the integrals

$$\int_{-\infty}^{\infty} |F(x+it)|\, dt$$

are uniformly bounded for $x > 0$. The space H^1 under the norm

$$\|F\|_1 = \sup\Big\{\int_{-\infty}^{\infty} |F(x+it)|\, dt\colon\ x > 0\Big\}$$

is a Banach space ([1], [2]).

Since all the results [8] and [9] for the space H^1 hold, so in particular 4.2 in formulation for H^1 decides that the function $F \in H^1$ has non-tangential limits in almost every point of the imaginary axis and its boundary function $F(i\cdot)$ belongs to the space L^1, and 4.5 that the identity $\|F\|_1 = \|F(i\cdot)\|_1$ holds (also see [1] and [2]).

By H^∞ we denote the class of all functions F analytic and bounded in Ω. The space H^∞ under the norm

$$\|F\|_\infty = \sup\{|F(w)|:\ w \in \Omega\}$$

is a Banach space ([1], [2]).

It is obvious that the map U defined by the formula

$$(UF)(z) = F\left(\frac{1+z}{1-z}\right) \quad \text{for } z \in D,$$

transforms isometrically and isomorphically the space H^∞ in Ω into the space $H^\infty(D)$ (see 3.4).

II. The space H_N

1.1. We denote

$$H_N = \bigcap_{\psi} H^{\psi},$$

where the product extends over all N-functions ψ, and H^ψ is the Hardy–Orlicz class in Ω.

We study properties of the class H_N.

1.2. Lemma. *If $F \in H_N$, then the identity holds*

$$\sup\{|F(w)|:\ w \in \Omega\} = \operatorname{ess\,sup}\{|F(it)|:\ t \in (-\infty, \infty)\}.$$

Proof. Let $F \in H_N$. Then, according to 4.3 of Section I for a function F the Poisson's integral formula holds

$$F(x+iy) = \frac{1}{\pi}\int_{-\infty}^{\infty} F(it)\frac{x}{x^2+(y-t)^2}\,dt,$$

where $F(i\cdot)$ is a boundary function of an analytic function F. Hence and from the fact that

$$\frac{1}{\pi}\int_{-\infty}^{\infty} \frac{x}{x^2+(y-t)^2}\,dt = 1$$

we obtain

$$|F(w)| \leqslant \operatorname{ess\,sup}\{|F(it)|:\ t \in (-\infty, \infty)\}.$$

Therefore on the ground of the Maximum Principle the identity must hold.

1.3. LEMMA. $H_N \dot{\subset} H^\infty$.

Proof. Let $F \in H_N$. Let us suppose that $F \notin H^\infty$. We write

$$E_n = \{t\colon\ n \leqslant |F(it)|\}, \qquad k_n = \operatorname{mes} E_n \qquad \text{for } n = 1, 2, \ldots$$

The $\{k_n\}$ is a non-increasing sequence of positive numbers tending to zero, because the function $F(i\cdot)$ is measurable but one the ground of Lemma 1.2 it is not essentially bounded. We take

$$p(t) = \begin{cases} 2k_1^{-1}t & \text{for } \ 0 \leqslant t < \frac{1}{2}, \\ k_n^{-1} & \text{for } \ \frac{1}{2}n \leqslant t < \frac{1}{2}(n+1),\ n = 1, 2, \ldots \end{cases}$$

and next

$$\psi(u) = \int_0^u p(t)\,dt \qquad \text{for } u \geqslant 0.$$

Since the function p is non-decreasing for $t \geqslant 0$ also $p(t) \to 0$ as $t \to 0^+$ and $p(t) \to \infty$ as $t \to \infty$, so in view of 1.2 of Section I the function ψ is a N-function. We state that

$$\begin{aligned} \int_{-\infty}^{\infty} \psi(|F(it)|)\,dt \geqslant \psi(n)\cdot \operatorname{mes} E_n &\geqslant \int_{\frac{n}{2}}^{n} p(t)\,dt \cdot k_n \\ &\geqslant \tfrac{1}{2}n\cdot p(\tfrac{1}{2}n)\cdot k_n = \tfrac{1}{2}n \qquad \text{for } n = 1, 2, \ldots \end{aligned}$$

Hence we get

$$\int_{-\infty}^{\infty} \psi(|F(it)|)\,dt = \infty,$$

what yields, in view of 4.5 of Section I, that $F \notin H^\psi$ and further $F \notin H_n$. This contradicts the assumption.

