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Pseudotopologies for modular spaces

Abstract. For generalized modular spaces ([6]) a linear pseudotopology ([1], [2], [4]) is defined. A class of linear-pseudotopological spaces satisfying a certain condition (called *condition* (M)) and including generalized modular spaces is introduced. Balanced linear-pseudotopological spaces satisfying condition (M) are considered. Criteria for linear-pseudotopological spaces satisfying (M) to be linear-topological are given. So-called Orlicz and Wiweger topologies are constructed for linear-pseudotopological spaces satisfying condition (M).

When discussing the Orlicz spaces (for instance, [5]) we introduce a certain functional having the properties which suggest the definition of a more general notion called a *pseudomodular* (in particular a *modular*). There are some versions (for example, in spaces with or without a partial order) and special cases of this notion ([9], [10] etc.). We shall use linear spaces without an order. Using a pseudomodular, we can define a modular convergence. In general, this convergence has no topological character. A pseudomodular generates a filter which has a countable base. This filter is like the neighbourhood filter of zero in a linear-topological space. The filters (and their bases) generated by pseudomodulars have the algebraical and set-theoretical properties which are needed in definitions of new notions called a *modular base* ([6], [7]) and a *modular filter*. A modular filter needs not have any countable bases. Further generalized modular spaces are defined. In modular spaces F-pseudonorms or pseudonorms are introduced and, analogously, in the generalized modular spaces certain linear topologies can be constructed. The modular convergence can have no topological character but in fundamental cases ([6], [7] and [8] with certain changes) it has a pseudotopological description and the corresponding spaces may be regarded as special cases of linear-pseudotopological spaces.

1. Pseudotopological spaces. Let X be a non-empty set and let

$$2^X = \{A\colon A \subset X\}.$$

1.1. A non-void family $\mathscr{F} \subset 2^X$ is called a *filter* in X if the following conditions are satisfied:

$\emptyset \notin \mathscr{F}$;

if $F \in \mathscr{F}$, $F \subset G \subset X$, then $G \in \mathscr{F}$;

if F, $G \in \mathscr{F}$, then $F \cap G \in \mathscr{F}$.

Here the symbol $\emptyset$ denotes the empty set.

1.2. A non-void family $\mathscr{B} \subset 2^X$ is called a *filter-base* in X if the following conditions are satisfied:

$\emptyset \notin \mathscr{B}$;

if A, $B \in \mathscr{B}$, then there exists a set $C \in \mathscr{B}$ such that $C \subset A \cap B$.

A filter in X is a filter-base in X (and in every $Y \supset X$). If $\mathscr{B}$ is a filter-base in X, then the family

$$[\mathscr{B}] = \{A \subset X\colon \text{there exists a set } B \in \mathscr{B} \text{ such that } A \supset B\} = [\mathscr{B}]_X$$

is a filter in X. If $\mathscr{F} = [\mathscr{B}]$, then we say that $\mathscr{B}$ is a *base of the filter* $\mathscr{F}$. A filter $[\{A\}]$ (defined in a set X), where $A \subset X$, $A \neq \emptyset$, will be denoted by $[A]$, a filter $[\{x\}]$, where $x \in X$ – by $[x]$.

1.3. The set $F(X)$ of all filters in $X \neq \emptyset$ can be partially ordered by the inclusion. Then for every family $\{\mathscr{F}_j\}_{j \in J} \subset F(X)$ (where $J \neq \emptyset$) $\inf\limits_{j \in J} \mathscr{F}_j \in F(X)$ exists and

$$\inf_{j \in J} \mathscr{F}_j = \bigcap_{j \in J} \mathscr{F}_j = \{\bigcup_{j \in J} F_j\colon F_j \in \mathscr{F}_j,\ j \in J\}.$$

Moreover, if the family $\{\mathscr{F}_j\}_{j \in J}$ satisfies the condition: for arbitrary sets $F_1, \ldots, F_n \in \bigcup\limits_{j \in J} \mathscr{F}_j$ (n is a positive integer) the set $F_1 \cap \ldots \cap F_n$ is non-empty, then $\sup\limits_{j \in J} \mathscr{F}_j \in F(X)$ exists and

$$\sup_{j \in J} \mathscr{F}_j = \{F_1 \cap \ldots \cap F_n\colon n \in \mathbf{N},\ F_1, \ldots, F_n \in \bigcup_{j \in J} \mathscr{F}_j\},$$

where $\mathbf{N}$ denotes the set of all positive integers.

