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On interpolation inequalities with mixed norm

1. Introduction. The aim of this paper is to prove a generalization of the interpolation inequality and the Poincaré inequality in the case of the spaces $H^{m,p}(\Omega)$ with mixed norm. The results contain some of the results from [1], [5], [7]. Other results on this problem can be found in [4], for bibliography and results see also [3], p. 236.

2. The index i runs through $1, \ldots, n$, unless otherwise stated. Let R be the set of real numbers and $k_i > 0$ an integer, $1 \leqslant p_i < \infty$, $\lambda_i \geqslant 0$. In the following we shall use vector notations, i.e., $x = (x_1, \ldots, x_n)$, $p = (p_1, \ldots, p_n)$, etc. Let Ω_i be an open, connected, bounded subset of the real Euclidean space R^{k_i}. Let $\Omega = \mathop{\mathrm{P}}\limits_{i=1}^{n} \Omega_i$, $\bar{\Omega} = \mathop{\mathrm{P}}\limits_{i=1}^{n} \bar{\Omega}_i$, $R^N = \mathop{\mathrm{P}}\limits_{i=1}^{n} R^{k_i}$. The measure means always Lebesgue measure. To simplify the notation we shall write, for example,

$$\int_{\Omega} |f(x)|\,dx = \int_{\Omega_n} \ldots \int_{\Omega_1} |f(x)|\,dx_1 \ldots dx_n,$$

$$\int_{b\Omega} |f(x)|^p\,dx = \|f\|^{p_n}_{L^p(\Omega)} = \int_{\Omega_n} \Big[\ldots\Big(\int_{\Omega_1} |f(x)|^{p_1}\,dx_1\Big)^{p_2/p_1} dx_2 \ldots\Big]^{p_n/p_{n-1}} dx_n,$$

$$\int_{b\Omega} \sum_{i=1}^{n} |f_i(x)|^p\,dx = \int_{\Omega_n} \Big[\ldots \int_{\Omega_2} \Big(\int_{\Omega_1} \sum_{i=1}^{n} |f_i(x)|^{p_1}\,dx_1\Big)^{p_2/p_1} dx_2 \ldots\Big]^{p_n/p_{n-1}} dx_n.$$

Let $|l| = \sum_{i=1}^{N} l_i$, $\sum_{i=1}^{n} k_i = N$, $x^l = x_1^{1\,l_1^1} \ldots x_{k_n}^{n\,l_{k_n}^n}$. Let $D^l f(x)$ denote the strong derivative of the function $f(x)$ and let

$$D^l f(x) = \frac{\partial^{|l|} f(x)}{\partial x_1^{1\,l_1^1} \ldots \partial x_{k_n}^{n\,l_{k_n}^n}} \quad \text{if} \quad |l| > 0, \qquad D^l f(x) = f(x) \quad \text{if} \quad l = 0.$$

DEFINITION. The set $\Omega_i \subset R^{k_i}$ is said to have the *cone property* if there exists a finite cone C^i (the intersection of an open ball in R^{k_i} centered at the origin with a set of the form

$$\{\lambda x_i \colon \lambda > 0,\ x_i \in R^{k_i},\ |x_i - y_i| < r_i\},$$

where $r_i > 0$ and y_i is a fixed point in R^{k_i}, $|y_i| > r_i$) such that every point $x_i \in \bar{\Omega}_i$ is the vertex of a finite cone $C^i_{x_i}$ congruent to C^i and contained in Ω_i.

DEFINITION. We say that Ω_i has the *restricted cone property* if $\partial\Omega_i$ has a locally finite open covering $\{O^i_j\}$ and corresponding cones $\{C^i_j\}$ with vertices at the origin and the property that $x_i + C^i_j \subset \Omega_i$ for $x_i \in \Omega_i \cap O^i_j$.

For example of the sets satisfying above definitions see, among others [3], p. 118, [6], p. 300.

Let $E_{\bar{\Omega}}(R^N)$ denote the set containing all restrictions to $\bar{\Omega}$ of the functions in $C^\infty(R^N)$. We apply in this paper the definition of the space $L^p(\Omega)$ with mixed norm from [2].

DEFINITION. We say that $f \in H^{m,p}(\Omega)$ if and only if $f \in L^p(\Omega)$ and there exists a sequence $\{f_n\}_{n=1}^\infty \subset E_{\bar{\Omega}}(R^N)$, such that for arbitrary $\alpha = (\alpha_1, \ldots, \alpha_N)$, $\sum_{i=1}^N \alpha_i \leqslant m$, $(D^\alpha f_n)_{n=1}^\infty$ is a Cauchy sequence in $L^p(\Omega)$, $\lim_{n\to\infty} \|f_n - f\|_{L^p(\Omega)} = 0$.

