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Variation and compactness

Abstract. Some relations are investigated between relative compactness of the sums $\sum_{1}^{n} \eta_i x_i$ and perfect convergence (subseries convergence) in either a Banach space or an F-space under some additional assumptions.

The condition $O(X, Y)$ is introduced as a generalization of the condition O which appears in the theory of perfectly convergent series.

Applications of these notions to the study of some properties of vector valued functions of bounded weak variation are given.

1. Let $(X, \|\ \|)$ denote an F-space. The series $\sum_{1}^{\infty} x_i, x_i \in X$, is called *perfectly bounded in* X if the set S of sums of the form $\sum_{i=1}^{n} \eta_i x_i$, $n = 1, 2, \ldots$, $\eta_i = 0, 1$, is bounded in $(X, \|\ \|)$. The series $\sum_{1}^{\infty} x_i$ is *perfectly convergent* (*subseries convergent*) in $(X, \|\ \|)$ if every series of the form $\sum_{1}^{\infty} \eta_i x_i$, $\eta_i = 0, 1$, is convergent in $(X, \|\ \|)$. We say that $(X, \|\ \|)$ *satisfies the condition O* if every perfectly bounded series $\sum_{1}^{\infty} x_i$ is also perfectly convergent. We are going to give some generalization of the condition O. Let $(X, \|\ \|)$, $(Y, \|\ \|^*)$ be F-spaces, $X \subset Y$. The space X is said to *satisfy the condition* $O(X, Y)$ if every perfectly bounded series $\sum_{1}^{\infty} x_i$, $x_i \in X$ in X is perfectly convergent with respect to the norm $\|\ \|^*$. In connection with the condition O we give in Section 1 of this article a generalization of Gelfand's criterion for the perfectly bounded series $\sum_{1}^{\infty} x_i$ to be perfectly convergent. In Section 2 we give an application of the conditions O and $O(X, Y)$ in the theory of vector valued functions of bounded weak variation.

1.1. LEMMA 1. *Let* $(X, \|\ \|)$ *be either an F-space with a basis or a B-space.*

If the set of elements $x_i \in X$ is relatively compact in X and the series $\sum_1^\infty x_i$ is perfectly bounded, then $x_i \to 0$.

Proof. Let us assume that the elements u_i constitute a basis in $(X, \|\ \|)$ so that $x = \sum_1^\infty a_j(x) u_j$, $x \in X$, where $a_j(x)$ are linear functionals continuous in $(X, \|\ \|)$. Let $A_m(x) = \sum_{j=1}^m a_j(x) u_j$ for $m = 1, 2, \ldots$ Since the set $y_n = \sum_{i=1}^n \eta_i x_i$, $n = 1, 2, \ldots, \eta_i = 0, 1$ is bounded in $(X, \|\ \|)$ so the sequence $a_j(y_n) = \sum_{i=1}^n \eta_i a_j(x_i)$ is bounded for $j = 1, 2, \ldots$ and arbitrarily chosen $\eta_i = 0, 1$. In consequence

$$\text{(A)} \qquad \sum_{i=1}^\infty |a_j(x_i)| < \infty \qquad \text{for } j = 1, 2, \ldots$$

Thus $a_j(x_i) \to 0$ for $j = 1, 2, \ldots$ and so $A_m(x_i) \to 0$ when $i \to \infty$. One can extract from (x_i) a partial sequence (x_{k_i}) such that for some $x \in X$, $\|x_{k_i} - x\| \to 0$. Thus $A_m(x_{k_i}) \to A_m(x)$. On the other hand $A_m(x_{k_i}) \to 0$ so $A_m(x) = 0$ for $m = 1, 2 \ldots$ and since $A_m(x) \to x$ so $x = 0$ and consequently $x_{k_i} \to 0$. Because the above reasoning can be carried out for arbitrary partial sequence of (x_i) so $x_i \to 0$. In the case where $(X, \|\ \|)$ is a Banach space, the perfect boundedness of the series $\sum_1^\infty x_i$ implies $\sum_1^\infty |\xi(x_i)| < \infty$ for arbitrary functional from the dual space of $(X, \|\ \|)$. For some partial sequence (x_{k_i}) there exists an x such that $x_{k_i} \to x$ so $\xi(x_{k_i}) \to \xi(x)$. Since, on the other hand $\xi(x_{k_i}) \to 0$ so $\xi(x) = 0$, $x = 0$ or $x_{k_i} \to 0$. Applying this reasoning to arbitrary partial sequence of the sequence (x_i) we conclude $x_i \to 0$.

1.2. THEOREM 1. *If $(X, \|\ \|)$ is an F-space with a basis, then the series $\sum_1^\infty x_i$ is perfectly convergent if and only if the set S is relatively compact in $(X, \|\ \|)$ (i.e., the closure of S is compact in $(X, \|\ \|)$).*

Proof. To show the necessity let us choose an arbitrary sequence $y_k = \sum_1^{n_k} \eta_i^{(k)} x_i = \sum_1^\infty \eta_i^{(k)} x_i$, where $\eta_i^{(k)} = 0$ for $i > n_k$. Using the diagonal process we can extract a sequence of indices (k_ν) such that $\eta_i^{(k_\nu)} \to \eta_i^{(0)}$ for every i as $k_\nu \to \infty$. Any perfectly convergent series in an arbitrary F-space has the following property: for every $\varepsilon > 0$ there exists an index N such that $\|\sum_{i=p}^q \eta_i x_i\| \leqslant \varepsilon$ for $q \geqslant p \geqslant N$ and arbitrary $\eta_i = 0, 1$. The sequence $y_{k_\nu} = \sum_1^\infty \eta_i^{(k_\nu)} x_i$ converges to $\sum_1^\infty \eta_i^{(0)} x_i$. Indeed, choose arbitrarily $\varepsilon > 0$, then

$\sum_{1}^{N-1} \eta_i^{(k_\nu)} x_i \to \sum_{1}^{N-1} \eta_i^{(0)} x_i$ and also $\|\sum_{N}^{\infty} \eta_i^{(k_\nu)} x_i\| < \varepsilon$ for all k_ν. In consequence we get

$$\|\sum_{1}^{\infty} \eta_i^{(k_\nu)} x_i - \sum_{1}^{\infty} \eta_i^{(0)} x_i\| \leqslant \|\sum_{1}^{N-1} (\eta_i^{(k_\nu)} - \eta_i^{(0)}) x_i\| + \|\sum_{N}^{\infty} (\eta_i^{(k_\nu)} - \eta_i^{(0)}) x_i\| \leqslant 3\varepsilon$$

for sufficiently large k_ν.