1.4. THEOREM. $H_N = H^1 \cap H^\infty$.

Proof. We shall show first that the inclusion $H_N \subset H^1 \cap H^\infty$ holds.

Let $F \in H_N$. Then, in virtue of Lemma 1.3, we have $F \in H^\infty$. Thus there exists the constant $M > 0$ such that $|F(it)| \leqslant M$ for almost all t from the interval $(-\infty, \infty)$. Let us suppose that $F \notin H^1$ and let us denote

$$E_n = \left\{t\colon\ \frac{M}{n+1} < |F(it)| \leqslant \frac{M}{n}\right\} \qquad \text{and} \qquad k_n = \operatorname{mes} E_n \qquad \text{for } n = 1, 2, \ldots$$

From the assumption we get

$$\infty = \int_{-\infty}^{\infty} |F(it)|\,dt \leqslant \sum_{n=1}^{\infty} \frac{M}{n}\cdot k_n.$$

It is a well-known fact that one can then take a non-increasing sequence of positive numbers $\{\alpha_n\}$ tending to zero such that

$$\sum_{n=1}^{\infty} \alpha_n \cdot \frac{M}{n} \cdot k_n = \infty .$$

Let us remark that a sequence $\{(1+1/n)\alpha_n\}$ decreases to zero. We take

$$p(t) = \begin{cases} 0 & \text{for} \quad t = 0 \\ 2\left(1+\frac{1}{n}\right)\alpha_n & \text{for} \quad \frac{M}{2(n+1)} \leqslant t < \frac{M}{2n}, \quad n = 1, 2, \dots \\ \frac{8\alpha_1}{M} \cdot t & \text{for} \quad \frac{M}{2} \leqslant t \end{cases}$$

and further

$$\psi(u) = \int_0^u p(t)\,dt \quad \text{for } u \geqslant 0.$$

The function p is non-decreasing for $t \geqslant 0$ and $p(t) \to 0$ as $t \to 0^+$ also $p(t) \to \infty$ as $t \to \infty$. Therefore on the ground of 1.2 of Section I the function ψ is a N-function. We state that

$$\int_{-\infty}^{\infty} \psi(|F(it)|)\,dt$$

$$\geqslant \sum_{n=1}^{\infty} \psi\left(\frac{M}{n+1}\right) \cdot k_n \geqslant \sum_{n=1}^{\infty} \int_{\frac{M}{2(n+1)}}^{\frac{M}{n+1}} p(t)\,dt \cdot k_n$$

$$\geqslant \sum_{n=1}^{\infty} p\left(\frac{M}{2(n+1)}\right) \cdot \frac{M}{2(n+1)} \cdot k_n = \sum_{n=1}^{\infty} 2\left(1+\frac{1}{n}\right)\alpha_n \cdot \frac{M}{2(n+1)} \cdot k_n$$

$$= \sum_{n=1}^{\infty} \alpha_n \cdot \frac{M}{n} \cdot k_n = \infty .$$

Hence we deduce, in virtue of 4.5 of Section I, that $F \notin H^{\psi}$ and further $F \notin H_N$ what contradicts the assumption. Thus the inclusion $H_N \subset H^1$ holds. Hence and from Lemma 1.3 we get that the inclusion holds

$$(*) \qquad H_N \subset H^1 \cap H^{\infty}.$$

We shall now show that the other inclusion holds. Let $F \in H^1 \cap H^{\infty}$. Then, in view of $F \in H^{\infty}$, there exists the positive constant M such that $|F(w)| \leqslant M$ for each $w \in \Omega$. From $F \in H^1$ we get now for an arbitrary

N-function ψ and $x > 0$ the inequality

$$\int_{-\infty}^{\infty} \psi(|F(x+it)|)\,dt = \int_{-\infty}^{\infty} \psi\left(\frac{|F(x+it)|}{M}\cdot M\right)dt$$
$$\leqslant \frac{\psi(M)}{M} \int_{-\infty}^{\infty} |F(x+it)|\,dt \leqslant \frac{\psi(M)}{M}\|F\|_1.$$

Hence we deduce that $F \in H^{\psi}$ under an arbitrary N-function ψ. Therefore the inclusion

$$(**) \qquad H^1 \cap H^{\infty} \subset H_N$$

holds.