The space $H^{m,p}(\Omega)$ is identified with the class of functions $f \in L^p(\Omega)$, which have the strong derivative of order up to m.

The expression

$$\|f\|_{H^{m,p}(\Omega)} = \Big\{ \sum_{|\alpha| \leqslant m} \|D^\alpha f\|^{p_n}_{L^p(\Omega)} \Big\}^{1/p_n} \tag{1}$$

is a norm in $H^{m,p}(\Omega)$.

By $H^{m,p}_0(\Omega)$ we shall denote the subset of $H^{m,p}(\Omega)$ consisting of functions having compact support contained in Ω.

We shall apply the next four inequalities given for scalar p. They are simple generalizations of Theorem 3.1 and its conclusions [1], p. 17–19, for $p_i = 2$ for their simplicity we omit the proofs.

If $f \in H^{2,p}(a, b)$, then

$$\begin{aligned}(2)\quad & \int_a^b |f'(x)|^p dx \\ & \leqslant 2^{2p+1}(p+1)^{p+1} p^{-p}(b-a)^{-p} \int_a^b |f(x)|^p dx + 2^p (b-a)^p \int_a^b |f''(x)|^p dx.\end{aligned}$$

If $f \in H^{2,p}(a, b)$ and if $0 < \varepsilon \leqslant 1$, then

$$\int_a^b |f'(x)|^p dx \leqslant C_1(a, b, p)\Big\{\varepsilon^{-1} \int_a^b |f(x)|^p dx + \varepsilon \int_a^b |f''(x)|^p dx\Big\}. \tag{3}$$

$$|f'(x)| \leqslant 8(b-a)^{-1} \int_a^b |f(x)|\, dx + (b-a) \int_a^b |f''(x)|\, dx \tag{4}$$

for every $f \in H^{2,1}(a, b)$,

(5) $$|f'(x)| \leqslant C_2(a, b)\big[\varepsilon^{-1}\int_a^b |f(x)|\,dx+\varepsilon\int_a^b |f''(x)|\,dx$$

for every $f\in H^{2,1}(a, b)$, $0<\varepsilon\leqslant 1$.

The notations and some parts of the proofs are analogous to those of Theorem 3.2 and Lemma 7.3 in [1], for the sake of completeness we outline these results here.

3. THEOREM 1. *Let Ω_i be a bounded, open set with the restricted cone property; then there exists a constant $c_3(p, \Omega)$, such that for $f\in H^{2,p}(\Omega)$*

$$\sum_{|\alpha|=1} \|D^\alpha f\|^{p_n}_{L^{P}(\Omega)} \leqslant C_3\{\varepsilon \sum_{|\alpha|=2} \|D^\alpha f\|^{p_n}_{L^{P}(\Omega)}+\varepsilon^{-1}\|f\|^{p_n}_{L^{P}(\Omega)}\}$$

for every $0<\varepsilon\leqslant 1$.

Proof. Let $\{O_j'^i\}$ and $\{C_j'^i\}$ be the covering of $\partial\Omega_i$ and the set of corresponding cones, respectively, as guaranteed by the restricted cone property. Since Ω_i is bounded, $\{O_j'^i\}$ is finite. Let h_j^i denote the height of $C_j'^i$, and let $h_i=\min h_j^i$. Let $\{O_j''^i\}$ be a finite open covering of $\partial\Omega_i$ with spheres $\{O_j''^i\}$ whose diameters are less than $h_i/2$. Let $\{O_j^i\}$ be the collection of all sets of the form $O_j^i=O_l'^i\cap O_s''^i$; to any set O_j^i assign the cone $C_j^i=O_j'^i$. Then $\{O_j^i\}$, $\{C_j^i\}$ is a covering of $\partial\Omega_i$, together with the set of corresponding cones as in the definition of the restricted cone property, and $\{O_j^i\}$ has the additional property that the diameter of O_j^i, $d(O_j^i)<h_j^i$. Let $\{O_j^i\}$ be a finite collection of cubes such that $\Omega_i-\bigcup_j O_j^i\subset\bigcup_j Q_j^i$, $\overline{\bigcup_j Q_j^i}\subset\Omega_i$, $\bigcup_j Q_j^i$ is open, $\overline{\bigcup_j Q_j^i}$ is compact, and edges of each cube Q_j^i are parallel to the coordinate axes in the space R^{k_i}. Let us notice that $\Omega_i-\bigcup_j Q_j^i\subset\bigcup_j Q_j^i$ and $\Omega_i\cap(\bigcup_j O_j^i)=\bigcup_j(\Omega_i\cap O_j^i)$. Let ξ_i be a unit vector which is a positive multiple of some vector in the cone C_j^i.