Sufficiency follows from Lemma 1 if we consider the remark that any sequence of sums $\sum_{n_i}^{m_i} \eta_i x_i$, $n_i < m_i < n_{i+1}$, is perfectly bounded and relatively compact.

1.3. THEOREM 2. *Let $(X, \|\ \|)$ be a Banach space. The series $\sum_{1}^{\infty} x_i$ is perfectly convergent in $(X, \|\ \|)$ if and only if the set S is relatively compact in $(X, \|\ \|)$* (cf. [5] for a different proof).

Proof. Necessity follows from Theorem 1. Sufficiency is implied by an application of Lemma 1 like for an F-space.

Let $(X, \|\ \|)$ be a Banach space, Ξ its dual. The set $\Xi_0 \subset \Xi$ is called *fundamental* or *norming* if the set $\{\|\xi\|\}$ is bounded for $\xi \in \Xi_0$ and, with some $c > 0$, $\|x\| c \leqslant \sup_{\xi \in \Xi_0} |\xi(x)|$ for $x \in X$. The set $\Xi_1 \subset \Xi$ is relatively $*$-w-sequentially compact in Ξ if from every sequence $\xi_n \in \Xi_1$ one can extract a partial sequence (ξ_{n_i}) $*$-w-convergent, that is to say, $\xi_{n_i}(x) \to \xi(x)$ for $x \in X$. The set $X_0 \subset X$ is relatively weakly sequentially compact with respect to Ξ_2 if from every sequence $x_n \in X_0$ one can extract a sequence (x_{n_i}), Ξ_2-weakly convergent to some element x, that is to say, $\xi(x_{n_i}) \to \xi(x)$ for $\xi \in \Xi_2$.

THEOREM 3. *Let $(X, \|\ \|)$ be a Banach space. Assume that the series $\sum_{1}^{\infty} x_i$ has the following properties: the series is perfectly bounded; for some fundamental set of functionals Ξ_0 which is relatively $*$-w-sequentially compact, the set S is relatively weakly sequentially compact with respect to Ξ_0. Then the series $\sum_{1}^{\infty} x_i$ is perfectly convergent.*

Proof. Perfect boundedness of $\sum_{1}^{\infty} x_i$ implies $\sum_{1}^{\infty} |\xi(x_i)| < \infty$ for $\xi \in \Xi_0$. For arbitrary $\sum_{1}^{\infty} \eta_i x_i$, $\eta_i = 0, 1$, there exists a partial sequence $\sum_{1}^{n_k} \eta_i x_i$ such that $\sum_{1}^{n_k} \eta_i \xi(x_i) \to \xi(x_\eta)$ for $\xi \in \Xi_0$ (x_η independent of ξ). Therefore

(a) $$\sum_{1}^{\infty} \eta_i \xi(x_i) = \xi(x_\eta) \quad \text{for } \xi \in \Xi_0,\ \eta_i = 0, 1.$$

By virtue of Theorem 1 in [9] the perfect convergence of $\sum_1^\infty x_i$ follows.

Remark. If in the theorem above we set $\Xi_0 = \Xi$, $\|\xi\| \leqslant 1$ then the perfect boundedness is a consequence of relative weak sequential compactness of S with respect to Ξ_0. The elements x_n of (a) belong to the separable linear space X_0 spanned over S and so Ξ is relatively $*$-w-sequentially compact in X_0 and in this case the above theorem may be applied. (a) means here the assumption of the Orlicz–Pettis theorem.

1.4. Let $(X, \| \ \|)$ be a Banach space, $(Y, \| \ \|^*)$ an F-space, $X \subset Y$. The unit ball $X_s = \{x \in X: \|x\| \leqslant 1\}$ endowed with the metric $d(x, y) = \|x-y\|^*$ provided it makes X_s a complete metric space, is called *Saks space* and denoted $(X_s, \| \ \|, \| \ \|^*)$ (see [8]).

THEOREM 4. *Let $(X, \| \ \|)$ be a Banach space, $(Y, \| \ \|^*)$ an F-space having a Schauder basis or a Banach space, $X \subset Y$. If $(X_s, \| \ \|, \| \ \|^*)$ is a relatively compact Saks space, then X satisfies the condition $O(X, Y)$.*

Proof. Indeed, if the series $\sum_1^\infty x_i$ is perfectly bounded in $(X, \| \ \|)$, then the set S is bounded and for some $\lambda > 0$, $\lambda S \subset X_s$. It suffices to apply Theorems 1 and 2.

1.5. We are going to give two applications of Theorem 4.

(i) Let $\omega(u)$: $\langle 0, \infty) \to \langle 0, \infty)$ denote a continuous strictly increasing function, $\omega(0) = 0$. Denote by $X = H^\omega$ the vector space of real functions in $\langle a, b\rangle$ taking 0 for $t = a$ and satisfying the condition

$$(*) \qquad |x(t_1)-x(t_2)| \leqslant k\omega(|t_1-t_2|) \quad \text{for } t_1, t_2 \in \langle a, b\rangle.$$