On the ground of inclusions $(*)$ and $(**)$ we obtain the desired quality.

1.5. THEOREM. *In the space* H_N *the formula*

$$\|F\|_N = \sup\{\|F\|_1, \|F\|_{\infty}\} \quad \textit{for } F \in H_N$$

defines the homogeneous norm.

Proof. Because of the finiteness of $\|F\|_1$ and $\|F\|_{\infty}$ we deduce that $\|F\|_N$ takes finite values for $F \in H_N$. Moreover, let us remark that, for $F \in H_N$ and $F \neq 0$, $\|F\|_N > 0$. Since for an arbitrary complex number α we have $\|\alpha F\|_1 = |\alpha|\cdot\|F\|_1$ and $\|\alpha F\|_{\infty} = |\alpha|\cdot\|F\|_{\infty}$, so

$$\|\alpha F\|_N = |\alpha|\cdot\|F\|_N.$$

Next, let F_1 and F_2 be arbitrary functions from H_N. Since

$$\|F_1+F_2\|_1 \leqslant \|F_1\|_1+\|F_2\|_1 \leqslant \|F_1\|_N+\|F_2\|_N$$

and similarly

$$\|F_1+F_2\|_{\infty} \leqslant \|F_1\|_{\infty}+\|F_2\|_{\infty} \leqslant \|F_1\|_N+\|F_2\|_N,$$

so

$$\|F_1+F_2\|_N \leqslant \|F_1\|_N+\|F_2\|_N.$$

1.6. THEOREM. *The space* H_N *is complete with respect to the norm* $\|\cdot\|_N$.

Proof. Let $\{F_n\}$ be an arbitrary sequence of elements from H_N satisfying the Cauchy's condition, i.e., such a sequence that for each $\varepsilon > 0$ there exists n_0 such that, for $n, m \geqslant n_0$, $\|F_n - F_m\|_N < \varepsilon$. Then we have

$$|F_n(w) - F_m(w)| \leqslant \|F_n - F_m\|_{\infty} \leqslant \|F_n - F_m\|_N < \varepsilon$$

for $n, m \geqslant n_0$ and $w \in \Omega$. From this inequality we deduce that the sequence $\{F_n(w)\}$ at fixed $w \in \Omega$ satisfies the Cauchy condition, thus it is convergent: let $F_n(w) \to F(w)$ as $n \to \infty$. The function $F(w)$ is analytic in Ω, because, from the same inequality it follows that the sequence $\{F_n\}$ is uniformly convergent in Ω. From $\|F_n - F_m\|_N < \varepsilon$ for n, $m \geqslant n_0$, we have

$$\text{(I)} \qquad \int_{-\infty}^{\infty} |F_n(x+iy) - F_m(x+iy)|\, dy < \varepsilon, \quad \text{where } x > 0$$

and

$$\text{(II)} \qquad |F_n(w) - F_m(w)| < \varepsilon, \quad \text{where } w \in \Omega$$

for n, $m \geqslant n_0$. Letting in inequalities (I) and (II), m tend to ∞, we get

$$\text{(III)} \qquad \int_{-\infty}^{\infty} |F_n(x+iy) - F(x+iy)|\, dy \leqslant \varepsilon, \quad \text{where } x > 0$$

and

$$\text{(IV)} \qquad |F_n(w) - F(w)| \leqslant \varepsilon, \quad \text{where } w \in \Omega$$

for $n \geqslant n_0$. Hence we deduce that in particular $F_{n_0} - F \in H_N$. Since $F_{n_0} \in H_N$, so $F \in H_N$. Further, from inequalities (III) and (IV), we get $\|F_n - F\|_N \leqslant \varepsilon$ for $n \geqslant n_0$, what proves that $\|F_n - F\|_N \to 0$, as $n \to \infty$.