We shall give an inequality for the directional derivative in the direction of ξ_i; we must be sure that the domain of integration is connected along each parallel to ξ_i. Let us introduce the set

$$\Omega_j^{\xi_i}=\{x_i\colon\ x_i=y_i+t_i\xi_i,\ y_i\in\Omega_i\cap O_j^i,\ 0\leqslant t_i\leqslant h_j^i\}.$$

Then $(\Omega_i\cap O_j^i)\subset\Omega_j^{\xi_i}\subset\Omega_i$, by the cone property. It is easy to see that the intersection with $\Omega_j^{\xi_i}$ of any line parallel to ξ_i is either void or is a line segment of length d_i, $h_j^i\leqslant d_i\leqslant 2h_j^i$. Let $D\xi_i$ be the operation of differentiation in the direction of ξ_i. If v is a function in Ω, let us define $(v)_j^i$ by $(v)_j^i=v$ in $\Omega_1\times\Omega_2\times\ldots\times\Omega_i\times\Omega_j^{\xi_i}\times\Omega_{i+1}\times\ldots\times\Omega_n=A_i$, $(V)_j^i=0$ in $P_{i=1}^n R^{k_i}-A_i$. Let L_{y_i} be a line in the direction ξ_i passing through a point $y_i\in R^{k_i}$.

Consider a cube $Q_j^i=\{a_s^i<x_s^i<b_s^i\}$, $s=1,\ldots,k_i$, $Q_j^i\subset R^{k_i}$. For every

$s = 1, \ldots, k_i$ we have by (3)

$$(6) \qquad \int_{a_s^i}^{b_s^i} |D^{1}_{x_s^i} f|^{p_i} dx_s^i \leqslant C_4 \big[\varepsilon \int_{a_s^i}^{b_s^i} |D^{2}_{x_s^i} f|^{p_i} dx_s^i + \varepsilon^{-1} \int_{a_s^i}^{b_s^i} |f|^{p_i} dx_s^i\big],$$

$i = 1, \ldots, n$. If we write the above inequality for $i = 1$ and integrating on the rest edges of the cube Q_j^1 we get

$$\int_{Q_j^1} |D^1_{x_s^1} f|^{p_1} dx_s^1 \leqslant C_4 \big[\varepsilon \int_{Q_j^1} |D^2_{x_s^1} f|^{p_1} dx_s^1 + \varepsilon^{-1} \int_{Q_j^1} |f|^{p_1} dx_s^1\big].$$

Then, sucessively, we sum up on $s = 1, \ldots, k_1$, sum up on j and, taking into account that $\overline{\bigcup_j Q_j^2} \subset \Omega_i$,

$$\int_{\bigcup_j Q_j^i} \sum_{s=1}^{k_1} |D^1_{x_s^1} f|^{p_1} dx_s^1 \leqslant C_4 \big[\xi \int_{\Omega_1} |D^2_{x_s^1} f|^{p_1} dx_s^1 + \varepsilon^{-1} \int_{\Omega_1} |f|^{p_1} dx_s^1\big].$$

Rising up both sides of this inequality to the power p_i/p_{i-1}, integrating with respect to x_i on Ω_i for $i = 2, \ldots, n$ and next applying for its right-hand side the Minkowski inequality for mixed norm we get

$$(7) \qquad \int_{\Omega_n} \big[\ldots \int_{\Omega_2} \big(\int_{\bigcup_j Q_j^1} \sum_{s=1}^{k_1} |D^1_{x_s^1} f|^{p_1} dx_1\big)^{p_2/p_1} dx_2 \ldots\big]^{p_n/p_{n-1}} dx_n \leqslant C_5 \big\{\varepsilon \int_{b\Omega} \sum_{s=1}^{k_1} |D^2_{x_s^1} f|^p dx + \varepsilon^{-1} \int_{b\Omega} |f|^p dx\big\}.$$

From observation concerning $\Omega_j^{\xi_i}$, $(V)_j^i$ and from (6) we have for $i = 1$

$$\int_{L_{y_1}} |(D_{\xi_1} f)_j^1|^{p_1} ds_1 \leqslant C_6 \big\{\varepsilon \int_{L_{y_1}} |(D^2_{\xi_1} f)_j^1| ds_1 + \varepsilon^{-1} \int_{L_{y_1}} |(f)_j^1|^{p_1} ds_1\big\}.$$

