ANNALES SOCIETATIS MATHEMATICAE POLONAE
Series I: COMMENTATIONES MATHEMATICAE XXV (1985)
ROCZNIKI POLSKIEGO TOWARZYSTWA MATEMATYCZNEGO
Seria I: PRACE MATEMATYCZNE XXV (1985)

Z. KAMONT (Gdańsk)

Existence of solutions of first order partial differential-functional equations

Assume that a function f of the variables (x, y, z, u, q), where $y = (y_1, \ldots, y_n)$, $u = (u_1, \ldots, u_m)$, $q = (q_1, \ldots, q_n)$, is defined on $\Gamma = \{(x, y, z, u, q)\colon x \in [0, a),\ y, q \in \boldsymbol{R}^n,\ z \in \boldsymbol{R},\ u \in \boldsymbol{R}^m\}$, $a > 0$. ($\boldsymbol{R}^k$ is the k-dimensional Euclidean space.) Suppose that α is a function of class C^1 on $\Omega_0 = \{(x, y)\colon x \in (p_0, 0],\ y \in \boldsymbol{R}^n\}$, where $p_0 < 0$ (in particular, it may be $p_0 = -\infty$). Let $C_\alpha^1[0, a)$ be a set of all functions which are of class C^1 on $\Omega_0 \cup ([0, a) \times \boldsymbol{R}^n)$ and are identical with α on Ω_0. Let $V_i\colon C_\alpha^1[0, a) \to C_\alpha^1[0, a)$, $i = 1, \ldots, m$. For an element $z \in C_\alpha^1[0, a)$ we define $(Vz)(x, y) = \big((V_1 z)(x, y), \ldots, (V_m z)(x, y)\big)$ and $z_y(x, y) = \big(z_{y_1}(x, y), \ldots, z_{y_n}(x, y)\big)$.

In this note we shall deal with the Cauchy problem for the non-linear partial differential-functional equation of the first order

$$
(1) \qquad \begin{aligned} &z_x(x, y) = f\big(x, y, z(x, y), (Vz)(x, y), z_y(x, y)\big),\\ &z(x, y) = \alpha(x, y) \quad \text{for } (x, y) \in \Omega_0. \end{aligned}
$$

The paper of Myshkis and Šlopak [7] initiated investigations of first order partial differential-functional equations. At the present moment there exists a numerous literature on this subject. Detailed bibliographical information can be found in [2], [4]. The problems of existence of solutions were considered by many authors and under various assumptions. Certain types of differential-integral equations were considered in [11]. In papers [2], [4] the method of successive approximations in considered for partial equations with a retarded argument. A global existence of solutions of certain non-linear class of differential-functional equations was investigated in [9], [10]. Generalized solutions of an initial-boundary value problem for an almost linear equation were investigated in [1].

In this paper we consider the problem of the global existence and estimations of the existence domain for non-linear differential-functional equations. This will be a generalization of the results published in [2], [4], [11]. As a special case we shall obtain theorems on the existence of solutions for equations with a retarded argument and for differential-integral equations. Our results are obtained by using the method of successive approximations.

I. Notations and assumptions. Let $\Delta = \{x, y_1, \ldots, y_n\}$, $\tilde{\Delta} = \{x, y_1, \ldots, y_n, z, u_1, \ldots, u_m, q_1, \ldots, q_n\}$. We introduce

ASSUMPTION H_1. Suppose that:

1° the function f of the variables (x, y, z, u, q) is of class C^2 on Γ and there exists a constant A such that $|f(x, y, z, u, q)| \leqslant A$ on Γ and for $\tau, \eta \in \tilde{\Delta}$ we have

$$(2) \qquad |f_\tau(x, y, z, u, q)| \leqslant A, \quad |f_{\tau\eta}(x, y, z, u, q)| \leqslant A \quad \text{on } \Gamma,$$

(f_τ is the first order partial derivative of f with respect to τ, $\tau \in \tilde{\Delta}$, $f_{\tau\eta}$ is the second derivative);

2° the initial function α of the variables (x, y) is of class C^2 on Ω_0 and for $s, t \in \Delta$ we have

$$(3) \qquad |\alpha_s(x, y)| \leqslant B, \quad |\alpha_{st}(x, y)| \leqslant B \quad \text{on } \Omega_0$$

and $|\alpha(0, y)| \leqslant B$ for $y \in \boldsymbol{R}^n$.

Let $\tilde{\Omega} = [0, a) \times \boldsymbol{R}^n$. For $z \in C^1_\alpha[0, a)$ we denote by $z|_{\tilde{\Omega}}$ the restriction of the function z to the set $\tilde{\Omega}$. Let $\tilde{C}^2_\alpha[0, a)$ be a set of all functions $z \in C^1_\alpha[0, a)$ which are of class C^2 on $\Omega_0 \cup \tilde{\Omega}$ and such that $z|_{\tilde{\Omega}}$ and the derivatives $z_x|_{\tilde{\Omega}}$, $z_{y_i}|_{\tilde{\Omega}}$, $z_{xx}|_{\tilde{\Omega}}$, $z_{xy_i}|_{\tilde{\Omega}}$, $z_{y_i y_j}|_{\tilde{\Omega}}$, $i, j = 1, \ldots, n$, are bounded on $\tilde{\Omega}$.

Suppose that v is a function of the variables (x, y) defined on $\tilde{\Omega}$. If v is of class C^2 on $\tilde{\Omega}$, then we denote

$$(D_0 v)(x, y) = v_x(x, y), \quad (D_i v)(x, y) = v_{y_i}(x, y), \quad i = 1, \ldots, n,$$
$$(D_{ij} v)(x, y) = D_i(D_j v)(x, y), \quad i, j = 0, 1, \ldots, n.$$

If v is bounded on $\tilde{\Omega}$, then for $\xi \in (0, a]$ we define

$$\|v\|_{[0,\xi)} = \sup_{(x,y) \in [0,\xi) \times \boldsymbol{R}^n} |v(x, y)|.$$

ASSUMPTION H_2. Suppose that

1° V_i: $C^1_\alpha[0, a) \to C^1_\alpha[0, a)$, $i = 1, \ldots, m$, and if $z \in \tilde{C}^2_\alpha[0, a)$, then $V_i z \in \tilde{C}^2_\alpha[0, a)$, $i = 1, \ldots, m$;

2° there exist constants $L_i \geqslant 0$, $i = 1, \ldots, m$, such that if $z, \tilde{z} \in C^1_\alpha[0, a)$ and $\|z - \tilde{z}\|_{[0,x)}$ is bounded, then

$$(4) \qquad \|V_i z - V_i \tilde{z}\|_{[0,x)} \leqslant L_i \|z - \tilde{z}\|_{[0,x)}, \quad x \in (0, a], \ i = 1, \ldots, m,$$