1.4. Let non-empty sets X, Y and a mapping $f\colon X \to Y$ be given. Then for a filter $\mathscr{F} \in F(X)$ the filter

$$f(\mathscr{F}) = [\{f(F)\colon F \in \mathscr{F}\}]$$

can be defined in Y ($f(F) = \{f(x)\colon x \in F\}$).

1.5. The mapping $\tau\colon X \to 2^{F(X)}$ is called a *pseudotopology* in X ([1], [2], [3]) if the following conditions are satisfied:

if $F(X) \ni \mathscr{F} \supset \mathscr{G} \in \tau(x)$, then $\mathscr{F} \in \tau(x)$;

if $\mathscr{F}, \mathscr{G} \in \tau(x)$, then $\mathscr{F} \cap \mathscr{G} \in \tau(x)$;

$[x] \in \tau(x)$

for every $x \in X$.

If τ is a pseudotopology in X, then the pair (X, τ) (also X) is called a *pseudotopological space*. The filters from $\tau(x)$ are called *convergent to* x in the space (X, τ) (or with respect to the pseudotopology τ).

1.6. Let $\mathscr{V}_x$ be the neighbourhood filter of $x \in X$ for a topology $\mathscr{T}$ in X and let

$$\tau_{\mathscr{T}}(x) = \{\mathscr{F} \in F(X): \mathscr{F} \supset \mathscr{V}_x\}, \qquad \text{where } x \in X.$$

Then $\tau_{\mathscr{T}}$ is a pseudotopology in X.

If σ is a pseudotopology in X and there exists a topology $\mathscr{T}$ in X with $\tau_{\mathscr{T}} = \sigma$, then we say that σ is a *topology*. We shall identify the topology $\mathscr{T}$ with the pseudotopology $\tau_{\mathscr{T}}$. Hence, one may write $\tau_{\mathscr{T}}$ instead of $\mathscr{T}$ and $(X, \tau_{\mathscr{T}})$ instead of $(X, \mathscr{T})$.

The condition

$$\inf \tau(x) \in \tau(x), \qquad x \in X,$$

is necessary for the space (X, τ) to be topological.

1.7. Let (X, σ), (Y, τ) be pseudotopological spaces. We say that a mapping $f\colon X \to Y$ (or $f\colon (X, \sigma) \to (Y, \tau)$) is *continuous at a point* $x \in X$ if for every $\mathscr{F} \in \sigma(x)$ the filter $f(\mathscr{F})$ belongs to $\tau\big(f(x)\big)$.

For topological spaces this definition is equivalent to the classical one.

If a mapping $f\colon (X, \sigma) \to (Y, \tau)$ is continuous at every $x \in X$, then we say that f is *continuous* (in X, in (X, σ)).

1.8. The set $P(X)$ of all pseudotopologies in X can be partially ordered in the following way:

$\sigma \leqslant \tau$ if and only if $\sigma(x) \supset \tau(x)$ for every $x \in X$; $\sigma, \tau \in P(X)$.

For σ, $\tau \in P(X)$, $\sigma \leqslant \tau$, one may write $(X, \sigma) \leqslant (X, \tau)$.

Let $\mathscr{T}_1$, $\mathscr{T}_2$ be topologies in X. Then $\mathscr{T}_1 \leqslant \mathscr{T}_2$ (or $\tau_{\mathscr{T}_1} \leqslant \tau_{\mathscr{T}_2}$) means that $\mathscr{T}_1 \subset \mathscr{T}_2$.

The Cartesian product of pseudotopological spaces is defined similarly as for topological spaces.

1.9. Let (X, τ) be a pseudotopological space. We say that an MS-sequence $S = (x_j)_{j \in J}$ consisting of elements of X is convergent to $x \in X$ with respect to the pseudotopology τ (or in a space (X, τ)) if the filter

$$\mathscr{F}(S) = \big[\{\{x_j\colon j \geqslant j_0\}\colon j_0 \in J\}\big]$$

belongs to $\tau(x)$.