Letting y_1 in $k_1 - 1$ dimensional space R^{k_1-1} orthogonal to ξ_1, integrating both sides of the last inequality with respect to y_1, taking into account that $\Omega_1 \cap O_j^1 \subset \Omega_j^{\xi_1} \subset \Omega_1$ and summing on j, we get

$$\int_{\bigcup_j (\Omega_1 \cap O_j^1)} |(D_{\xi_1} f)_j^1|^{p_1} dx_1 \leqslant C_6 \big\{\varepsilon \int_{\Omega_1} |(D^2_{\xi_1} f)_j^1|^{p_1} dx_1 + \varepsilon^{-1} \int_{\Omega_1} |(f)|_j^1|^{p_1} dx_1\big\}.$$

Repeating the procedure with the above inequality as we did before inequality (7), we get

$$\int_{\Omega_n} \big[\ldots \int_{\Omega_2} \big(\int_{\Omega_1} |D_{\xi_1} f|^{p_1} dx_1\big)^{p_2/p_1} dx_2 \ldots\big]^{p_n/p_{n-1}} dx_n \leqslant C_7 \big\{\varepsilon \int_{b\Omega} |D^2_{\xi_1} f|^p dx + \varepsilon^{-1} \int_{b\Omega} |f|^p dx\big\}.$$

Let $\{\xi_1^i, \ldots, \xi_k^i\}$ be linearly independent set of vectors, each of which is a positive multiple of some vector in the cone C_j^i. Then the inequality above holds for ξ_s^i, $s = 1, \ldots, k_i$. Since any differentiation operator can be written as a linear combination of $D_{\xi_1^i}, \ldots, D^i_{\xi_{k_i}}$ it follows on adding that for $i = 1$

$$(8) \qquad \int_{\Omega_n} [\ldots \int_{\Omega_2} (\int_{\Omega_1} \sum_{s=1}^{k_1} |D^1_{\xi^1_s} f|^{p_1} dx_1)^{p_2/p_1} dx_2 \ldots]^{p_n/p_{n-1}} dx_n$$
$$\leqslant C_7 \{\varepsilon \int_{b\Omega} \sum_{s=1}^{k_1} |D^2_{\xi^1_s} f|^p dx + \varepsilon^{-1} \int_{b\Omega} |f|^p dx\}.$$

From (7) and (8) it follows

$$(9) \qquad \int_{b\Omega} \sum_{s=1}^{k_1} |D^1_{\xi^1_s} f|^p dx \leqslant C_8 \{\varepsilon \int_{b\Omega} \sum_{s=1}^{k_1} |D^2_{\xi^1_s} f|^p dx + \varepsilon^{-1} \int_{b\Omega} |f|^p dx\}.$$

Then we want to get the above inequality but for derivatives of variables from R^{k_2}. Let us write inequality (5) for the variable x_s^2, where $s = 1, \ldots, k_2$, $Q_j^2 = \{a_s^2 < x_s^2 < b_s^2\}$,

$$(10) \qquad |D^1_{x^2_s} f| \leqslant C_9 \{\varepsilon \int_{a_s^2}^{b_s^2} |D^2_{x^2_s} f| dx_s^2 + \varepsilon^{-1} \int_{a_s^2}^{b_s^2} |f| dx_s^2\}.$$

Rising up both sides of this inequality to the power p_1, applying the fact $(a+b)^{p_1} \leqslant 2^{p_1}(a^{p_1} + b^{p_1})$, integrating with respect to x_1 on Ω_1, taking power $1/p_1$ and applying, for the both parts of the right-hand side, the generalized Minkowski inequality, see (10), [3], p. 22, we get

$$(\int_{\Omega_1} |D^1_{x^2_s} f|^{p_1} dx_1)^{1/p_1}$$
$$\leqslant 2C_9 \{\varepsilon \int_{a_s^2}^{b_s^2} (\int_{\Omega_1} |D^2_{x^2_s} f|^{p_1} dx_1)^{1/p_1} dx_s^2 + \varepsilon \int_{a_s^2}^{b_s^2} (\int_{\Omega_1} |f|^{p_1} dx_1)^{1/p_1} dx_s^2\}.$$