3° there exist constants $C_i^{(j)}$, $i = 0, 1, 2$, $j = 1, \ldots, m$, such that

$$(5)\quad \|D_i(V_j z)\|_{[0,x)} \leqslant C_0^{(j)} + C_1^{(j)} \|z\|_{[0,x)} + C_2^{(j)} \sum_{k=0}^{n} \|D_k z\|_{[0,x)}, \quad x \in (0, a],\ i = 0, 1, \ldots, n,\ j = 1, \ldots, m;$$

for each $z \in \tilde{C}_\alpha^2 [0, a)$;

4° there exist constants $E_k^{(i)}$, $k = 0, 1, 2, 3$, $i = 1, \ldots, m$, such that for $z \in \tilde{C}_\alpha^2 [0, a)$, $x \in (0, a]$ we have

$$(6)\quad \|D_{ij}(V_k z)\|_{[0,x)} \leqslant E_0^{(k)} + E_1^{(k)} \|z\|_{[0,x)} + E_2^{(k)} \sum_{l=0}^{n} \|D_l z\|_{[0,x)} + E_3^{(k)} \sum_{l,r=0}^{n} \|D_{lr} z\|_{[0,x)},$$
$$i, j = 0, 1, \ldots, n,\ k = 1, \ldots, m.$$

Remark 1. It follows from (4) that V_i, $i = 1, \ldots, m$, satisfy the following Volterra condition: if $z, \tilde{z} \in C_\alpha^1 [0, a)$ and $z(\xi, \eta) = \tilde{z}(\xi, \eta)$ for $(\xi, \eta) \in [0, x) \times \boldsymbol{R}^n$, then $(V_i z)(x, y) = (V_i \tilde{z})(x, y)$, $i = 1, \ldots, m$.

ASSUMPTION H_3. Suppose that the consistency conditions

$$\alpha_x(0, y) = f(0, y, \alpha(0, y), (V\alpha)(0, y), \alpha_y(0, y)),$$

$$\alpha_{xx}(0, y) = \frac{\partial}{\partial x} f(x, y, z(x, y), (Vz)(x, y), z_y(x, y))\Big|_{\substack{z(x,y) = \alpha(x,y) \\ x = 0}}$$

are satisfied for $y \in \boldsymbol{R}^n$.

We adopt the following notations:

$$L = \sum_{i=1}^{m} L_i, \quad \bar{C} = \sum_{i=1}^{m} C_0^{(i)}, \quad C = 1 + \sum_{i=1}^{m} C_1^{(i)} + (1+n) \sum_{i=1}^{m} C_2^{(i)},$$

$$\bar{E} = \sum_{i=1}^{m} E_0^{(i)}, \quad E = \sum_{i=1}^{m} E_1^{(i)} + (1+n) \sum_{i=1}^{m} E_2^{(i)} + (1+n)^2 \sum_{i=1}^{m} E_3^{(i)},$$

$$\tilde{p} = \max\{[(1+\bar{C})^2 + 1 + \bar{E}]^{1/2},\ 1 + \bar{C} + \tfrac{1}{2} E C^{-1}\},$$

$$c = \varepsilon \big[A(C+n)(\tilde{p} + B(C+n))\big]^{-1}, \quad 0 < \varepsilon < 1.$$

Let functions λ and $\tilde{\mu}$ be defined by

$$\lambda(x) = [B + (1+\bar{C}) C^{-1}] \exp(ACx) - (1+\bar{C}) C^{-1},$$

$$\tilde{\mu}(x) = \frac{B + \tilde{p} A [\tilde{p} + B(C+n)]\, x}{1 - A(C+n)[\tilde{p} + B(C+n)]\, x}$$

and

$$\tilde{r} = \max(\lambda(c), \tilde{\mu}(c)), \qquad N = A[1+\bar{C}+\tilde{r}(C-1)]^2 + A(\bar{E}+\tilde{r}E),$$

$$0 < \tilde{q} < \frac{1}{N}\log\left[1+\frac{\log 3}{2n(1+B(1+n))}\right].$$

Let $\Omega = [0, b) \times R^n$, where $b = \min(a, c, \tilde{q})$.

Let us define a sequence $\{z^{(k)}\}$ by the relations:
$z^{(0)}$ is an arbitrary function such that $u^{(0)}$ defined by

$$u^{(0)}(x, y) = \begin{cases} z^{(0)}(x, y) & \text{for } (x, y) \in \Omega, \\ \alpha(x, y) & \text{for } (x, y) \in \Omega_0, \end{cases} \tag{7}$$

is of class C^2 on $\Omega_0 \cup \Omega$ and

$$\begin{aligned} \|z^{(0)}\|_{[0,x)} &\leqslant \lambda(x), & \|D_i z^{(0)}\|_{[0,x)} &\leqslant \lambda(x), \\ \|D_{ij} z^{(0)}\|_{[0,x)} &\leqslant \tilde{\mu}(x), & i, j &= 0, 1, \ldots, n, \ x \in (0, b]. \end{aligned}$$

If $z^{(k)}$ is a known function, then $z^{(k+1)}$ is a solution of the initial problem

$$z_x(x, y) = F^{(k)}(x, y, z(x, y), z_y(x, y)), \qquad z(0, y) = \omega(y), \qquad y \in R^n, \tag{9}$$

where

$$F^{(k)}(x, y, z, q) = f(x, y, z, (Vu^{(k)})(x, y), q), \qquad \omega(y) = \alpha(0, y) \tag{10}$$

and

$$u^{(k)}(x, y) = \begin{cases} z^{(k)}(x, y) & \text{for } (x, y) \in \Omega, \\ \alpha(x,y) & \text{for } (x, y) \in \Omega_0. \end{cases} \tag{11}$$

II. The existence of the sequence of successive approximations.

Lemma 1. *If Assumptions* H_1–H_3 *are satisfied, then for an arbitrary index* k, $z^{(k)}$ *is defined and is of class* C^2 *on* Ω *and for* $(x, y) \in \Omega$, $k = 0, 1, 2, \ldots$, *we have*

$$\|D_i z^{(k)}\|_{[0,x)} \leqslant \lambda(x), \ i = 0, 1, \ldots, n, \tag{12}$$

$$\|D_{ij} z^{(k)}\|_{[0,x)} \leqslant \tilde{\mu}(x), \qquad i, j = 0, 1, \ldots, n. \tag{13}$$