Let us define a standard norm in X, $\|x\|_\omega = \inf k$ (k from inequality $(*)$). $(H^\omega, \| \ \|_\omega)$ is a Banach space. Note that if $\omega(u)/u \to 0$, $u \to 0$, then H^ω is reduced to the function $x(t) = 0$ for $t \in \langle a, b\rangle$. Let $Y = (H^{\omega_1}, \| \ \|_{\omega_1})$, where ω_1 satisfies the conditions from the definition of ω and the additional one: $\omega(u)/\omega_1(u) \to 0$ where $u \to 0$. It is easy to see that $X \subset Y$. Let us form the Saks space $(X_s, \| \ \|_\omega, \| \ \|_{\omega_1})$. Its completeness follows directly from the observation that $\|x_n - x_0\|_{\omega_1} \to 0$ implies $x_n(t) - x_0(t) \to 0$ uniformly in $\langle a, b\rangle$, and $\liminf_{m\to\infty} \|x_n - x_m\|_{\omega_1} \geqslant \|x_n - x_0\|_{\omega_1}$, $\liminf_{n\to\infty} \|x_n\|_\omega \geqslant \|x_0\|_\omega$. We shall prove that $(X_s, \| \ \|_\omega, \| \ \|_{\omega_1})$ is compact. Let $x_n \in X_s = \{x \in H^\omega: \|x\|_\omega \leqslant 1\}$. From the Arzelà theorem there follows the existence of a partial sequence (x_{n_i}) such that $x_{n_i}(t) \to x_0(t)$, $x_0 \in X_s$, uniformly in $\langle a, b\rangle$. Put $y_i(t) = x_{n_i}(t) - x_0(t)$, then

$$|y_i(t_1) - y_i(t_2)| \leqslant 2\omega(|t_1-t_2|) = 2\frac{\omega(|t_1-t_2|)}{\omega_1(|t_1-t_2|)}\omega_1(|t_1-t_2|).$$

Taking $u_0 > 0$ sufficiently small we have, for $|t_1 - t_2| \leqslant u_0$,

$|y_i(t_1)-y_i(t_2)| \leqslant \varepsilon\omega_1(|t_1-t_2|)$, $i = 1, 2, \ldots$ For arbitrary t_1, t_2 for which $|t_1-t_2| > u_0$ we have $|y_i(t_1)-y_i(t_2)| \leqslant \varepsilon\omega_1(u_0) \leqslant \varepsilon\omega_1(|t_1-t_2|)$ for $i \geqslant i_0$. In consequence $\|x_{n_i}-x_0\|_{\omega_1} \to 0$, while $x_0 \in X_s$.

Putting in particular $\omega(u) = u^\alpha$, $\omega_1(u) = u^\beta$ and denoting in this case $H^\omega = H^\alpha$, $H^{\omega 1} = H^\beta$, $\|\ \|_\omega = \|\ \|_\alpha$, $\|\ \|_{\omega_1} = \|\ \|_\beta$ we can see that the Saks spaces $(X_s, \|\ \|_\alpha, \|\ \|_\beta)$ are compact for arbitrary $0 < \beta < \alpha \leqslant 1$.

In connection with this theorem, let us notice that H^α, H^β fail the condition O for $0 < \alpha \leqslant 1$. For $\alpha < 1$ this follows from a theorem of Ciesielski [4] about the linear isomorphism between H^α and l^∞; for $\alpha = 1$ this is a consequence of linear isometry of H^1 and $L^\infty\langle a, b\rangle$. However, Theorem 4 insures that, for $0 < \beta < \alpha \leqslant 1$, $O(H^\alpha, H^\beta)$ holds.

(ii) Let $\alpha \geqslant 1$. The real function x in $\langle a, b\rangle$ is called *function of bounded α-variation* if

$$v_\alpha(x) = \sup \sum_{i=1}^{n} |x(t_i)-x(t_{i-1})|^\alpha < \infty,$$

where the supremum is taken over all partitions π: $a = t_0 < t_1 < \ldots < t_n = b$.

The set V^α of functions of bounded α-th variation, taking 0 for $t = a$, is a Banach space with the usual operations and with the norm $\|x\|_{v,\alpha} = (v_\alpha(x))^{1/\alpha}$. It is known that V^1 satisfies the condition O whereas it is not known whether this condition still holds for $\alpha > 1$ [1], [7].

Let $\alpha > 1$, ω the same as in 1.5 (i), $X = H^\omega \cap V^\alpha$, $Y = V^\beta$, where $\alpha < \beta$. For X take the B-norm $\|x\| = \sup(\|x\|_\omega, \|x\|_{v,\alpha})$. Then $X \subset Y$, $(X_s, \|\ \|, \|\ \|_{v,\beta})$ is a compact Saks space.

For the proof let us take a sequence $x_n \in X_s = \{x \in X: \|x\| \leqslant 1\}$. Since $\|x_n\|_\omega \leqslant 1$ so there exists a partial sequence (x_{n_i}) such that $x_{n_i}(t) \to x_0(t)$ uniformly in $\langle a, b\rangle$. Since $\liminf_{i\to\infty} \|x_{n_i}\| \geqslant \|x_0\|$ so $\|x_0\| \leqslant 1$. Put $y_i(t) = x_{n_i}(t)-x_0(t)$ and choose partitions π_i: $a = t_1^i < t_2^i < \ldots < t_{n(i)}^i = b$ in such a way that

$$(1.1) \qquad \tfrac{1}{2}(\|y_i\|_{v,\beta})^\beta \leqslant \sum_{j=1}^{n(i)} |y_i(t_j^i)-y_i(t_{j-1}^i)|^\beta \quad \text{for } i = 1, 2, \ldots$$

The functions y_i are equicontinuous in $\langle a, b\rangle$; hence taking $u_0 > 0$ sufficiently small, we have $|y_i(t_j^i)-y_i(t_{j-1}^i)| \leqslant \varepsilon^{1/\beta-\alpha}$ for such intervals in the partition π_i, where $|t_j^i-t_{j-1}^i| < u_0$. Let the set of these intervals of π_i be denoted by $A(\pi_i)$ and the remaining ones by $B(\pi_i)$. Let k_i denote the number of intervals in $B(\pi_i)$. Of course, $k_i < (b-a)/u_0$. In view of uniform convergence of y_i to 0, summing over intervals of $B(\pi_i)$, we get

$$(1.2) \qquad \sum_{B(\pi_i)} |y_i(t_j^i)-y_i(t_{j-1}^i)|^\beta < \varepsilon \quad \text{for } i \geqslant i_0.$$