2. Linear-pseudotopological spaces. Let K be the field of real or complex numbers and let X be a linear space over K. We shall use the following notations:

$I_\delta = \{\lambda \in K\colon |\lambda| \leqslant \delta\}$, where $\delta \geqslant 0$;

$I = I_1$;

V – the neighbourhood filter of $0 \in K$ (the field K will be equipped with the usual topology);

$A \pm B = \{a \pm b: a \in A, b \in B\}$, where $A, B \subset X$;

$A \pm b = A \pm \{b\}$, where $A \subset X$, $b \in X$;

$\Lambda \cdot A = \{\lambda a: \lambda \in \Lambda, a \in A\}$, where $\Lambda \subset K$, $A \subset^{\bullet} X$;

$\Lambda \cdot a = \Lambda \cdot \{a\}$, where $\Lambda \subset K$, $a \in X$;

$\lambda \cdot A = \{\lambda\} \cdot A$, where $\lambda \in K$, $A \subset X$;

$\mathscr{F}_1 \pm \mathscr{F}_2 = [\{F_1 \pm F_2: F_1 \in \mathscr{F}_1, F_2 \in \mathscr{F}_2\}]$, where $\mathscr{F}_1, \mathscr{F}_2 \in F(X)$;

$\mathscr{F} \pm x = \mathscr{F} \pm [x]$, where $\mathscr{F} \in F(X)$, $x \in X$;

$\mathscr{G} \cdot \mathscr{F} = [\{G \cdot F: G \in \mathscr{G}, F \in \mathscr{F}\}]$, where $\mathscr{G} \in F(K)$, $\mathscr{F} \in F(X)$;

$\Lambda \cdot \mathscr{F} = [\Lambda] \cdot \mathscr{F}$, where $\emptyset \neq \Lambda \subset K$, $\mathscr{F} \in F(X)$;

$\lambda \cdot \mathscr{F} = [\lambda] \cdot \mathscr{F}$, where $\lambda \in K$, $\mathscr{F} \in F(X)$;

$\mathscr{G} \cdot x = \mathscr{G} \cdot [x]$, where $\mathscr{G} \in F(K)$, $x \in X$.

2.1. We say that $\tau \in P(X)$ is a *linear pseudotopology* in a linear space X ([1], [2], [4]) if the algebraic operations

$$+: X \times X \to X, \qquad \cdot : K \times X \to X$$

are continuous (where the space X is equipped with the pseudotopology τ). If τ is a linear pseudotopology in X, then the pair (X, τ) is called a *linear-pseudotopological space.*

Every linear pseudotopology τ in a linear space X satisfies the following conditions:

(α) if $F(X) \ni \mathscr{F} \supset \mathscr{G} \in \tau(0)$, then $\mathscr{F} \in \tau(0)$;

(β) if $\mathscr{F}, \mathscr{G} \in \tau(0)$, then $\mathscr{F} \cap \mathscr{G} \in \tau(0)$;

(1) if $\mathscr{F}, \mathscr{G} \in \tau(0)$, then $\mathscr{F} + \mathscr{G} \in \tau(0)$;

(2) if $\lambda \in K$, $\mathscr{F} \in \tau(0)$, then $\lambda \cdot \mathscr{F} \in \tau(0)$;

(3) if $\mathscr{F} \in \tau(0)$, then $V \cdot \mathscr{F} \in \tau(0)$;

(4) if $x \in X$, then $V \cdot x \in \tau(0)$;

$$\tau(x) = \tau(0) + x = \{\mathscr{F} + x: \mathscr{F} \in \tau(0)\} \quad \text{for } x \in X.$$

2.2. If a set $\tau(0) \subset F(X)$ (where X is a linear space) satisfies conditions (α), (β), (1)–(4) and if $\tau(x) = \tau(0) + x$ for $x \in X$, then τ is a linear pseudotopology in X.