1.7. THEOREM. *The space $\langle H_N, \|\cdot\|_N\rangle$ is not separable.*

Proof. In the interval $(-\infty, \infty)$ we construct a sequence of intervals E_n

$$E_n = \left(\frac{1}{2^{n+1}}, \frac{1}{2^n}\right) \quad \text{for } n = 1, 2, \ldots$$

We define a sequence $\{g_n\}$ of real functions:

$$g_n(t) = \begin{cases} 1 & \text{for } t \in E_n, \\ 0 & \text{elsewhere in } (-\infty, \infty). \end{cases}$$

Subsequently, we define the family of real functions

$$g_\eta(t) = \sum_{n=1}^{\infty} \eta_n g_n(t) \quad \text{for } t \in (-\infty, \infty),$$

where $\eta = \{\eta_n\}$ is an arbitrary sequence of terms 0 and 1. Next, we take a sequence $\{f_\eta\}$ of real functions

$$f_\eta(t) = g_\eta(t) + 1/(1+t^2).$$

Let us observe that

$$\|f_\eta\|_1 = \int_{-\infty}^{\infty} f_\eta(t)\, dt = \int_{-\infty}^{\infty} g_\eta(t)\, dt + \int_{-\infty}^{\infty} \frac{dt}{1+t^2} \leqslant \sum_{n=1}^{\infty} \int_{-\infty}^{\infty} g_n(t)\, dt + \pi$$

$$= \sum_{n=1}^{\infty} \frac{1}{2^{n+1}} + \pi = \tfrac{1}{2} + \pi$$

and

$$\|f_\eta\|_\infty = \sup\{f_\eta(t)\colon\ t\in(-\infty,\infty)\}$$

$$\leqslant \sup\{g_\eta(t)\colon\ t\in(-\infty,\infty)\}+\sup\left\{\frac{1}{1+t^2}\colon\ t\in(-\infty,\infty)\right\}\leqslant 1+1=2.$$

We shall show now that there exists the integral

$$\int_{-\infty}^{\infty}\frac{|\ln f_\eta(t)|}{1+t^2}\,dt.$$

With this in view, let us denote

$$A=\{t\colon\ |f_\eta(t)|\geqslant 1\},\qquad B=\{t\colon\ |f_\eta(t)|<1\}.$$

Since for $u\geqslant 1$, the inequality $\ln u\leqslant u$ holds so that for each η

$$\int_A\frac{|\ln f_\eta(t)|}{1+t^2}\,dt=\int_A\frac{\ln f_\eta(t)}{1+t^2}\,dt\leqslant\int_A f_\eta(t)\,dt\leqslant\int_{-\infty}^{\infty}f_\eta(t)\,dt<\infty.$$

Since for $u\geqslant 1$, the inequality $\ln u\leqslant 4u^{1/4}$ holds and for $t\in B$

$$\frac{1}{1+t^2}\leqslant f_\eta(t)\leqslant 1,$$

which gives

$$1\leqslant\frac{1}{f_\eta(t)}\leqslant 1+t^2$$

so for each η

$$\int_B\frac{|\ln f_\eta(t)|}{1+t^2}\,dt=\int_B\frac{\ln(1/f_\eta(t))}{1+t^2}\,dt\leqslant\int_B\frac{\ln(1+t^2)}{1+t^2}\,dt$$

$$\leqslant 4\int_B\frac{(1+t^2)^{1/4}}{1+t^2}\,dt\leqslant 4\int_{-\infty}^{\infty}\frac{dt}{(1+t^2)^{3/4}}=8\int_0^{\infty}\frac{dt}{(1+t^2)^{3/4}}$$

$$\leqslant 8\Big(\int_0^1 dt+\int_1^{\infty}dt/t^{3/2}\Big)=24<\infty.$$

So we obtain for each η

$$\int_{-\infty}^{\infty}\frac{|\ln f_\eta(t)|}{1+t^2}\,dt=\int_A\frac{|\ln f_\eta(t)|}{1+t^2}\,dt+\int_B\frac{|\ln f_\eta(r)|}{1+t^2}\,dt<\infty.$$