However, for $i = 1, 2, \ldots,$ we have

$$(1.3)\qquad \sum_{A(\pi_i)} |y_i(t_j^i)-y_i(t_{j-1}^i)|^\beta \leqslant \varepsilon \sum_{A(\pi_i)} |y_i(t_j^i)-y_i(t_{j-1}^i)|^\alpha \leqslant \varepsilon v_\alpha(y_i) \leqslant \varepsilon(1+2^\alpha).$$

Inequalities (1.1)–(1.3) yield $\|y_i\|_{v,\beta} \leqslant (2\varepsilon(1+2^\alpha))^{1/\beta}$ for $i \geqslant i_0$. So $\|y_i\|_{v,\beta} \to 0$. (It is apparent that compactness of X_s implies completeness of the Saks space $(X_s, \|\ \|, \|\ \|_{v,\beta})$.)

1.6. THEOREM 5. *Let $x_i \in C$ for $i = 1, 2, \ldots$ and for the series $\sum_1^\infty x_i$ let the set S be relatively compact in C with respect to convergence in measure. Then $\sum_1^\infty x_i$ is perfectly convergent in C.*

Proof. Since the set S is relatively compact in C with respect to convergence in measure so it is relatively compact in L^0 and thereby, bounded in L^0. From this there follows the perfect boundedness of the series $\sum_1^\infty x_i$ which implies, by virtue of Theorem 1 of [6] its perfect convergence in L^0. For a given series $\sum_1^\infty \eta_i x_i$ some partial sequence $\sum_1^{i_k} \eta_i x_i$ converges in measure to $x_\eta \in C$, which means $\sum_1^\infty \eta_i x_i = x_\eta$ with respect to convergence in measure for arbitrary $\eta_i = 0, 1$. By virtue of Theorem 7 of [9] the perfect convergence of $\sum_1^\infty x_i$ in C follows.

2. In this section we shall be interested in the condition $O(X, Y)$ for some spaces of sequences. Let l^0 denote the vector space of sequences with real terms $x = (t_i)$. Let φ be a φ-function, i.e., φ: $\langle 0, \infty) \to \langle 0, \infty)$, $\varphi(0) = 0$, $\varphi(u) > 0$ for $u > 0$, φ continuous and nondecreasing, $\varphi(u) \to \infty$ for $u \to \infty$. In l^0 a modular $\varrho_\varphi(x) = \sum_1^\infty \varphi(|t_i|)$ is defined. Write $l^{*\varphi} = \{x \in l^0\colon \varrho_\varphi(\lambda x) < \infty$ for some $\lambda < 0\}$, $l_f^{*\varphi} = \{x \in l^0\colon \varrho_\varphi(\lambda x) < \infty$ for all $\lambda > 0\}$ – the space of finite elements. $l^{*\varphi}$ is an F-space with respect to the generated norm $\|x\|_\varphi = \inf\{\varepsilon > 0\colon \varrho_\varphi(x/\varepsilon) \leqslant \varepsilon\}$, and $l_f^{*\varphi}$ its complete subspace. $\|x_n\|_\varphi \to 0$ is equivalent to $\varrho_\varphi(\lambda x_n) \to 0$ for each $\lambda > 0$. Recall also that φ satisfies the condition Δ_2 for small u if for some k, $u_0 > 0$, $0 \leqslant u \leqslant u_0$, $\varphi(2u) \leqslant k\varphi(u)$.

2.1. LEMMA 2. *If $l_f^{*\varphi}$ satisfies the condition O, then φ satisfies Δ_2 for small u.*

Proof. Suppose Δ_2 fails. Then there exists a numerical sequence $u_n \to 0$ such that $\varphi(2u_n) \geqslant 2^n \varphi(u_n)$, $\varphi(u_n) \leqslant 1/2^n$, $\varphi(2u_n) \leqslant 1$ for $n = 1, 2, \ldots$ Determine integers k_n in such a way that $1/2^{n+1} \leqslant k_n \varphi(u_n) \leqslant 1/2^n$ and define sequences $x_1 = (u_1, u_1, \ldots, u_1, 0, 0, \ldots), \ldots, x_n = (0, 0, 0, u_n, u_n, \ldots, u_n, \ldots), \ldots$, where the i-th term of x_n is equal to u_n for $k_1 + \ldots + k_{n-1} < i \leqslant k_1 + \ldots + k_n$, and x_1 has u_1 at i, $1 \leqslant i \leqslant k_1$. Let us choose a sequence $\lambda_i \to 0$. We have $x_n \in l_f^{*\varphi}$,

$$\varrho_\varphi\big(\lambda_i(\eta_1 x_1 + \ldots + \eta_k x_k)\big) \leqslant \sum_{n=1}^{\infty} k_n \varphi(\lambda_i u_n) < \varepsilon \quad \text{for } i \geqslant i(\varepsilon),\ k = 1, 2, \ldots, \eta_i = 0, 1,$$

because $\sum_{n=1}^{\infty} k_n \varphi(u_n) \leqslant 1$. Suppose $\|x_n\|_\varphi \to 0$; then $\varrho_\varphi(2x_n) \to 0$ but $\varrho_\varphi(2x_n) \geqslant k_n \varphi(2u_n) \geqslant \frac{1}{2}$ – a contradiction. So the sequence (x_n) is not convergent in $l^{*\varphi}$ which means that the series $\sum_1^{\infty} x_n$ is not perfectly convergent, being at the same time perfectly bounded.