Proof. It follows from (7), (8) that $z^{(0)}$ satisfies all the conditions of our lemma. Suppose that Lemma 1 is true for a certain fixed k. Let us consider the Cauchy problem (9), where $F^{(k)}$ and ω are defined by (10). Since $\|z^{(k)}\|_{[0,x)} \leqslant \lambda(x)$ for $x \in (0, b]$ then we obtain by (12), (13) and by Assumptions H_1, H_2 that

$$|F^{(k)}_{\tau}(x, y, z, q)| \leqslant N, \qquad |F^{(k)}_{\tau\eta}(x, y, z, q)| \leqslant N \qquad \text{on } \tilde{\Gamma}, \tag{14}$$

where $\tau, \eta \in \{x, y_1, \ldots, y_n, z, q_1, \ldots, q_n\}$ and $\tilde{\Gamma} = [0, b) \times \boldsymbol{R}^{n+1+n}$. From Assumption H_1 and from (14) it follows that there exists a solution $z^{(k+1)}$ of (9). This solution is defined and is of class C^2 on $\tilde{\Theta} = [0, \tilde{b}) \times \boldsymbol{R}^n$, where $\tilde{b} = \min(b, \tilde{q})$ (see [5], Chapter II). Since $\tilde{b} = b$, we have $\tilde{\Theta} = \Omega$.

Now we prove that for $x \in (0, b]$

$$(15) \qquad \|D_i z^{(k+1)}\|_{[0,x)} \leqslant \lambda(x), \qquad i = 0, 1, \ldots, n.$$

Let $T_\eta = \{(x, y)\colon x \in [0, b), |y_i| \leqslant \eta - Ax, i = 1, \ldots, n\}$, where $Ab < \eta$ and for a certain fixed i, $0 \leqslant i \leqslant n$, $\tilde{v}(x, y) = (D_i z^{(k+1)})_{(x,y)}$. In virtue of hypotheses of our lemma we obtain that the differential inequality

$$|\tilde{v}_x(x, y)| \leqslant A[1 + |\tilde{v}(x, y)| + \bar{C} + (C-1)\lambda(x)] + A \sum_{j=1}^{n} |\tilde{v}_{y_j}(x, y)|, \qquad (x, y) \in T_\eta,$$

and the initial inequality $|\tilde{v}(0, y)| \leqslant B$ for $(0, y) \in T_\eta$ are satisfied. This implies, by comparison theorems for partial differential inequalities (see [6], Chapter 9; [8], Chapter 7) that $|\tilde{v}(x, y)| \leqslant \tilde{\lambda}(x)$ for $(x, y) \in T_\eta$, where $\tilde{\lambda}$ is the solution of the initial problem

$$v'(x) = A[1 + v(x) + \bar{C} + (C-1)\lambda(x)], \qquad v(0) = B.$$

Since $\tilde{\lambda}(x) = \lambda(x)$ for $x \in [0, b)$ then we have the estimation

$$(16) \qquad |D_i z^{(k+1)}(x, y)| \leqslant \lambda(x), \qquad i = 0, 1, \ldots, n,$$

for $(x, y) \in T_\eta$. Because for each point $(x, y) \in \Omega$ there exists a $\eta > 0$ such that $(x, y) \in T_\eta$, then we have (16) on Ω. Since λ is monotone, we have, because of relations (16), inequality (15).

Now we prove that

$$(17) \qquad \|D_{ij} z^{(k+1)}\|_{[0,x)} \leqslant \tilde{\mu}(x), \qquad i, j = 0, 1, \ldots, n, \; x \in (0, b].$$

First we shall prove that

$$(18) \qquad |z^{(k+1)}_{y_i y_j}(x, y) \leqslant \tilde{\mu}(x), \qquad i, j = 1, \ldots, n,$$

for $(x, y) \in \bar{T}(b_0, \eta)$, where

$$\bar{T}(b_0, \eta) = \{(x, y)\colon x \in [0, b_0], |y_i| \leqslant \eta - Ax, \quad i = 1, \ldots, n\}$$

and $0 < b_0 < b$, $Ab < \eta$.

Consider the functions

$$(19) \quad v^{(ij)}(x, y, h) = \frac{1}{h}[z^{(k+1)}_{y_j}(x, y + \tau_i h) - z^{(k+1)}_{y_j}(x, y)], \qquad i, j = 1, \ldots, n,$$

where $\tau_i = (0, \ldots, 0, 1, 0, \ldots, 0)$, 1 standing on the i-th place.

We will prove (18) using theorems on differential inequalities. First we prove that

$$
\begin{aligned}
(20)\quad & |v_x^{(ij)}(x, y, h)| \\
& \leqslant A\Big[1+\bar{C}+C\tilde{\mu}(x)+\sum_{l=1}^{n} |v^{(il)}(x, y, h)|\Big] \times \\
& \quad \times\Big[1+\bar{C}+C\tilde{\mu}(x)+\sum_{l=1}^{n} |v^{(il)}(x, y, h)|+\delta(h)\Big]+ \\
& \quad +A\big(\bar{E}+E\tilde{\mu}(x)\big)+A|v^{(ij)}(x, y, h)|+A\sum_{l=1}^{n} |v_{y_l}^{(ij)}(x, y, h)|, \\
& \qquad\qquad (x, y)\in \bar{T}(b_0, \eta),
\end{aligned}
$$

where δ is a continuous and non-negative function and

$$(21)\qquad \lim_{h\to 0} \delta(h) = 0.$$

Substituting $z^{(k+1)}(x, y)$ and $z^{(k+1)}(x, y+\tau_i h)$ into (9) and differentiating the identities thus obtained with respect to y_j we get

$$
\begin{aligned}
(22)\quad |v_x^{(ij)}(x, y, h)| \leqslant{} & \left|\frac{1}{h}\big[F_{y_j}^{(k)}(Q^{(k)}(x, y+\tau_i h))-F_{y_j}^{(k)}(Q^{(k)}(x, y))\big]\right|+ \\
& +\left|\frac{1}{h}\big[F_z^{(k)}(Q^{(k)}(x, y+\tau_i h))-F_z^{(k)}(Q^{(k)}(x, y))\big]\right| |z_{y_j}^{(k+1)}(x, y+\tau_i h)|+ \\
& +|F_z^{(k)}(Q^{(k)}(x, y))|\,|v^{(ij)}(x, y, h)|+ \\
& +\sum_{l=1}^{n}\left|\frac{1}{h}\big[F_{q_l}^{(k)}(Q^{(k)}(x, y+\tau_i h))-F_{q_l}^{(k)}(Q^{(k)}(x, y))\big]\right| |z_{y_i y_l}^{(k+1)}(x, y+\tau_i h)|+ \\
& +\sum_{l=1}^{n} |F_{q_l}^{(k)}(Q^{(k)}(x, y))|\,|v_{y_l}^{(ij)}(x, y, h)|,
\end{aligned}
$$

where $Q^{(k)}(x, y) = \big(x, y, z^{(k+1)}(x, y), z_y^{(k+1)}(x, y)\big)$.