Rising up both sides of this inequality to the power p_2, applying the Hölder inequality to every component of the right-hand side, integrating with respect to x_s^2 on (a_s^2, b_s^2), we get

$$\int_{a_s^2}^{b_s^2} (\int_{\Omega_1} |D^1_{x^2_s} f|^{p_1} dx_1)^{p_2/p_1} dx_s^2$$
$$\leqslant 2^{p_2} C_9 (b_s^2 - a_s^2)^{p_2} \{\varepsilon^{p_2} \int_{a_s^2}^{b_s^2} (\int_{\Omega_1} |D^2_{x^2_s} f|^{p_1} dx_1)^{p_2/p_1} dx_s^2 +$$
$$+ \varepsilon^{-p_2} \int_{a_s^2}^{b_s^2} (\int_{\Omega_1} |f|^{p_1} dx_1)^{p_2/p_1} dx_s^2\}.$$

Now we integrate both sides of this inequality on the rest of edges of the cube Q_j^2, $k_2 - 1$ times, next we sum on $s = 1, \ldots, k_2$, apply simple inequality for sums and then we sum over j. Hence

$$\int_{\bigcup_j Q_j^2} \Big(\int_{\Omega_1} \sum_{s=1}^{k_2} |D^1_{x^2_s} f|^{p_1} dx_1\Big)^{p_2/p_1} dx_2$$
$$\leqslant C_{10} \Big\{\varepsilon^{p_2} \int_{\bigcup_j Q_j^2} \Big(\int_{\Omega_1} \sum_{s=1}^{k_2} |D^2_{x^2_s} f|^{p_1} dx_1\Big)^{p_2/p_1} dx_2 + $$
$$+\varepsilon^{-p_2} \int_{\bigcup_j Q_j^2} \Big(\int_{\Omega_1} |f(x)|^{p_1} dx_1\Big)^{p_2/p_1} dx_2\Big\}.$$

Rising up both sides of the above inequality to the power p_i/p_{i-1}, integrating with respect to x_i suitably on Ω_i for $i = 3, \ldots, n$, with $\bigcup_j Q_j^2 \subset \Omega_2$, we have

$$(11) \quad \int_{\Omega_n} \Big\{\ldots \int_{\Omega_3} \Big[\int_{\bigcup_j Q_j^2} \Big(\int_{\Omega_1} \sum_{s=1}^{k_2} |D^1_{x^2_s} f|^{p_1} dx_1\Big)^{p_2/p_1} dx_2\Big] \ldots\Big\}^{p_n/p_{n-1}} dx_n$$
$$\leqslant C_{11} \Big\{\varepsilon^{p_n} \int_{b\Omega} \sum_{s=1}^{k_1} |D^2_{x^2_s} f|^p dx + \varepsilon^{-p_n} \int_{b\Omega} |f|^p dx\Big\}.$$

We know, by introducing notations, that (10) holds in the direction $\xi_2 \in R^{k_2}$

$$|(D^1_{\xi_2} f)^2_j| \leqslant C_{12} \Big\{\varepsilon \int_{L_{y_2}} |(D^2_{\xi_2} f)^2_j| \, ds_2 + \varepsilon^{-1} \int_{L_{y_2}} |(f)^2_j| \, ds_2\Big\}.$$

Repeating to this inequality the procedure mentioned immediately after (10) we get

$$\Big[\int_{\Omega_1} |(D^1_{\xi_2} f)^2_j|^{p_1} dx_1\Big]^{1/p_1}$$
$$\leqslant 2C_{12} \Big\{\xi \int_{L_{y_2}} \Big[\int_{\Omega_1} |(D^2_{\xi_2} f)^2_j|^{p_1} dx_1\Big]^{1/p_1} d\xi_2 + \varepsilon^{-1} \int_{L_{y_2}} \Big[\int_{\Omega_1} |(f)^2_j|^{p_1} dx_1\Big]^{1/p_1} d\xi_2\Big\}.$$

Proceeding analogously as we did with inequality (10) but integrating with respect to ξ_2 on L_{y_2} we get

$$\int_{L_{y_2}} \Big[\int_{\Omega_1} |(D^1_{\xi_2})^2_j|^{p_1} dx_1\Big]^{p_2/p_1} d\xi_2$$
$$\leqslant C_{13} \Big\{\varepsilon^{p_2} \int_{L_{y_2}} \Big[\int_{\Omega_1} |(D^2_{\xi_2} f)^2_j|^{p_1} dx_1\Big]^{p_2/p_1} d\xi_2 + \varepsilon^{-p_2} \int_{L_{y_2}} \Big[\int_{\Omega_1} |(f)^2_j|^{p_1} dx_1\Big]^{p_2/p_1} d\xi_2\Big\}.$$

Letting y_2 vary in $k_2 - 1$ dimensional space orthogonal to ξ_2, integrating the last inequality with respect to y_2, taking into account that