2.2. LEMMA 3. *The space $l_f^{*\varphi}$ has a basis consisting of unit vectors e_i (Δ_2 is not assumed).*

Proof. If $x \in l_f^{*\varphi}$ then, for arbitrary $\lambda > 0$, $\sum_1^{\infty} \varphi(\lambda t_i) < \infty$, $\sum_{i=k+1}^{\infty} \varphi(\lambda t_i) \to 0$, that is, for the sequence $x_k = (0, 0, \ldots, t_{k+1}, t_{k+2}, \ldots)$, $\|x_k\|_\varphi \to 0$ whenever $k \to \infty$, $\|s_k - x\|_\varphi \to 0$, where $s_k = \sum_1^k t_j e_j$, and $x = \sum_1^{\infty} t_j e_j$.

2.3. THEOREM 6. *Let*

$$(*) \qquad \varrho_\varphi\Big(\sum_1^n \eta_k x_k\Big) \leqslant K \quad \textit{for } n = 1, 2, \ldots, \eta_k = 0, 1.$$

Then

$$(**) \qquad \varrho_\varphi\Big(\tfrac{1}{4}\Big(\sum_1^n \eta_k x_k - x_\eta\Big)\Big) \to 0 \quad \textit{for arbitrary } \eta_k = 0, 1,$$

where $\varrho_\varphi(\frac{1}{8} x_\eta) \leqslant K$.

Proof. Let us introduce some notations: χ_e – the characteristic function on the set N of natural numbers; if $e = \langle p, q\rangle$ then we write χ_{pq} instead of χ_e; $\chi_{\langle p, \infty)}$ will have parallel meaning. We assert that for each $\varepsilon > 0$ there exists a p such that

$$(2.1) \qquad \varrho_\varphi(\tfrac{1}{4} x_k \chi_{\langle p, \infty)}) < \varepsilon \quad \text{for } k = 1, 2, \ldots$$

If 2.1 does not hold, then there exists a partial sequence of (x_k) denoted (y_l) and a sequence of intervals of natural numbers $p_l, q_l, p_l < q_l < p_{l+1}$, for $l = 1, 2, \ldots$ and $\varepsilon_0 > 0$ such that

$$(2.2) \qquad \varrho_\varphi(\tfrac{1}{4} y_l \chi_{\langle p_l q_l\rangle}) \geqslant \varepsilon_0, \quad l = 1, 2, \ldots$$

Let $y_l = (t_i^l)\cdot(*)$ implies

$$\varphi\Big(\sum_{l=1}^{n} \eta_l t_i^l\Big) \leqslant K \quad \text{for arbitrary } \eta_l = 0, 1. \tag{2.3}$$

Let $\varphi(u) > K$ for $u > u_0$. From (2.3) we get

$$\sum_{l=1}^{\infty} |t_i^l| \leqslant 2u_0 \quad \text{for } i = 1, 2, \ldots \tag{2.3'}$$

Let $y_\eta = \eta_1 y_1 + \eta_2 y_2 + \ldots + \eta_n y_n$, $y_\eta = (t_i^\eta)$. Define sets $e_l' = \{i \in \langle p_l, q_l\rangle : |t_i^\eta| \leqslant \frac{1}{4}|t_i^l|\}$, $e_l'' = \langle p_l, q_l\rangle \setminus e_l'$. Whenever $l \to \infty$ then $p_l, q_l \to \infty$ so $\varrho_\varphi(y_\eta \chi_{p_l q_l}) \to 0$, and since

$$\varrho_\varphi(y_\eta \chi_{p_l q_l}) \geqslant \varrho_\varphi(y_\eta \chi_{e_l''}) \geqslant \varrho_\varphi(\tfrac{1}{4} y_l \chi_{e_l''})$$

so, for sufficiently large l, $\varrho_\varphi(\frac{1}{4} y_l \chi_{e''}) \leqslant \frac{1}{4}\varepsilon_0$, that is

$$\varrho_\varphi(\tfrac{1}{4} y_l \chi_{e_l'}) = \varrho_\varphi(\tfrac{1}{4} y_l \chi_{\langle p_l q_l\rangle}) - \varrho_\varphi(\tfrac{1}{4} y_l \chi_{e_l''}) \geqslant \tfrac{3}{4}\varepsilon_0,$$

and there is

$$|t_i^\eta| \leqslant \tfrac{1}{4}|t_i^l| \quad \text{when } i \in e_l'. \tag{2.4}$$

Let a_l' be such a subset of e_l' on which $|t_i^l| > 0$ and let $\liminf_{i \in a_l'} |t_i^l| = d_l$. Clearly

$$\varrho_\varphi(\tfrac{1}{4} y_l \chi_{a_l'}) \geqslant \tfrac{3}{4}\varepsilon_0, \tag{2.5}$$

and furthermore (2.4) holds for $i \in a_l'$. Having fixed some l for which (2.4), (2.5) $i \in a_l'$ is valid we can always find an l_1 in such a way that

$$|t_i^{l_1}| \leqslant \eta d_l \quad \text{for } i \in a_l', \tag{2.6}$$

where η is given in advance. The possibility of choosing such an l_1 is implied directly by (2.3′). Using the remarks above we are in a position to define a partial sequence (y_{l_k}), $y_{l_k} = (t_i^{l_k})$ and a sequence of sets (a_k) satisfying the conditions:

1° $\varrho_\varphi(\frac{1}{4} y_{l_k} \chi_{a_k}) \geqslant \frac{3}{4}\varepsilon_0$,

2° $\Big|\sum_{j=1}^{k-1} t_i^{l_j}\Big| \leqslant \frac{1}{4}|t_i^{l_k}|$ for $i \in a_k$,

3° $\frac{1}{4}|t_i^{l_k}| \geqslant \sum_{j=k+1}^{\infty} |t_i^{l_j}|$ for $i \in a_k$,

4° $a_{k'} \cap a_{k''} = \emptyset$ when $k' \neq k''$.