In virtue of Assumptions H_1, H_2 and (12) we have for $\xi\in\tilde{\Delta}$

$$
\begin{aligned}
(23)\quad & \left|\frac{1}{h}\big[f_\xi(P^{(k)}(x, y+\tau_i h))-f_\xi(P^{(k)}(x, y))\big]\right| \\
& \qquad \leqslant A\Big[1+\bar{C}+C\lambda(x)+\sum_{l=1}^{n} |v^{(il)}(x, y, h)|\Big],
\end{aligned}
$$

where $P^{(k)}(x, y) = \big(x, y, z^{(k+1)}(x, y), (Vu^{(k)})(x, y), z_y^{(k+1)}(x, y)\big)$.

These estimates, together with (10) and (12), (13) lead to the inequalities

$$\tag{24} \begin{aligned} &\left|\frac{1}{h}\left[F^{(k)}_{y_j}\left(Q^{(k)}(x, y+\tau_i h)\right)-F^{(k)}_{y_j}\left(Q^{(k)}(x, y)\right)\right]\right| \\ &\quad \leqslant A\left[1+\bar{C}+C\lambda(x)+\sum_{l=1}^{n}|v^{(il)}(x, y, h)|\right]\left[1+\bar{C}+(C-1)\lambda(x)\right]+ \\ &\qquad\qquad +A\left(\bar{E}+E\tilde{\mu}(x)\right), \quad j=1, \ldots, n. \end{aligned}$$

It follows from (19) that there exists a continuous and non-negative function δ_0 such that

$$\tag{25} \lim_{h\to 0} \delta_0(h) = 0$$

and

$$\tag{26} \begin{aligned} |z^{(k+1)}_{y_i y_j}(x, y)| &\leqslant |v^{(ij)}(x, y, h)|+\delta_0(h), \\ |v^{(ij)}(x, y, h)| &\leqslant |z^{(k+1)}_{y_i y_j}(x, y)|+\delta_0(h), \end{aligned} \qquad i, j = 1, \ldots, n,$$

for $(x, y)\in \bar{T}(b_0, \eta)$.

The inequality $\lambda(x) \leqslant \tilde{\mu}(x)$ for $x\in[0, b)$ together with (22)–(24), (26) lead to the differential inequalities (20) with $\delta(h) = n\delta_0(h)$.

Since $|v^{(ij)}(0, y, h)| \leqslant B+\delta_0(h)$ for $(0, y)\in \bar{T}(b_0, \eta)$, then from (20) and by comparison theorems for partial inequalities it follows that for $(x, y)\in \bar{T}(b_0, \eta)$ we have

$$\tag{27} |v^{(ij)}(x, y, h)| \leqslant \bar{u}^{(h)}_{ij}(x), \quad i, j = 1, \ldots, n,$$

where the functions $\bar{u}^{(h)}_{ij}$, $i, j = 1, \ldots, n$, satisfy the system of differential equations

$$\tag{28} \begin{aligned} u'_{ij}(x) = A&\left[1+\bar{C}+C\tilde{\mu}(x)+\sum_{l=1}^{n} u_{il}(x)\right]\left[1+\bar{C}+C\tilde{\mu}(x)+\right. \\ &\left.+\sum_{l=1}^{n} u_{il}(x)+\delta(h)\right]+A\left(\bar{E}+E\tilde{\mu}(x)\right)+Au_{ij}(x), \quad i, j = 1, \ldots, n, \end{aligned}$$

and the initial conditions

$$\tag{29} u_{ij}(0) = B+\delta_0(h), \quad i, j = 1, \ldots, n.$$

Let $\bar{u}_{ij}$, $i, j = 1, \ldots, n$, be a solution of (28), (29) with $\delta(h) = 0$, $\delta_0(h) = 0$. It follows from (21), (25) that

$$\tag{30} \lim_{h\to 0} \bar{u}^{(h)}_{ij}(x) = \bar{u}_{ij}(x), \quad i, j = 1, \ldots, n,$$

uniformly with respect to $x\in[0, b_0]$. In virtue of (19), (27), (30) we get, making h in (27) tend to zero,

$$\tag{31} |z^{(k+1)}_{y_i y_j}(x, y)| \leqslant \bar{u}_{ij}(x), \quad (x, y)\in \bar{T}(b_0, \eta), \quad i, j = 1, \ldots, n.$$

Since

$$\bar{u}'_{ij}(x) \leqslant A\,[\tilde{p}+C\tilde{\mu}(x)+n\bar{u}_{ij}(x)]^2, \qquad i, j = 1, \ldots, n, \ x \in [0, b),$$

we obtain $\bar{u}_{ij}(x) \leqslant \tilde{u}(x)$ for $x \in [0, b)$, $i, j = 1, \ldots, n$, where $\tilde{u}$ is a solution of the initial problem

$$u'(x) = A\,[\tilde{p}+C\tilde{\mu}(x)+nu(x)]^2, \qquad u(0) = B.$$

In view of $\tilde{u}(x) = \tilde{\mu}(x)$ for $x \in [0, b)$ we have (18) for $(x, y) \in \bar{T}(b_0, \eta)$. For each point $(x, y) \in \Omega$ we can choose b_0, η so large that $(x, y) \in \bar{T}(b_0, \eta)$ and $0 < b_0 < b$, $Ab < \eta$, therefore inequalities (18) are satisfied on Ω.

In a similar way we can prove that

$$(32) \qquad |z_{xx}^{(k+1)}(x, y)| \leqslant \tilde{\mu}(x), \qquad |z_{xy_i}^{(k+1)}(x, y)| \leqslant \tilde{\mu}(x), \qquad i = 1, \ldots, n,$$

for $(x, y) \in \Omega$. Since $\tilde{\mu}$ is monotone on $[0, b)$ we get from (18), (32) the estimation (17).

Now we obtain Lemma 1 by induction.