$\Omega_2 \cap O_j^2 \subset \Omega_j^{\xi 2} \subset \Omega_2$ and then summing over j we get

$$\int_{\Omega_2 \cap \bigcup_j O_j^2} \Big(\int_{\Omega_1} \sum_{s=1}^{k_2} |D^1_{\xi^2_s} f|^{p_1} dx_1\Big)^{p_2/p_1} dx_2$$
$$\leqslant C_{13} \Big\{\varepsilon^{p_2} \int_{\Omega_2} \Big(\int_{\Omega_1} \sum_{s=1}^{k_2} |D^2_{\xi^2_s} f|^{p_1} dx_1\Big)^{p_2/p_1} dx_2 + \varepsilon^{-p_2} \int_{\Omega_2} \Big(\int_{\Omega_1} |f|^{p_1} dx_1\Big)^{p_2/p_1} dx_2\Big\}.$$

Repeating the procedure written immediately before inequality (7) for $i = 3, \ldots, n$, then applying the Minkowski inequality for mixed norm to the components of the right-hand side and taking into account the considerations described just before (8) but for $i = 2$ we have

$$(12) \quad \int_{\Omega_n} \Big\{\ldots \int_{\Omega_3} \Big[\int_{\Omega_2 \cap \bigcup_j Q_j^2} \Big(\int_{\Omega 1} \sum_{s=1}^{k_2} |D^1_{\xi^2_s} f|^{p_1} dx_1\Big)^{p_2/p_1} dx_2 \ldots\Big]^{p_n/p_{n-1}} dx_n$$
$$\leqslant C_{14} \Big\{\varepsilon^{p_n} \int_{b\Omega} \sum_{s=1}^{k_2} |D^2_{\xi^2_s} f|^p dx + \varepsilon^{-p_n} \int_{b\Omega} |f|^p dx\Big\}.$$

It follows from (11) and (12), considering that ε is arbitrary

$$(13) \quad \int_{b\Omega} \sum_{s=1}^{k_2} |D^1_{\xi^2_s} f|^p dx \leqslant C_{15} \Big\{\varepsilon \int_{b\Omega} \sum_{s=1}^{k_2} |D^2_{\xi^2_s} f|^p dx + \varepsilon^{-1} \int_{b\Omega} |f|^p dx\Big\}.$$

We get, after repeating the described process, n of the inequalities of the type (11) and n of the inequalities of the type (12), and from these suitable n inequalities of the type (13). Summing the last ones over i for $i = 1, \ldots, n$ we get

$$\sum_{i=1}^{n} \int_{b\Omega} \sum_{s=1}^{k_i} |D^1_{\xi^i_s} f|^p dx \leqslant C_{16} \Big\{\varepsilon \sum_{i=1}^{n} \int_{b\Omega} \sum_{s=1}^{k_i} |D^2_{\xi^i_s} f|^p dx + \varepsilon^{-1} \int_{b\Omega} |f|^p dx\Big\}.$$

At last if we apply for the left- and right-hand side of the above inequality the generalizations of the two next inequalities

$$a^q + b^q \leqslant (a+b)^q \leqslant 2^q (a^q + b^q) \quad \text{for } q > 1 \quad \text{and} \quad 2^{-q}(a^q + b^q) \leqslant (a+b)^q \leqslant a^q + b^q \quad \text{for } 0 < q < 1,$$

we get the thesis.

Reasoning analogously as in [1], p. 24, we can get by induction

THEOREM 2. *Let Ω_i be a bounded, open set with the restricted cone property and $0 < \varepsilon \leqslant 1$. If $f \in H^{m,p}(\Omega)$ for some $m \geqslant 2$, $1 \leqslant j \leqslant m-1$, then*

$$(14) \quad \sum_{|\alpha| = j} \|D^\alpha f\|^{p_n}_{L^p(\Omega)} \leqslant C_{17} \Big\{\varepsilon^{m-j} \sum_{|\alpha| = m} \|D^\alpha f\|^{p_n}_{L^p(\Omega)} + \varepsilon^{-j} \|f\|^{p_n}_{L^p(\Omega)}\Big\},$$

where $C_{17} = C_{17}(m, p, \Omega)$.

4. We give here the generalization of so-called *Poincaré inequality* from [1].

Definition. We say that Ω_i has bounded width $\leqslant d_i$ if and only if there exists a line such that each line parallel to l_i intersects in a set whose diameter is not greater than d_i.

Theorem 3. *If Ω_i has bounded width $\leqslant d_i$, then*

(15)
$$\sum_{|\alpha|=j} \Big\{\int_{b\Omega} |D^\alpha f|^p dx\Big\}^{1/p_n} \leqslant C_{18}(n, p, m, \Omega)\, d^{m-j} \sum_{|\alpha|=m} \Big\{\int_{b\Omega} |D^\alpha f|^p dx\Big\}^{1/p_n},$$

for every $f \in H_0^{m,p}(\Omega)$, $0 \leqslant j \leqslant m-1$, $d = \max d_i$, $C_{18} = C_{18}(n, m, p, \Omega)$.