From 1°–3° we obtain

$$\varrho_\varphi\Big(\Big(\sum_{j=1}^{\infty} y_{l_j}\Big)\chi_{a_k}\Big) = \sum_{i \in a_k} \varphi\Big(\Big|\sum_{j=1}^{\infty} t_i^{l_j}\Big|\Big) \geqslant \sum_{i \in a_k} \varphi(\tfrac{1}{2}|t_i^{l_k}|) \geqslant \tfrac{3}{4}\varepsilon_0.$$

Hence by (*) and 4°, denoting $b_r = \bigcup_{k=1}^{r} a_k$, we have $K \geqslant \varrho_\varphi\big((\sum_{j=1}^{\infty} y_{l_j})\chi_{b_r}\big) \geqslant \frac{3}{4}\varepsilon_0 r$ so taking $r > 4K/3\varepsilon_0$ we are led to a contradiction.

Because of (2.1) there is $\sum_{i=p}^{\infty} \varphi(\frac{1}{4}|t_i^k|) \leqslant \varepsilon$ for sufficiently large p, $k = 1, 2, \ldots$, and (*) yields $\varphi(\sum_{k=1}^{\infty} |t_i^k|) \leqslant \varphi(2u_0)$ for $i = 1, 2, \ldots$ Thus for $k \geqslant k_0$ there is $\varrho_\varphi(\frac{1}{4}x_k) < 2\varepsilon$ that is $\varrho_\varphi(\frac{1}{4}x_k) \to 0$. Let us consider an arbitrary series $\sum_1^{\infty} \eta_k x_k$, $s_n = \sum_1^{n} \eta_k x_k$. The Cauchy modular condition

$$\varrho_\varphi\big(\tfrac{1}{4}(s_p - s_q)\big) \to 0 \quad \text{as } p, q \to \infty \tag{2.7}$$

is satisfied. Otherwise there would exist, for some $\varepsilon_0 > 0$, an increasing sequence of indices p_i, q_i, $p_i < q_i < p_{i+1}$ for $i = 1, 2, \ldots$ such that $\varrho_\varphi\big(\frac{1}{4}(s_{q_i} - s_{p_i})\big) \geqslant \varepsilon_0$. However, for the sequence $z_i = s_{q_i} - s_{p_i}$, by (*) we have $\varrho_\varphi(\sum_1^{n} \eta_i z_i) \leqslant K$ for $n = 1, 2, \ldots$ and arbitrary $\eta_i = 0, 1$. Using the previously proved lemma for z_i replacing x_i, we would have $\varrho_\varphi(\frac{1}{4}z_i) \to 0$ and we are led to a contradiction. From (2.7) there follows the existence of a sequence $x_\eta = (t_i^\eta)$ such that, for each i, $\sum_1^{n} \eta_k t_i^k \to t_i^\eta$, and $\liminf_{q\to\infty} \varrho_\varphi\big(\frac{1}{4}(s_p - s_q)\big) \geqslant \varrho_\varphi\big(\frac{1}{4}(s_p - x_\eta)\big)$. From this last inequality and (2.7) we get $\varrho_\varphi\big(\frac{1}{4}(s_p - x_\eta)\big) < \varepsilon$ for p sufficiently large and thus (**) holds. Since

$$\varrho_\varphi(\tfrac{1}{8}x_\eta) \leqslant \varrho_\varphi\big(\tfrac{1}{4}(s_p - x_\eta)\big) + \varrho_\varphi(\tfrac{1}{4}s_p) \leqslant \varepsilon + K$$

for p sufficiently large, so $\varrho_\varphi(\frac{1}{8}x_\eta) \leqslant K$.

2.4. THEOREM 7. *Let φ, ψ be φ-functions satisfying the condition: for each $\lambda > 0$ there exist $c_\lambda > 0$, $u(\lambda)$ such that*

$$\psi(\lambda u) \leqslant c_\lambda \varphi(u) \quad \textit{for } 0 \leqslant u \leqslant u(\lambda). \tag{*}$$

Then $l^{*\varphi} \subset l_f^{*\psi}$ and $O(l^{*\varphi}, l_f^{*\psi})$ holds.

Proof. The inclusion $l^{*\varphi} \subset l_f^{*\psi}$ is obvious. Let the series $\sum_1^{\infty} x_i$ be perfectly bounded in $l^{*\varphi}$. Then, for some λ_0, there is

$$\varrho_\varphi\big(\lambda_0(\sum_1^{n} \eta_i x_i)\big) \leqslant 1 \quad \text{for } n = 1, 2, \ldots, \ \eta_i = 0, 1. \tag{2.8}$$

Hence $\varphi(\lambda_0 \sum_{i=1}^{n} \eta_i t_k^i) \leqslant 1$, and for some u_0, $\lambda_0 \sum_{i=1}^{n} |t_k^i| \leqslant u_0$ for $k = 1, 2, \ldots$,

$n = 1, 2, \ldots$ However, $(*)$ implies that for some constant $\bar{c}_\lambda > 0$ $\psi(\lambda u) \leqslant \bar{c}_\lambda \varphi(u)$ for $0 \leqslant u \leqslant u_0$, so (2.8) yields

$$\varrho_\psi\left(\lambda\lambda_0 \sum_1^n \eta_i x_i\right) \leqslant \bar{c}_\lambda \varrho_\varphi\left(\lambda_0 \sum_1^n \eta_i x_i\right) \leqslant \bar{c}_\lambda. \tag{2.9}$$

From (2.9) and Theorem 4 we obtain for some x_η^λ

$$\varrho_\psi\left(\tfrac{1}{4}\lambda\lambda_0\left(\sum_{i=1}^n \eta_i x_i - x_\eta^\lambda\right)\right) \to 0 \quad \text{for each } \lambda, \tag{2.10}$$

but it is clear that $x_\eta^\lambda = x_\eta$ independently of λ, and in this way from (2.10) we get $\|\sum_1^n \eta_i x_i - x_\eta\|_\psi \to 0$.

Remark 1. In the above theorem we did not assume φ, ψ satisfied Δ_2 for small u.

2. If φ does satisfy Δ_2 for small u, then $(*)$ holds when we set $\varphi(u) = \psi(u)$. In this case, $l^{*\varphi} = l_f^{*\psi}$ and for the space $l^{*\varphi}$ the condition O is fulfilled.