In the sequel, we take a sequence $\{F_\eta\}$ of functions defined by the formula

$$F_\eta(w)=\exp\left(\frac{1}{\pi}\int_{-\infty}^{\infty}\frac{tw+i}{t+iw}\cdot\frac{\ln f_\eta(t)}{1+t^2}\,dt\right).$$

From above, in view of 4.8 of Section I, we state that functions F_η are analytic in Ω and such that $|F_\eta(it)| = f_\eta(t)$ for almost all t from the interval $(-\infty, \infty)$. Moreover, let us remark that $\|F_\eta\|_1 = \|f_\eta\|_1$ and $\|F_\eta\|_\infty = \|f\eta\|_\infty$. Hence and from the fact that $\|f_\eta\|_1 < \infty$ and $\|f_\eta\|_\infty < \infty$, in virtue of Theorem 1.4 we deduce that $F_\eta \in H_N$ for each η.

Now, let us take two different sequences $\eta' = \{\eta'_k\}$ and $\eta'' = \{\eta''_k\}$ of terms 0 and 1. Then there exists index n such that $\eta'_n \neq \eta''_n$. Since

$$\begin{aligned}\|F_{\eta'} - F_{\eta''}\|_\infty &= \|F_{\eta'}(i\cdot) - F_{\eta''}(i\cdot)\|_\infty \\ &= \sup\{|F_{\eta'}(it) - F_{\eta''}(it)|:\ t \in (-\infty, \infty)\} \\ &\geqslant \sup\{\big||F_{\eta'}(it)| - |F_{\eta''}(it)|\big|:\ t \in (-\infty, \infty)\} \\ &= \sup\{|f_{\eta'}(t) - f_{\eta''}(t)|:\ t \in (-\infty, \infty)\} \\ &= \sup\{|g_{\eta'}(t) - g_{\eta''}(t)|:\ t \in (-\infty, \infty)\} \\ &= \sup\{g_n(t):\ t \in (-\infty, \infty)\} = 1,\end{aligned}$$

so

$$\|F_{\eta'} - F_{\eta''}\|_N \geqslant 1.$$

Since the set of sequences with terms 0 and 1 is a continuum so there exists in H_N a continuum of elements whose distances are $\geqslant 1$. Hence the space $\langle H_N, \|\cdot\|_N\rangle$ is not separable.

1.8. THEOREM. *For an arbitrary N-function* ψ *the inclusion*

$$H_N \subset H^{0\psi}$$

holds and the inequality

$$\|F\|_\psi \leqslant c_\psi \|F\|_N \quad \textit{for } F \in H_N$$

is satisfied, where c_ψ *is a constant such that* $\psi(c_\psi^{-1}) = 1$.

Proof. If $F = 0$, then the theorem is obvious. Let F be an arbitrary function from the class H_N, different at zero. Then, on the ground of Lemma 1.2 and the definition of the norm $\|\cdot\|_N$, we get

$$|F(it)|/\|F\|_N \leqslant |F(it)|/\|F\|_\infty \leqslant 1$$

for almost all t from the interval $(-\infty, \infty)$. We state that for an arbitrary N-function ψ and for an arbitrary positive number k

$$\begin{aligned}\varrho_\psi(kF) = \varrho_\psi(kF(i\cdot)) &= \int_{-\infty}^{\infty} \psi\left(\frac{|F(it)|}{\|F\|_N}\cdot k\|F\|_N\right)dt \\ &\leqslant \int_{-\infty}^{\infty} \frac{|F(it)|}{\|F\|_N}\psi(k\|F\|_N)\,dt = \frac{\|F\|_1}{\|F\|_N}\psi(k\|F\|_N) \leqslant \psi(k\|F\|_N).\end{aligned}$$

Hence we conclude that $F \in H^{0\psi}$. Taking $k = (c_\psi \|F\|_N)^{-1}$ in the obtained inequality, where c_ψ is the constant such that $\psi(c_\psi^{-1}) = 1$, we get

$$\varrho_\psi(F/c_\psi \|F\|_N) \leqslant \psi(1/c_\psi) = 1.$$

Hence we obtain the inequality mentioned in the theorem.