2.3. Let σ, τ be linear pseudotopologies in X. Then $\sigma \leqslant \tau$ if and only if $\sigma(0) \supset \tau(0)$.

2.4. A linear pseudotopology τ is a linear topology if and only if $\inf \tau(0) \in \tau(0)$ (cf. with 1.6).

3. Balanced and equable linear-pseudotopological spaces. Let X be a real or complex linear space.

3.1. We say that a set $A \subset X$ is *balanced* if $\operatorname{bal} A = IA = A$.

3.2. Let us observe that:

if $A = \operatorname{bal} A$, $\lambda \in K$, then $\lambda A = \operatorname{bal}(\lambda A)$,

if $A = \operatorname{bal} A$, λ, $\mu \in K$, $|\lambda| \leqslant |\mu|$, then $\lambda A \subset \mu A$.

3.3. We say that a filter $\mathscr{F}$ in a linear space is *balanced* if $\operatorname{bal} \mathscr{F} = I\mathscr{F} = \mathscr{F}$.

Let us observe that for every filter $\mathscr{F}$ in a linear space the condition $I\mathscr{F} \subset \mathscr{F}$ holds.

3.4. If a filter $\mathscr{F}$ is balanced, λ, $\mu \in K$, $|\lambda| \leqslant |\mu|$, then $\mu\mathscr{F} \subset \lambda\mathscr{F}$ (cf. with 3.2).

3.5. A linear pseudotopology τ in a linear space X (and a space (X, τ)) is *balanced* ([4]) if the following condition is satisfied:

for every $\mathscr{F} \in \tau(0)$ there exists a $\mathscr{G} \in \tau(0)$ such that $\mathscr{F} \supset \mathscr{G} = I\mathscr{G}$ (an equivalent condition: if $\mathscr{F} \in \tau(0)$, then $I\mathscr{F} \in \tau(0)$).

3.6. For every filter $\mathscr{F}$ in a linear space the filter $I\mathscr{F}$ is contained in $V\mathscr{F}$. So, we have the following theorem:

Let a set $\tau(0) \subset F(X)$ (where X is a linear space) satisfy conditions (α), (β), (1), (2), (4) *from* 2.1 *and the condition*:

(3b) *if $\mathscr{F} \in \tau(0)$, then $I\mathscr{F} \in \tau(0)$.*

Moreover, let $\tau(x) = \tau(0) + x$ for $x \in X$. Then τ is a balanced linear pseudotopology in X.

3.7. We say that a filter $\mathscr{F}$ in a linear space is *equable* if $V\mathscr{F} = \mathscr{F}$ ([2], [4]).

3.8. We say that a linear pseudotopology τ in a linear space X is *equable* (and that a space (X, τ) is equable) if for every $\mathscr{F} \in \tau(0)$ there exists a filter $\mathscr{G} \in \tau(0)$ such that $\mathscr{F} \supset \mathscr{G} = V\mathscr{G}$ ([2], [4]).

3.9. Every equable filter is balanced; hence an equable linear-pseudotopological space is balanced.

The neighbourhood filter of zero in a linear-topological space is equable, so a linear-topological space is equable. There exist equable linear-pseudotopological spaces which are not linear-topological.

3.10. For a linear pseudotopology τ in a linear space X there exists a (unique) equable linear pseudotopology $\tau^{\#} \geqslant \tau$ such that for every equable linear pseudotopology $\sigma \geqslant \tau$ the condition $\tau^{\#} \leqslant \sigma$ holds. Moreover, we have

$$\tau^{\#}(0) = \{\mathscr{F} \in F(X): \text{there exists a filter } \mathscr{G} \in \tau(0) \text{ with } \mathscr{F} \supset V\mathscr{G}\}.$$

4. Generalized modular spaces. Let X be a linear space over the field K of real or complex numbers.

4.1. We say that a non-empty family $\mathscr{B} \subset 2^X$ is a *modular base* in X ([6], [7]) if the following conditions are satisfied:

(M1) for every sets $U_1, U_2 \in \mathscr{B}$ there exists a set $U \in \mathscr{B}$ such that $\Gamma(U) = \{\alpha x + \beta y\colon x, y \in U, |\alpha| + |\beta| \leqslant 1\} \subset U_1 \cap U_2$,

(M2) every set $U \in \mathscr{B}$ is absorbent in X (i.e., for each $x \in X$ there exists a number $\alpha \neq 0$ such that $\alpha x \in U$).