III. The convergence of the sequences $\{z^{(k)}\}$, $\{z_x^{(k)}\}$, $\{z_y^{(k)}\}$. We define

$$M = \sup_{(x,y)\in\Omega} |z_x^{(0)}(x, y) - f(x, y, z^{(0)}(x, y), (Vu^{(0)})(x, y), z_y^{(0)}(x, y))|,$$

$$S = ALe^{Ab}.$$

LEMMA 2. *If Assumptions* H_1–H_3 *are satisfied, then*

$$(33) \qquad \|z^{(k+1)} - z^{(k)}\|_{[0,x)} \leqslant \frac{M}{AL}\,\frac{(Sx)^{k+1}}{(k+1)!}, \qquad (x, y) \in \Omega, \ k = 0, 1, 2, \ldots$$

Proof. In virtue of assumptions of our lemma we have for $k = 0$

$$|z_x^{(1)}(x, y) - z_x^{(0)}(x, y)| \leqslant A\,|z^{(1)}(x, y) - z^{(0)}(x, y)| + M + \sum_{j=1}^{n} |z_{y_j}^{(1)}(x, y) - z_{y_j}^{(0)}(x, y)|, \ (x, y) \in T_\eta,$$

$$z^{(1)}(0, y) - z^{(0)}(0, y) = 0 \quad \text{for } (0, y) \in T_\eta,$$

where the set T_η is defined in II. Hence, by comparison theorems for partial inequalities we get

$$(34) \qquad |z^{(1)}(x, y) - z^{(0)}(x, y)| \leqslant \frac{M}{A}(e^{Ax} - 1) \leqslant \frac{M}{AL}\,\frac{Sx}{1!}$$

for $(x, y) \in T_\eta$. Since η is arbitrary, we have (34) on Ω and, as a consequence, we get (33) for $k = 0$.

Suppose now that for a certain fixed $k \geqslant 1$

$$(35) \qquad \|z^{(k)} - z^{(k-1)}\|_{[0,x)} \leqslant \frac{M}{AL} \frac{(Sx)^k}{k!}, \qquad (x, y) \in \Omega.$$

It follows from Assumptions H_1, H_2 and from (35) that the function $z^{(k+1)} - z^{(k)}$ satisfies the differential inequality

$$|z_x^{(k+1)}(x, y) - z_x^{(k)}(x, y)| \leqslant A|z^{(k+1)}(x, y) - z^{(k)}(x, y)| +$$
$$+ M\frac{(Sx)^k}{k!} + A\sum_{j=1}^{n} |z_{y_j}^{(k+1)}(x, y) - z_{y_j}^{(k)}(x, y)|, \qquad (x, y) \in T_\eta,$$

and the initial condition

$$z^{(k+1)}(0, y) - z^{(k)}(0, y) = 0 \quad \text{for } (0, y) \in T_\eta.$$

These estimates and comparison theorems lead to the inequality

$$|z^{(k+1)}(x, y) - z^{(k)}(x, y)| \leqslant v_k(x), \qquad (x, y) \in T_\eta,$$

where

$$v_k(x) = \frac{MS^k}{A^{k+1}} \left[e^{Ax} - 1 - \frac{Ax}{1!} - \frac{(Ax)^2}{2!} - \ldots - \frac{(Ax)^k}{k!} \right].$$

In view of

$$v_k(x) \leqslant \frac{M}{AL} \cdot \frac{(Sx)^{k+1}}{(k+1)!}, \qquad x \geqslant 0,$$

we get

$$(36) \qquad |z^{(k+1)}(x, y) - z^{(k)}(x, y)| \leqslant \frac{M\,(Sx)^{k+1}}{AL(k+1)!}$$

for $(x, y) \in T_\eta$. Since η is arbitrary, we have (36) on Ω, and as a consequence, we get

$$\|z^{(k+1)} - z^{(k)}\|_{[0,x)} \leqslant \frac{M}{AL} \frac{(Sx)^{k+1}}{(k+1)!}.$$

Now, we obtain Lemma 2 by induction.

LEMMA 3. *If Assumptions* H_1–H_3 *are satisfied, then the sequences* $\{z_x^{(k)}\}$ *and* $\{z_y^{(k)}\}$ *are uniformly convergent on* Ω.

Proof. It follows from Lemma 1 that

$$\|D_i z^{(k)}\|_{[0,b)} \leqslant \tilde{r}, \qquad \|D_{ij} z^{(k)}\|_{[0,b)} \leqslant \tilde{r},$$
$$i, j = 0, 1, \ldots, n, \; k = 0, 1, 2, \ldots$$

These estimates and Lemma 2 imply Lemma 3.

IV. Theorems on the existence of solutions. Lemmas 1–3 imply

THEOREM 1. *If Assumptions* H_1–H_3 *are satisfied, then there exists on* Ω *a solution* $\bar{z}$ *of the Cauchy problem* (1). *The sequence* $\{z^{(k)}\}$ *defined by* (7)–(11) *and the sequences of partial derivatives* $\{z_x^{(k)}\}$, $\{z_y^{(k)}\}$ *are uniformly convergent on* Ω *to the solution* $\bar{z}$ *and its derivatives* $\bar{z}_x$, $\bar{z}_y$ *respectively.*

Remark 2. If Assumptions H_1–H_3 are satisfied, then the Cauchy problem (1) admits at most one solution on $[0, a) \times R^n$. The uniqueness of the solution follows from [3] (see also [12], [13]).

The initial problem mentioned above is such that the initial set Ω_0 is the $n+1$-dimensional zone $(p_0, 0] \times R^n$ and the initial function α is a function of $n+1$ variables. We shall now consider a Cauchy problem for partial differential-functional equations with the initial set of the form $\tilde{\Omega}_0 = \{(x, y)\colon x = 0,\ y \in R^n\}$. The initial function α_0 is a function of n variables.

THEOREM 2. *Suppose that*

1° *the function* f *satisfies condition* 1° *of Assumption* H_1,

2° $V_i\colon C^1[0, a) \to C^1[0, a)$, $i = 1, \ldots, m$, *where* $C^1[0, a)$ *is the set of all function which are of class* C^1 *on* $[0, a) \times R^n$,

3° *if* $z \in \tilde{C}^2[0, a)$, *then* $V_i z \in \tilde{C}^2[0, a)$, $i = 1, \ldots, m$, *where* $\tilde{C}^2[0, a)$ *is the set of all functions* z *which are of class* C^2 *on* $[0, a) \times R^n$ *and* z *and the derivatives* z_x, z_y, z_{xx}, z_{xy_i}, $z_{y_iy_j}$, $i, j = 1, \ldots, n$, *are bounded on* $[0, a) \times R^n$,

4° *conditions* (4), (5), (6) *from Assumption* H_2 *are satisfied for* z, $\tilde{z} \in C^1[0, a)$ *and for* $z \in \tilde{C}^2[0, a)$ *respectively,*

5° *the initial function* α_0 *is of class* C^2 *on* R^n *and there exists a constant* B *such that* $|\alpha(y)| \leqslant B$, $|\alpha_{y_i}(y)| \leqslant B$,

$$|\alpha_{y_iy_j}(y)| \leqslant B \quad \textit{for } y \in R^n \textit{ and } i, j = 1, \ldots, n.$$

Under these assumptions there exists on the set Ω *defined in* I *a solution* $\bar{z}$ *of the initial problem*

$$z_x(x, y) = f\big(x, y, z(x, y), (Vz)(x, y), z_y(x, y)\big), \qquad z(0, y) = \alpha_0(y), \quad y \in R^n. \tag{37}$$

The proof of this theorem is similar to the proof of Theorem 1.