3. Theorems analogous to Theorems 6 and 7 for function spaces $L^{*\varphi}$, $L^{*\psi}$ have been given in [3].

3. Let $(X, \|\ \|)$ be an F-space, x: $\langle a, b\rangle \to X$. The function $x(\cdot)$ is called *function of bounded weak variation* whenever the set of sums

$$S(n, \pi, \eta) = \sum_{i=1}^n \eta_i\big(x(t_i) - x(t_{i-1})\big),$$

where π: $a = t_0 < t_1 < \ldots < t_n = b$ is an arbitrary partition of $\langle a, b\rangle$, $\eta_i = 0, 1$, $n = 1, 2, \ldots$ is bounded in $(X, \|\ \|)$.

Denote $V^w(X)$ the set of functions $x(t)$: $\langle a, b\rangle \to X$, which are of bounded weak variation and $x(a) = 0$. With the standard operations on functions it is a vector space. If X is a Banach space (s-normed complete space), then $x \in V^w(X)$ if and only if $v^w(x) = \sup\|S(n, \pi, \eta)\| < \infty$, where supremum is taken over all n, π, η. In this case $v^w(x)$ is called *weak variation* of x and $V^w(X)$ is a Banach space (s-normed complete space) if we set $\|x\|_v = v^w(x)$.

Let $(Y, \|\ \|^*)$ be an F-space, $X \subset Y$. We are interested in the following properties of functions from $V^w(X)$:

A. Let $x \in V^w(X)$. At each point of $\langle a, b\rangle$ there exist one-sided limits of x (right-sided at $t = a$ and left-sided at $t = b$), with respect to $\|\ \|^*$ convergence;

B. Let $x \in V^w(X)$. The function x is continuous in $\langle a, b\rangle$ except for a countable set and with respect to the norm $\|\ \|^*$.

THEOREM 8. *Let $(X, \|\ \|)$, $(Y, \|\ \|^*)$ be F-spaces, $X \subset Y$. If $x \in V^w(X)$ and has property* A, *then it has property* B.

Proof. Denote

$$\omega(t) = \limsup_{\delta\to 0} \{\|x(t')-x(t'')\|^*:\ t',\ t''\in\langle t-\delta,\ t+\delta\rangle\}.$$

We assert that the set $A_k = \{t:\ \omega(t) \geqslant 1/k\}$, $k = 1, 2, \ldots$, is finite. If the set A_k was infinite, then there would exist some t_0 and a sequence of different $t_i \to t_0$, $t_i \in A_k$; we may assume, for instance, $t_i > t_0$. Let us choose a sequence of disjoint intervals $\langle t_i-\delta_i,\ t_i+\delta_i\rangle$ and points $t_i',\ t_i''\in\langle t_i-\delta_i,\ t_i+\delta_i\rangle$ in such a way that $\|x(t_i')-x(t_i'')\|^* \geqslant 1/2k$ for $i = 1, 2, \ldots$ However, $t_i' \to t_0$, $t_i'' \to t_0$ so, by property A, $\|x(t_i')-x(t_i'')\|^* \to 0$ and we get a contradiction. The set $\bigcup_{k=1}^{\infty} A_k$ is at most countable and for $t\in\langle a, b\rangle\setminus\bigcup_1^\infty A_k$ the function x is continuous with respect to $\|\ \|^*$.

Theorem 9. *Let $(X, \|\ \|)$ be an F-space, $(Y, \|\ \|^*)$ either a Banach or an F-space with a basis and let $\|\ \|$ be not weaker than $\|\ \|^*$ in X, $X \subset Y$.*

A function $x\in V^w(X)$ has property A *if and only if the set of its values is relatively compact in $(Y, \|\ \|^*)$.*

Proof. Necessity is obvious: assuming A from every sequence (t_n) a partial sequence (t_{n_i}), $t_{n_i} \nearrow t_0$ (or $t_{n_i} \searrow t_0$) can be extracted such that $x(t_{n_i}) \to x_0$ with respect to $\|\ \|^*$.

To prove the sufficiency suppose that for $a < t_0 < b$ the left-sided limit of $x(t)$ in $(Y, \|\ \|^*)$ does not exist. Then there exists a sequence $t_i \nearrow t_0$ and a sequence of disjoint intervals $\langle t_{i_k},\ t_{j_k}\rangle$ such that $\|x(t_{i_k})-x(t_{j_k})\|^* > \varepsilon_0$ for $k = 1, 2, \ldots$ Since $x\in V^w(X)$ so the series $\sum_1^\infty \big(x(t_{i_k})-x(t_{j_k})\big)$ is perfectly bounded in $(X, \|\ \|)$ and thereby in $(Y, \|\ \|^*)$, using Lemma 1 for $(Y, \|\ \|^*)$ we get $\|x(t_{i_k})-x(t_{j_k})\|^* \to 0$ and a contradiction.

The following simple theorem has already been proved in [3] but as it concerns property A and for the reader's convenience we formulate it here again with the proof.

Theorem 10. *Let $(X, \|\ \|)$, $(Y, \|\ \|^*)$ be some F-spaces $X \subset Y$.*

Each function $x\in V^w(X)$ has property A *if and only if the condition $O(X, Y)$ holds.*

Proof. If $O(X, Y)$ fails, then there exists a series $\sum_1^\infty x_i$ perfectly bounded in $(X, \|\ \|)$ and such that $\|x_i\|^* \geqslant \varepsilon_0$ for $i = 1, 2, \ldots$ Let (w_i) be the sequence of rational numbers in $\langle a, b\rangle$ arbitrarily arranged. Define a function x setting $x(t) = x_i$ for $t = w_i$, $x(t) = 0$ elsewhere in $\langle a, b\rangle$. It is easy to notice that $x\in V^w(x)$ but neither left-sided nor right-sided limits exist at any point of $\langle a, b\rangle$ with respect to $\|\ \|^*$ convergence.

Sufficiency is obtained by reasoning analogous to the proof of the preceding theorem and application of $O(X, Y)$ instead of Lemma 1.