Proof. Let l_i' be a line parallel to l_i and assume that x_i^0 and $x_i^0+q_i$ are points of $l_i' \cap \partial\Omega_i$ such that $l_i' \cap \Omega_i$ is contained in the segment between x_i^0 and $x_i^0+q_i$. By defining f to vanish outside Ω_i, we can assume $f \in H_0^{m,p}(R^N)$.

Let $g(t_i) = f(x_1^0, \ldots, x_{i-1}^0, x_i^0+(t_i q_i)/|q_i|, x_{i+1}^0, \ldots, x_n^0)$; then $g(0) = 0$ for every $t_i = 0$, $i = 1, \ldots, n$, and hence $g(t_i) = \int_0^{t_i} g'(\tau_i)\, d\tau_i$ and

$$|g(t_i)| \leqslant \int_0^{d_i} |g'(\tau_i)|\, d\tau_i \leqslant \int_{-\infty}^{+\infty} |g'(\tau_i)|\, dt_i. \tag{16}$$

By the Hölder inequality,

$$|g(t_i)|^{p_i} \leqslant \Big(\int_0^{t_i} |g'(\tau_i)|\, d\tau_i\Big)^{p_i} \leqslant d_i^{p_i-1} \int_{-\infty}^{+\infty} |g'(t_i)|^{p_i}\, dt_i.$$

Integrating both sides of the above inequality with respect to $t_i \in (-\infty, +\infty)$ we get

$$\int_{-\infty}^{+\infty} |g(t_i)|^{p_i}\, dt_i = \int_0^{d_i} |g(t_i)|^{p_i}\, dt_i \leqslant d_i \int_{-\infty}^{+\infty} |g'(t_i)|^{p_i}\, dt_i.$$

Now express $\int_{\Omega_1} |f(x)|^{p_1} dx_1$ as an iterated integral with one of the integrations taken in the direction of l_1. From the above inequality for $i = 1$ it follows

$$\int_{\Omega_1} |f(x)|^{p_1}\, dx_1 = \int_{\Sigma_1} \Big(\int_{-\infty}^{+\infty} |f|^{p_1} dl_1\Big) d\sigma_1 \leqslant \int_{\Sigma_1} \Big(d_1^{p_1} \int_{-\infty}^{+\infty} |D_{l_1}^1 f|^{p_1} dl_1\Big) d\sigma_1,$$

where Σ_1 is the k_1-1 dimensional space with the axes orthogonal to each other and to l_1. The $D_{l_1}^1 f$ is the directional derivative of the function f in the space R^{k_1} with fixed variables $x_i \in R^{k_i}$, $i = 2, \ldots, n$, so that $|D_{l_1}^1 f|^{p_1} \leqslant \sum_{i=1}^{k_1} |D_{x_i^1}^1 f|^{p_1}$ and from the two above inequalities we get

$$\int_{\Omega_1} |f(x)|^{p_1}\, dx_1 \leqslant d_1^{p_1} \int_{\Omega_1} \sum_{i=1}^{k_1} |D_{x_i^1}^1 f|^{p_1}\, dx_1.$$

Rising up both sides of this inequality to the power p_i/p_{i-1}, integrating suitably with respect to x_i over Ω_i for $i = 2, \ldots, n$ and then rising up to the power $1/p_n$ we have

$$\{\int_{b\Omega} |f(x)|^p\, dx\}^{1/p_n} \leqslant d_1 \{\int_{b\Omega} \sum_{i=1}^{k_1} |D_{x_i^1}^1 f|^p\, dx\}^{1/p_n}.$$

From (16) it follows for $i = 1$ that

$$|f| \leqslant \int_0^{d_2} |D_{l_2}^1 f|\, dl_2,$$

where $D_{l_2}^1 f$ is the directional derivative of the function f in the space R^{k_2} with fixed variables $x_i \in R^{k_2}$, $i = 1, 3, \ldots, n$. The above inequality we rise up to the power p_1, integrate with respect to x_1 on Ω_1, rise up to the power $1/p_1$, and hence

$$(\int_{\Omega_1} |f|^{p_1}\, dx_1)^{1/p_1} \leqslant [\int_{\Omega_1} (\int_0^{d_2} |D_{l_2}^1 f|\, dl_2)^{p_1}\, dx_1]^{1/p_1}.$$