V. Some modifications of the existence theorem. In this section we give a theorem on the existence of solutions of (1) in the case where operators V_i, $i = 1, \ldots, m$, satisfy Assumption H_2 for $C_1^{(j)} = 0$, $E_1^{(j)} = 0$, $j = 1, \ldots, m$. We do not assume in this case that f and $\alpha|_{x=0}$ are bounded.

ASSUMPTION H_1'. Suppose that functions f and α are of class C^2 on Γ and Ω_0, respectively, and satisfy (2), (3).

Let $\tilde{C}^2_\alpha[0, a)$ be the set of all functions $z \in C^1_\alpha[0, a)$ which are of class C^2 on $\Omega_0 \cup \tilde{\Omega}$ and such that the derivatives $z_x|_{\tilde{\Omega}}$, $z_{y_i}|_{\tilde{\Omega}}$, $z_{xx}|_{\tilde{\Omega}}$, $z_{xy_i}|_{\tilde{\Omega}}$, $z_{y_iy_j}|_{\tilde{\Omega}}$,

$i, j = 1, \ldots, n$, are bounded on $\tilde{\Omega}$. Let $\bar{T}(b_0, \eta)$, $0 < b_0 < a$, $Aa < \eta$, be the set defined in II. For a function z continuous on $[0, a) \times \boldsymbol{R}^n$ we define

$$|z|_{b_0,\eta} = \sup_{(x,y)\in\bar{T}(b_0,\eta)} |z(x, y)|.$$

ASSUMPTION H_2'. Suppose that

1° V_i: $C_\alpha^1[0, a) \to C_\alpha^1[0, a)$, $i = 1, \ldots, m$, and if $z \in \bar{C}_\alpha^2[0, a)$, then $V_i z \in \bar{C}_\alpha^2[0, a)$, $i = 1, \ldots, m$,

2° for each $i = 1, \ldots, m$, $0 < b_0 < a$, $\eta > Aa$, there exists a constant $L_i^{(b_0,\eta)}$ such that

$$|V_i z - V_i \bar{z}|_{b_0,\eta} \leqslant L_i^{(b_0,\eta)} |z - \bar{z}|_{b_0,\eta},$$

3° there exist constants $C_0^{(j)}$, $C_2^{(j)}$, $j = 1, \ldots, m$, such that for each $z \in \bar{C}_\alpha^2[0, a)$ we have estimations (5) for $C_1^{(j)} = 0$, $j = 1, \ldots, m$,

4° there exist constants $E_0^{(j)}$, $E_2^{(j)}$, $E_3^{(j)}$, $j = 1, \ldots, m$, such that for $z \in \bar{C}_\alpha^2[0, a)$, $x \in (0, a]$ we have inequalities (6) with $E_1^{(k)} = 0$, $k = 1, \ldots, m$.

THEOREM 3. *Suppose that Assumptions* H_1', H_2', H_3 *are satisfied. Then there exists on* Ω *a solution* $\bar{z}$ *of* (1) (*the set* Ω *is defined in* I *with* $C_1^{(j)} = E_1^{(j)} = 0$, $j = 1, \ldots, m$). *In an arbitrary closed and bounded domain contained in* Ω *the sequence* $\{z^{(k)}\}$ *defined by* (7)–(11) *and the sequences* $\{z_x^{(k)}\}$, $\{z_y^{(k)}\}$ *are uniformly convergent to the solution* $\bar{z}$ *and its derivatives* $\bar{z}_x$, $\bar{z}_y$, *respectively.*

Proof. The existence on Ω of the sequence $\{z^{(k)}\}$ and relations (12), (13) follows by induction.

We define

$$M_{b_0,\eta} = \sup_{(x,y)\in T(b_0,\eta)} \left|z_x^{(0)}(x, y) - f\big(x, y, z^{(0)}(x, y), (Vu^{(0)})(x, y), z_y^{(0)}(x, y)\big)\right|,$$

$$L_{b_0,\eta} = \sum_{i=1}^{n} L_i^{(b_0,\eta)}, \qquad S_{b_0,\eta} = AL_{b_0,\eta}e^{Ab}.$$

It is easy to check that functions $z^{(k)}$, $k = 0, 1, \ldots$, satisfy the conditions

$$|z^{(k+1)} - z^{(k)}|_{b_0,\eta} \leqslant \frac{M_{b_0,\eta}}{AL_{b_0,\eta}} \frac{(S_{b_0,\eta}x)^{k+1}}{(k+1)!}, \qquad (x, y) \in \bar{T}(b_0, \eta),\ k = 0, 1, 2, \ldots \tag{38}$$

Relations (12), (13) and (38) imply the assertions of Theorem 3.

VI. Examples. As a particular case of (1) we obtain the initial problem for partial differential equations with a retarded argument (see [2], [4])

$$\begin{aligned} z_x(x, y) &= f\big(x, y, z(x, y), z(\varphi(x, y), \psi(x, y)), z_y(x, y)\big), \\ z(x, y) &= \alpha(x, y) \quad \text{for } (x, y) \in \Omega_0, \end{aligned} \tag{39}$$

where $z(\varphi(x, y), \psi(x, y)) = \big(z(\varphi^{(1)}(x, y), \psi^{(1)}(x, y)), \ldots, z(\varphi^{(m)}(x, y),$

$\psi^{(m)}(x, y))$ and $\varphi^{(i)}$, $\psi^{(i)} = (\psi^{(i1)}, \ldots, \psi^{(in)})$, $i = 1, \ldots, m$, are given functions.