1.9. THEOREM. *If the function F belongs to H^1 in Ω, then the function*

$$(TF)(z) = G(z) = \frac{2}{(1-z)^2} F\left(\frac{1+z}{1-z}\right) \quad (z \in D),$$

belongs to H^1 in D. Conversely, if the function G belongs to H^1 in D, then the function

$$F(w) = (T^{-1} G)(w) = \frac{2}{(1+w)^2} G\left(\frac{w-1}{w+1}\right) \quad (w \in \Omega),$$

belongs to H^1 in Ω. Moreover, $\|F\|_1 = \|G\|_1$.

This means that the space H^1 in Ω is isometrically isomorphic to H^1 in D: this isomorphism establishes the operator T (see [2], Chapter VII).

Proof. Let F belong to H^1 in Ω. Let us take a homographic transformation Ω into D:

$$z = \frac{w-1}{w+1}, \quad \text{where } w = x+iy.$$

Notice that under this transformation the lines $l_x = \{w \in C\colon \operatorname{Re} w = x\}$ are mapped into circles $\gamma_x = \{z \in C\colon |z - z_0| = R\}$, where $z_0 = x/(x+1)$ and $R = 1/(x+1)$, internally tangent to the unit circle C in the point $z = 1$. For the reciprocal map $w = (1+z)/(1-z)$ we have $dw = 2 \cdot dz/(1-z)^2$. Hence, from above, we get for $x > 0$

$$(*) \quad \int_{-\infty}^{\infty} |F(x+iy)\,dy = \int_{l_x} |F(w)| \cdot |dw| = \int_{\gamma_x} \left|F\left(\frac{1+z}{1-z}\right)\right| \cdot \frac{2|dz|}{|1-z|^2} = \int_{\gamma_x} |G(z)| \cdot |dz|.$$

Let C_r be a circle $C_r = \{z \in C\colon |z| = r\}$, where $0 < r < 1$. We estimate integrals from above

$$\int_0^{2\pi} |G(r \cdot e^{i\theta})|\, d\theta = \frac{1}{r} \int_{C_r} |G(z)| \cdot |dz| \quad \text{as } r \to 1^-.$$

We choose $x > 0$ so that the circle C_r could lie within the circle γ_x. Since F is an analytic function in Ω and the transformation $z = (w-1)/(w+1)$ has no singular points in Ω, consequently G is analytic within γ_x and is a continuous function in a closed disc limited by γ_x, maybe except for the point $z = 1$. We write

$$C_r^* = \{\zeta \in C\colon |\zeta + z_0| = r\}, \quad \gamma_x^* = \{\zeta \in C\colon |\zeta| = R\},$$

and

$$G_0(\zeta) = G(\zeta + z_0).$$

Applying the Poisson's integral formula for the disc $\{\zeta \in C\colon |\zeta| < R\}$, we obtain

$$G_0(r \cdot e^{i\theta}) = \frac{1}{2\pi} \int_0^{2\pi} G_0(e^{i\tau}) \cdot \frac{R^2 - r^2}{R^2 - 2Rr \cdot \cos(\theta - \tau) + r^2}\, d\tau$$

and further

$$|G_0(r \cdot e^{i\theta})| \leqslant \frac{1}{2\pi} \int_0^{2\pi} \frac{R^2 - r^2}{R^2 - 2Rr \cdot \cos(\theta - \tau) + r^2} |G(e^{i\tau})|\, d\tau,$$

i.e.,

$$|G_0(\alpha)| \leqslant \frac{1}{2\pi R} \int_{\gamma_x^*} \operatorname{Re} \frac{\zeta + \alpha}{\xi - \alpha} \cdot |G_0(\zeta)| \cdot |d\zeta|.$$