THEOREM 11. *Let $(X, \|\ \|)$ be a Banach space, Ξ_0 – a countable set in the dual space Ξ. If $x \in V^w(X)$ then, except for a countable set, x is $*$-w-continuous with respect to Ξ_0.*

Proof. Let $(\xi_n) = \Xi_0$. We introduce a B_0-pseudonorm in X

$$\|x\|^* = \sum_1^\infty \frac{1}{2^n} \frac{|\xi_n(x)|}{1+|\xi_n(x)|}.$$

Let $\omega(t)$, A_k, etc., have the same meaning as in the proof of Theorem 8. We are reasoning analogously to its arguments that if A_k is infinite then $\|x(t_i') - x(t_i'')\|^* > 1/2k$ for some t_0, t_i', $t_i'' \to t_0$ and disjoint $\langle t_i', t_i'' \rangle$. However, since the perfect boundedness of the series $\sum\limits_1^\infty (x(t_i') - x(t_i''))$ in $(X, \|\ \|)$ implies $\xi_n(x(t_i') - x(t_i'')) \to 0$ for $n = 1, 2, \ldots$, that is, $\|x(t_i') - x(t_i'')\|^* \to 0$, we have a contradiction. For $t \in \langle a, b \rangle \setminus \bigcup\limits_1^\infty A_k$, x is $*$-w-continuous with respect to Ξ_0.

THEOREM 12. *Let $(X, \|\ \|)$, $(Y, \|\ \|^*)$ be Banach spaces, $X \subset Y$, T_0 – a countable fundamental set in the dual space of $(Y, \|\ \|^*)$. Let the norm $\|\ \|$ be not weaker than $\|\ \|^*$. If $x \in V^w(X)$ and the set of its values is relatively $*$-w-compact with respect to T_0 in $(Y, \|\ \|^*)$, then x has property* A *for the $*$-w-convergence in $(Y, \|\ \|^*)$ with respect to T_0.*

Proof. Let $(\tau_n) = T_0$. Define a norm

$$\|x\|^0 = \sum_1^\infty \frac{1}{2^n} \frac{|\tau_n(x)|}{1+|\tau_n(x)|}.$$

Applying the reasoning analogous to the proof of Theorem 9 we find out that $\|x(t_p) - x(t_q)\|^0 \to 0$ as t_p, $t_q \to t$. However, the assumption of compactness of the set $\{x(t)\}$ in $(Y, \|\ \|^*)$ for the $*$-w-convergence with respect to T_0 means compactness with respect to the norm $\|\ \|^0$. This and the Cauchy condition yield $\|x(t_p) - x_0\|^0 \to 0$. Note that in Lemma 1 one can take a B_0 space instead of Banach space and restrict oneself to the fundamental set of functionals.

THEOREM 13. *Let $X = C\langle c, d \rangle$, $x \in V^w(X)$* so *$x_t(\cdot) = x(t, u)$ is continuous with respect to u for each t. $x = x(t, u)$ has the following properties:*

($*$) *for each u the function $x(t, u)$ is continuous except for a countable set (depending on u),*

($**$) *let $y_t^+(u)$, $y_t^-(u)$ denote, respectively, the right-sided and left-sided limit of $x(t, u)$ at t.*

There exists a countable set D such that, for $t_0 \in \langle a, b \rangle \setminus D$, $x(t_0, u) = y_{t_0}^+(u) = y_{t_0}^-(u)$ except for a set of measure 0 (depending on t_0), i.e., $x(\cdot, u)$ are continuous for almost all u on $\langle a, b \rangle \setminus D$.

Proof. From finiteness of $v^w(x)$ there follows $|x(t, u)| \leqslant K$ so $y_t^+(u)$ is a measurable and bounded function. Let us choose the set Ξ_0 of functionals over $C\langle c, d\rangle$ of the form $\int_c^d h(u)y(u)\,du$, $y \in C\langle c, d\rangle$, Ξ_0 countable and norming with $h(u)$ integrable. According to Theorem 11, except for a countable set, $\lim_{t\to t_0^+} \int_c^d x(t, u)h(u)\,du = \int_c^d x(t_0, u)h(u)\,du$. But there is also $\lim_{t\to t_0^+} \int_c^d x(t, u)h(u)\,du = \int_c^d y_{t_0}^+(u)h(u)\,du$. Hence $x(t_0, u) = y_{t_0}^+(u)$ for almost all u. Reasoning analogously for $t \to t_0^-$ we get $x(t_0, u) = y_{t_0}^-(u)$ except for a countable set and for almost all u.

3.1. THEOREM 14. *Let x: $\langle a, b\rangle \to C$, the set of sums $\sum_1^n \eta_i (x(t_i) - x(t_{i-1}))$ for arbitrary n, $\eta_i = 0, 1$ and any π be relatively compact in C with respect to convergence in measure. Then $x \in V^w(C)$ and x has the property* A *with respect to the norm in C.*

Proof. For some $\langle\alpha, \beta\rangle \subset \langle a, b\rangle$ denote by $v^w(x; \alpha, \beta)$ the weak variation of x (here $X = C$). We are going to show that $v^w(x; a, b) < \infty$. Suppose $v^w(x; a, b) = \infty$ and note the following lemma: if for some t_0, $a < t_0 < b$, $v^w(x; a, t_0) < \infty$, $v^w(x; t_0, b) < \infty$, then $v^w(x; a, b) < \infty$ (cf. [2]). The set $\{x(t)\}$ is bounded in C. Otherwise there would exist a sequence of non-overlapping intervals $\langle\alpha_i, \beta_i\rangle$ such that the series

$$\sum_1^\infty (x(\beta_i) - x(\alpha_i)) \tag{3.1}$$

would be divergent in C. Using the assumption on x and Theorem 5, we get a contradiction. Using the lemma given above and boundedness of the function x we can (cf. [2]), define a sequence of intervals $\langle\alpha_i, \beta_i\rangle$ for which series (3.1) is divergent in C and so, as before, Theorem 5 leads to a contradiction. Property A follows by a simple argument like in Theorem 9 and by repeated application of Theorem 5.

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