Remark 3. If

1° $\varphi^{(i)}$ and $\psi^{(i)}$, $i = 1, \ldots, m$, are of class C^2 on $[0, a) \times \boldsymbol{R}^n$ and $p_0 < \varphi^{(i)}(x, y) \leqslant x$, $i = 1, \ldots, m$, for $(x, y) \in [0, a) \times \boldsymbol{R}^n$,

2° there exist constants $C_2^{(i)}$, $E_1^{(i)}$ such that

$$|D_j \varphi^{(i)}(x, y)|,\ |D_j \psi^{(ik)}(x, y)| \leqslant C_2^{(i)}, \qquad (x, y) \in [0, a) \times \boldsymbol{R}^n,$$
$$j = 0, 1, \ldots, n,\ i = 1, \ldots, m,\ k = 1, \ldots, n,$$

and

$$|D_{jl} \varphi^{(i)}(x, y)|,\ |D_{jl} \psi^{(ik)}(x, y)| \leqslant E_1^{(i)}, \qquad (x, y) \in [0, a) \times \boldsymbol{R}^n,$$
$$i = 1, \ldots, m,\ j, l = 0, 1, \ldots, n,\ k = 1, \ldots, n,$$

then Assumption H_2 is satisfied for

$$(V_i z)(x, y) = z\big(\varphi^{(i)}(x, y), \psi^{(i)}(x, y)\big), \qquad i = 1, \ldots, m,$$

with $L_i = 1$, $C_0^{(i)} = 0$, $C_1^{(i)} = 0$, $E_0^{(i)} = 0$, $E_1^{(i)} = 0$, $E_2^{(i)} = (C_2^{(i)})^2$.

The second example concerns partial differential-integral equations. For $\mu = (\mu_0, \mu_1, \ldots, \mu_n)$, where $\mu_i = 0$ or $\mu_i = 1$, we define $I_\mu = \{i\colon \mu_i = 1\}$ and $|I_\mu| = \mu_0 + \mu_1 + \ldots + \mu_n$. Suppose that $\varphi^{(\mu)}, \psi^{(\mu)}\colon [0, a) \times \boldsymbol{R}^n \to \boldsymbol{R}^{|I_\mu|}$ where $\varphi^{(\mu)} = (\varphi_{i_0}^{(\mu)}, \ldots, \varphi_{i_k}^{(\mu)})$, $\psi^{(\mu)} = (\psi_{i_0}^{(\mu)}, \ldots, \psi_{i_k}^{(\mu)})$ and $0 \leqslant i_0 < i_1 < \ldots < i_k \leqslant n$, $i_0, i_1, \ldots, i_k \in I_\mu$. For $(\xi, \eta) \in [0, a) \times \boldsymbol{R}^n$ we define $\mu \cdot (\xi, \eta) = (\mu_0 \xi, \mu_1 \eta_1, \ldots, \mu_n \eta_n)$. Let $1 - \mu = (1 - \mu_0,\ 1 - \mu_1, \ldots, 1 - \mu_n)$ and $(1 - \mu)(\xi, \eta) = \big((1 - \mu_0)\xi, (1 - \mu_1)\eta_1, \ldots, (1 - \mu_n)\eta_n\big)$. Suppose that

$$\mu d\xi\, d\eta = \begin{cases} d\xi\, d\eta_{i_1} \ldots d\eta_{i_k} & \text{if } 0 \in I_\mu,\ i_1, \ldots, i_k \in I_\mu, \\ d\eta_{i_0} d\eta_{i_1} \ldots d\eta_{i_k} & \text{if } 0 \notin I_\mu,\ i_0, i_1, \ldots, i_k \in I_\mu. \end{cases}$$

We define an operator V_μ in the following way

$$(V_\mu z)(x, y) = \int_{\varphi^{(\mu)}(x,y)}^{\psi^{(\mu)}(x,y)} z\big(\mu(\xi, \eta) + (1 - \mu)(x, y)\big)\, \mu d\xi\, d\eta.$$

$\int \mu d\xi d\eta$ is the $|I_\mu|$-dimensional Riemann integral with respect to the variables $\xi, \eta_{i_1}, \ldots, \eta_{i_k}$ if $0 \in I_\mu$, $i_1, \ldots, i_k \in I_\mu$ and it is the integral with respect to $\eta_{i_0}, \ldots, \eta_{i_k}$ if $0 \notin I_\mu$, $i_0, i_1, \ldots, i_k \in I_\mu$.

Consider the Cauchy problem for a differential-integral equation

$$z_x(x, y) = f\big(x, y, z(x, y), (Vz)(x, y), z_y(x, y)\big),$$
$$z(x, y) = \alpha(x, y) \quad \text{for } (x, y) \in \Omega_0,$$

where $Vz = (V_{(1,\dots,1)}z,\ V_{(0,1,\dots,1)}z,\ V_{(1,0,1,\dots,1)}z,\ \dots,\ V_{(1,\dots,1,0)}z,\ V_{(0,0,1,\dots,1)}z,\ \dots$ $\dots,\ V_{(1,\dots,1,0,0)}z,\ \dots,\ V_{(1,0,\dots,0)}z)$.

We shall use the following notations. If $\alpha^{(\mu)} = (\alpha_{i_0}^{(\mu)}, \dots, \alpha_{i_k}^{(\mu)})$ and i_0, $i_1, \dots, i_k \in I_\mu$, then for $i_s \in I_\mu$ we define

$$\prod_{j\in I_\mu}^{(i_s)} \alpha_j^{(\mu)} = \alpha_{i_0}^{(\mu)} \cdot \ldots \cdot \alpha_{i_{s-1}}^{(\mu)} \cdot 1 \cdot \alpha_{i_{s+1}}^{(\mu)} \cdot \ldots \cdot \alpha_{i_k}^{(\mu)}.$$

If $|I_\mu| > 1$ and $i_s, i_t \in I_\mu$, then

$$\prod_{j\in I_\mu}^{(i_s,i_t)} \alpha_j^{(\mu)} = \alpha_{i_0}^{(\mu)} \cdot \ldots \cdot \alpha_{i_{s-1}}^{(\mu)} \cdot 1 \cdot \alpha_{i_{s+1}}^{(\mu)} \cdot \ldots \cdot \alpha_{i_{t-1}}^{(\mu)} \cdot 1 \cdot \alpha_{i_{t+1}}^{(\mu)} \cdot \ldots \cdot \alpha_{i_k}^{(\mu)}.$$

We introduce

ASSUMPTION H_4. Suppose that

1° the functions $\varphi^{(\mu)}$, $\psi^{(\mu)}$: $[0, a) \times \boldsymbol{R}^n \to \boldsymbol{R}^{|I_\mu|}$ are of class C^2 and $p_0 < \varphi_{i_0}^{(\mu)}(x, y) \leqslant x$, $p_0 < \psi_{i_0}^{(\mu)}(x, y) \leqslant x$, $(x, y) \in [0, a) \times \boldsymbol{R}^n$,