Integrating this inequality longwise the circle C_r^* and changing the order of integration on the right-hand side, we get

$$\frac{1}{r} \int_{C_r^*} |G_0(\alpha)| \cdot |d\alpha| \leqslant \frac{1}{2\pi r R} \int_{\gamma_x^*} \int_{C_r^*} \operatorname{Re} \frac{\zeta + \alpha}{\zeta - \alpha} |d\alpha| \cdot |G_0(\zeta)| \cdot |d\zeta|.$$

But $\operatorname{Re}[(\zeta + \alpha)/(\zeta - \alpha)]$ is a harmonic function of α, so, by the mean-value Theorem for harmonic functions, we obtain

$$\frac{1}{2\pi r} \int_{C_r^*} \operatorname{Re} \frac{\zeta + \alpha}{\zeta - \alpha} |d\alpha| = \operatorname{Re} \frac{\zeta + \alpha}{\zeta - \alpha}\bigg|_{\alpha = -z_0} = \operatorname{Re} \frac{\zeta - z_0}{\zeta + z_0},$$

thus

$$\frac{1}{r} \int_{C_r^*} |G_0(\alpha)| \cdot |d\alpha| \leqslant \frac{1}{R} \int_{\gamma_x^*} \operatorname{Re} \frac{\zeta - z_0}{\zeta + z_0} |G_0(\zeta)| \cdot |d\zeta|.$$

Since

$$\operatorname{Re} \frac{\zeta - z_0}{\zeta + z_0} = \frac{|\zeta|^2 - |z_0|^2}{|\zeta + z_0|^2} \leqslant \frac{R^2 - |z_0|^2}{(R - |z_0|)^2} = \frac{R + |z_0|}{R - |z_0|},$$

so the inequality

$$\frac{1}{r} \int_{C_r^*} |G_0(\alpha)| \cdot |d\alpha| \leqslant \frac{1}{R} \cdot \frac{R + |z_0|}{R - |z_0|} \int_{\gamma_x^*} |G_0(\zeta)| \cdot |d\zeta|$$

holds. By making a substitution $\zeta = z - z_0$, then $d\zeta = dz$. The circle C_r^* passes into a circle C_r, whereas a circle γ_x^* passes into a circle γ_x translated by z_0.

Then

$$\frac{1}{r}\int_{C_r^*}|G_0(\zeta)|\cdot|d\zeta| = \frac{1}{r}\int_{C_r}|G(z)|\,|dz|$$

and

$$\frac{1}{R}\int_{\gamma_x^*}|G_0(\zeta)|\cdot|d\zeta| = \frac{1}{R}\int_{\gamma_x}|G(z)|\,|dz|.$$

From above

$$\frac{1}{r}\int_{C_r}|G(z)|\,|dz| \leqslant \frac{1}{R}\cdot\frac{R+|z_0|}{R-|z_0|}\int_{\gamma_x}|G(z)|\,|dz|.$$

Hence and from inequality $(*)$ as well as from the fact that for $0 < x < \frac{1}{2}$, we have

$$\frac{1}{R}\cdot\frac{R+|z_0|}{R-|z_0|} = \frac{(1+x)^2}{1-x} \leqslant \tfrac{9}{2},$$

we get

$$\int_0^{2\pi}|G(r\cdot e^{i\theta})|\,d\theta = \frac{1}{r}\int_{C_r}|G(z)|\,|dz| \leqslant \tfrac{9}{2}\int_{\gamma_x}|G(z)|\,|dz|$$

$$= \tfrac{9}{2}\int_{-\infty}^{\infty}|F(x+iy)|\,dy \leqslant \tfrac{9}{2}\|F\|_1.$$

Hence we deduce that the function G belongs to $\dot{H}^1$ in D.