Applying to the right-hand side of this inequality the Minkowski inequality (10), [3], p. 22, next applying the Hölder inequality, rising up to the power p_2 and integrating both sides of the obtained inequality with respect to $t_2 \in R^{k_2}$, we get

$$\int_{-\infty}^{+\infty} (\int_{\Omega_1} |f|^{p_1}\, dx_1)^{p_2/p_1}\, dt_2 = \int_0^{d_2} (\int_{\Omega_1} |f|^{p_1}\, dx_1)^{p_2/p_1}\, dt_2$$

$$\leqslant d_2^{p_2} \int_{-\infty}^{+\infty} (\int_{\Omega_1} |D_{l_2}^1 f|^{p_1}\, dx_1)^{p_2/p_1}\, dt_2.$$

Now express $\int_{\Omega_2} (\int_{\Omega_1} |f|^{p_1}\, dx_1)^{p_2/p_1}\, dx_2$ as an iterated integral with one of the integrations taken in the direction of l_2. From the above inequality and from the fact that $|D_{l_2}^1 f|^{p_1} \leqslant \sum_{i=1}^{k_1} |D_{x_i^2}^1 f|^{p_1}$ it follows that

$$\int_{\Omega_2} (\int_{\Omega_1} |f|^{p_1}\, dx_1)^{p_2/p_1}\, dx_2 \leqslant d_2^{p_2} \int_{\Omega_2} (\int_{\Omega_1} \sum_{i=1}^{k_2} |D_{x_i^2}^1 f|^{p_1}\, dx_1)^{p_2/p_1}\, dx_2.$$

Rising up both sides of the above inequality to the power p_i/p_{i-1}, integrating suitably with respect to x_i over Ω_i successively for $i = 3, \ldots, n$ and rising up to the power $1/p_n$ we get

$$\Big\{\int_{b\Omega} |f|^p\,dx\Big\}^{1/p_n} \leqslant d_2 \Big\{\int_{b\Omega} \sum_{i=1}^{k_2} |D^1_{x^2_i} f|^p\,dx\Big\}^{1/p_n}.$$

Repeating this procedure with the function f and its derivatives in the spaces $R^{k_3}, \ldots, R^{k_n}$ we get n integrals of the form

$$\Big\{\int_{b\Omega} |f|^p\,dx\Big\}^{1/p_n} \leqslant d_j \Big\{\int_{b\Omega} \sum_{i=1}^{k_j} |D^1_{x^j_i} f|^p\,dx\Big\}^{1/p_n}, \quad j = 1, \ldots, n. \tag{17}$$

Summing over j we get

$$\Big\{\int_{b\Omega} |f|^p\,dx\Big\}^{1/p_n} \leqslant \frac{1}{n} \sum_{j=1}^{n} d_j \Big\{\int_{b\Omega} \sum_{i=1}^{k_j} |D^1_{x^j_i} f|^p\,dx\Big\}^{1/p_n}.$$

If we set $d = \max d_j$, $j = 1, \ldots, n$, then

$$\Big\{\int_{b\Omega} |f|^p\,dx\Big\}^{1/p_n} \leqslant C_{19}(p, \Omega)\frac{d}{n} \sum_{|\alpha|=1} \Big\{\int_{b\Omega} |D^\alpha f|^p\,dx\Big\}^{1/p_n}.$$

Applying this inequality to $D_k f$, we have

$$\Big\{\int_{b\Omega} |D^k f|^p\,dx\Big\}^{1/p_n} \leqslant C_{20}(p, \Omega)\frac{d}{n} \sum_{|\alpha|=1} \Big\{\int_{b\Omega} |D^\alpha (D^k f)|^p\,dx\Big\}^{1/p_n}.$$

Summing over k, $k = 1, \ldots, N$,

$$\sum_{|\alpha|=1} \Big\{\int_{b\Omega} |D^\alpha f|^p\,dx\Big\}^{1/p_n} \leqslant C_{21}(n, p, \Omega)\, d \sum_{|\alpha|=2} \Big\{\int_{b\Omega} |D^\alpha f|^p\,dx\Big\}^{1/p_n}.$$

Proceeding in this manner

$$\sum_{|\alpha|=j} \Big\{\int_{b\Omega} |D^\alpha f|^p\,dx\Big\}^{1/p_n} \leqslant C_{22}(n, p, \Omega, j)\, d \sum_{|\alpha|=j+1} \Big\{\int_{b\Omega} |D^\alpha f|^p\,dx\Big\}^{1/p_n}$$

for $0 \leqslant j \leqslant m-1$, hence (15) follows.

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