2° there exist constants $d_0^{(\mu)}$, $d_1^{(\mu)}$, $d_2^{(\mu)}$ such that for $(x, y) \in [0, a) \times \boldsymbol{R}^n$ we have

$$\prod_{j\in I_\mu} |\psi_j^{(\mu)}(x, y) - \varphi_j^{(\mu)}(x, y)| \leqslant d_0^{(\mu)},$$

$$\sum_{k\in I_\mu} \prod_{j\in I_\mu}^{(k)} |\psi_j^{(\mu)}(x, y) - \varphi_j^{(\mu)}(x, y)| \leqslant d_1^{(\mu)}$$

and for $|I_\mu| > 1$

$$\sum_{k,l\in I_\mu} \prod_{j\in I_\mu}^{(k,l)} |\psi_j^{(\mu)}(x, y) - \varphi_j^{(\mu)}(x, y)| \leqslant d_2^{(\mu)},$$

3° there exist constants $c_1^{(\mu)}$, $c_2^{(\mu)}$ such that for $(x, y) \in [0, a) \times \boldsymbol{R}^n$ we have

$$|D_k \varphi_j^{(\mu)}(x, y)|,\ |D_k \psi_j^{(\mu)}(x, y)| \leqslant d_1^{(\mu)}, \quad j \in I_\mu,\ k = 0, 1, \dots, n,$$

and

$$|D_{kl} \varphi_j^{(\mu)}(x, y)|,\ |D_{kl} \psi_j^{(\mu)}(x, y)| \leqslant d_2^{(\mu)}, \quad j \in I_\mu,\ k, l = 0, 1, \dots, n.$$

Now we have the following

LEMMA 4. *If Assumption* H_4 *is satisfied, then the integral operators* V_μ *satisfy the following conditions*:

1° V_μ: $C_\alpha^1[0,a) \to C_\alpha^1[0, a)$ *and if* $z \in \tilde{C}_\alpha^2[0, a)$, *then* $V_\mu z \in \tilde{C}^2[0, a)$,

2° *if* z, $\tilde{z} \in C_\alpha^1[0, a)$ *and* $\|z - \tilde{z}\|_{[0,x)}$ *is bounded for* $x \in (0, a]$, *then*

$$\|V_\mu z - V_\mu \tilde{z}\|_{[0,x)} \leqslant d_0^{(\mu)} \|z - \tilde{z}\|_{[0,x)},$$

3° *if* $z \in C^1_\alpha[0, a)$ *and* $\|z\|_{[0,x)}$, $\|D_i z\|_{[0,x)}$, $i = 0, 1, \ldots, n$, *are bounded for* $x \in (0, a]$, *then*

$$\|D_j V_\mu z\|_{[0,x)} \leqslant d_0^{(\mu)} \sum_{i=0}^{n} \|D_i z\|_{[0,x)} + 2d_1^{(\mu)} c_1^{(\mu)} \|z\|_{[0,x)},$$

4° *if* $z \in \tilde{C}^2_\alpha[0, a)$, $x \in (0, a]$, *then for* $i, k = 0, 1, \ldots, n$ *we have*

$$\|D_{ij} V_\mu z\|_{[0,x)} \leqslant E_1^{(\mu)} \|z\|_{[0,x)} + E_2^{(\mu)} \sum_{l=0}^{n} \|D_l z\|_{[0,x)} + E_3^{(\mu)} \sum_{l,r=0}^{n} \|D_{lr} z\|_{[0,x)},$$

where

$$E_1^{(\mu)} = 2d_1^{(\mu)} c_2^{(\mu)} + 4d_2^{(\mu)} (c_1^{(\mu)})^2, \qquad E_2^{(\mu)} = 2d_1^{(\mu)} c_1^{(\mu)} + 2d_1^{(\mu)} e^{(\mu)} c_1^{(\mu)},$$
$$E_3^{(\mu)} = d_0^{(\mu)}, \qquad e^{(\mu)} = \max(1, c_1^{(\mu)}).$$

We omit the simple proof of this lemma.

Remark 4. The results obtained in this paper can be extended to hyperbolic systems of the form

$$\begin{aligned} &z_x^{(i)}(x, y) = f^{(i)}(x, y, z(x, y), (Vz)(x, y), z_y^{(i)}(x, y)), \\ &z^{(i)}(x, y) = \alpha^{(i)}(x, y) \quad \text{for } (x, y) \in (p_0, 0] \times \mathbf{R}^n, \end{aligned} \qquad i = 1, \ldots, m,$$

where $z = (z_1, \ldots, z_m)$, $Vz = (V_1 z, \ldots, V_k z)$.

References

[1] W. E. Abolina, A. D. Myshkis, *A mixed problem for almost linear hyperbolic systems on a plane* (in Russian), Math. Sb. 50 (92) (1960), 423–442.

[2] Z. Kamont, *On the estimation of the existence domain for solutions of a non-linear partial differential-functional equations of the first order*, Glasnik Mat. 13 (1978), 277–291.

[3] —, *On the Cauchy problem for system of first order partial differential-functional equations*, Serdica Bulg. Math. Publ. 5 (1979), 327–339.

[4] —, *On the Cauchy problem for non-linear partial differential-functional equations of the first order*, Math. Nachr. 88 (1979), 13–29.

[5] E. Kamke, *Differentialgleichungen*, vol. 2, Leipzig 1965.

[6] A. Lakshmikantham, S. Leela, *Differential and Integral Inequalities*, New York, London 1969.

[7] A. D. Myshkis, A. S. Šlopak, *A mixed problem for systems of differential-functional equations with partial derivatives and with operators of type Volterra* (in Russian), Mat. Sb. 41 (83) (1957), 239–256.

[8] J. Szarski, *Differential Inequalities*, Warszawa 1967.

[9] —, *Generalized Cauchy problem for differential-functional equations with first order partial derivatives*, Bull. Acad. Polon. Sci., Sér. Sci. Math. Astr. Phys. 24 (1976), 575–580.

[10] —, *Cauchy problem for an infinite system of differential-functional equations with first order partial derivatives*, Comment. Math. tom. spec. I (1978), 293–300.

[11] H. Ugowski, *On the generalized Cauchy problem for integro-differential equations*, Zesz. Nauk. Politechniki Gdańskiej, Mat. 7 (1973), 3–14.

[12] K. Zima, *Sur un système d'équations différentielles avec dérivée à gauche*, Ann. Polon. Math. 22 (1969), 37–47.

[13] —, *Sur les équations aux dérivées partielles du premier ordre à argument fonctionnel*, ibidem 22 (1969), 49–59.

INSTYTUT MATEMATYKI UNIWERSYTETU GDAŃSKIEGO