Now, let G belong to H^1 in D. Considering the reciprocal map to $z = (w-1)/(w+1)$ of disc D into the half-plane Ω we conclude that $F(w) = \dfrac{2}{(1+w)^2}G\left(\dfrac{w-1}{w+1}\right)$ is an analytic function in Ω and we state that

$$(**)\qquad \int_{\gamma_x}|G(z)|\,|dz| = \int_{\gamma_x}\left|G\left(\frac{w-1}{w+1}\right)\right|\frac{2|dw|}{|1+w|^2}$$

$$= \int_{l_x}|F(w)|\,|dw| = \int_{-\infty}^{\infty}|F(x+iy)|\,dy, \qquad x > 0.$$

Let us denote by γ_x^ϱ the circle with the centre in the point $z_0 = x/(x+1)$ and the radius $0 < \varrho < R = 1/(x+1)$. Applying the Poisson's integral formula for the unit circle C, we obtain

$$G(\alpha) = \frac{1}{2\pi}\int_C \operatorname{Re}\frac{\zeta+\alpha}{\zeta-\alpha}\,G(\zeta)\,|d\zeta|.$$

Hence

$$|G(\alpha)| \leqslant \frac{1}{2\pi}\int_C \operatorname{Re}\frac{\zeta+\alpha}{\zeta-\alpha}|G(\zeta)|\cdot|d\zeta|.$$

Integrating this inequality longwise the circle γ_x^ϱ and changing the order of integration on the right-hand side also taking from the mean-value Theorem for of a harmonic function, we get

$$\begin{aligned}\frac{1}{\varrho}\int_{\gamma_x^\varrho}|G(\alpha)|\cdot|d\alpha| &\leqslant \int_C \frac{1}{2\pi\varrho}\int_{\gamma_x^\varrho}\operatorname{Re}\frac{\zeta+\alpha}{\zeta-\alpha}|d\alpha|\cdot|G(\zeta)|\cdot|d\zeta| \\ &= \int_C \operatorname{Re}\frac{\zeta+z_0}{\zeta-z_0}|G(\zeta)|\cdot|d\zeta| = \int_C \frac{1-|z_0|^2}{|\zeta-z_0|^2}|G(\zeta)|\cdot|d\zeta| \\ &\leqslant \frac{1-|z_0|^2}{(1-|z_0|)^2}\int_C |G(\zeta)|\cdot|d\zeta| = \frac{1+|z_0|}{1-|z_0|}\int_0^{2\pi}|G(e^{i\theta})|\,d\theta \\ &= \frac{1+|z_0|}{1-|z_0|}\|G\|_1.\end{aligned}$$

Taking the supremum with respect to ϱ satisfying the inequality $0 \leqslant \varrho < R$ on the ground of 3.6 of Section I, we get

$$\frac{1}{R}\int_{\gamma_x}|G(\alpha)|\,|d\alpha| \leqslant \frac{1+|z_0|}{1-|z_0|}\|G\|_1.$$

Hence and from inequality (**) and the fact that

$$R\cdot\frac{1+|z_0|}{1-|z_0|} = \frac{1}{x+1}\cdot\frac{1+x/(x+1)}{1-x/(x+1)} = \frac{2x+1}{x+1} = 2-\frac{1}{x+1} \leqslant 2,$$

we obtain

$$\int_{-\infty}^{\infty}|F(x+iy)|\,dy = \int_{\gamma_x}|G(z)|\,|dz| \leqslant R\cdot\frac{1+|z_0|}{1-|z_0|}\|G\|_1 \leqslant 2\|G\|_1, \qquad x > 0.$$

Hence we deduce that the function F belongs to H^1 in Ω. Moreover, in virtue of the fact, that the homographic transformation $z = (w-1)/(w+1)$ of the half-plane Ω into the disc D has on the boundary, the form $e^{i\theta} = (it-1)/(it+1)$, in view of 3.6 of Section I and 4.10 of Section I, we get

$$\|F\|_1 = \int_{-\infty}^{\infty}|F(it)|\,dt = \int_0^{2\pi}\left|F\left(\frac{1+e^{i\theta}}{1-e^{i\theta}}\right)\right|\cdot\frac{2\cdot d\theta}{|1-e^{i\theta}|^2} = \int_0^{2\pi}|G(e^{i\theta})|\,d\theta = \|G\|_1.$$

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