A modular base in X is a filter-base in X. If $\mathscr{B}$ is a modular base in X, then $[\mathscr{B}]$ is called a *modular filter* in X. Every base of a modular filter in X is a modular base in X.

4.2. Let $M(X)$ be the set of all modular bases in X. We define in $M(X)$ the following relations:

$\mathscr{B}_1 \prec \mathscr{B}_2$ if and only if there exists a number $\alpha \neq 0$ such that $[\mathscr{B}_1] \subset \alpha[\mathscr{B}_2]$,

$\mathscr{B}_1 \sim \mathscr{B}_2$ if and only if there exist numbers $\alpha_1, \alpha_2 \neq 0$ such that $\alpha_1[\mathscr{B}_2] \subset [\mathscr{B}_1] \subset \alpha_2[\mathscr{B}_2]$ (i.e., if and only if $\mathscr{B}_1 \prec \mathscr{B}_2$ and $\mathscr{B}_2 \prec \mathscr{B}_1$);

$\mathscr{B}_1, \mathscr{B}_2 \in M(X)$.

The relation $\prec$ is reflexive and transitive, $\sim$ is an equivalence relation. Let us observe that $\mathscr{B} \sim [\mathscr{B}]$ for every modular base $\mathscr{B}$.

4.3. Let $\mathscr{B} \in M(X)$ be given. We define

$$\mathscr{B}^{\sim} = \{\mathscr{B}_1 \in M(X)\colon \mathscr{B}_1 \sim \mathscr{B}\}.$$

If $\mathscr{B} \in M(X)$, then the pair $(X, \mathscr{B}^{\sim})$ is called a *generalized modular space* (shortly: a *modular space*).

4.4. Let us define in a generalized modular space a linear pseudotopology. We have the following theorem:

If $(X, \mathscr{B}^{\sim})$ is a generalized modular space, $\mathscr{B}_1 \in \mathscr{B}^{\sim}$,

$$\tau(0) = \{\mathscr{F} \in F(X)\colon \textit{there exists a } \lambda \in K \textit{ such that } \mathscr{F} \supset \lambda[\mathscr{B}_1]\},$$

$$\tau(x) = \tau(0) + x \quad \textit{for } x \in X,$$

then τ is a balanced linear pseudotopology in X; moreover, all bases from $\mathscr{B}^{\sim}$ generate the same pseudotopology τ.

4.5. In [6] a convergence of MS-sequences in modular spaces is defined in the following way: An MS-sequence $S = (x_j)_{j \in J}$ consisting of elements of a space X is called *convergent to* $x \in X$ with respect to the modular base $\mathscr{B}$ in X if there is a number $\alpha \neq 0$ such that for every $U \in \mathscr{B}$ there exists a $j_0 \in J$ such that for every $j \in J$, $j \geqslant j_0$, the condition $\alpha(x_j - x) \in U$ holds. Moreover, if $\mathscr{B}_1 \sim \mathscr{B}$, then S converges to x with respect to $\mathscr{B}$ if and only if S converges to x with respect to $\mathscr{B}_1$. So we introduce the definition: An MS-sequence S is *convergent to x in a space* $(X, \mathscr{B}^{\sim})$ if S converges to x with respect to $\mathscr{B}$ (or with respect to any base $\mathscr{B}_1 \in \mathscr{B}^{\sim}$).

We have constructed a pseudotopology in a modular space. Now let us observe that the above definition of a convergence of MS-sequences is equivalent to the one given in 1.9.

In the sequel a generalized modular space $(X, \mathscr{B}^{\sim})$ will be identified with the linear-pseudotopological space (X, τ) with τ defined as in 4.4.

5. Condition (M). Let X be a linear space over the filed K of real or complex numbers.

5.1. We say that a linear pseudotopology τ in X (and a space (X, τ)) *satisfies condition* (M) if there exists a filter $\mathscr{F} \in \tau(0)$ such that for every $\mathscr{G} \in \tau(0)$ the condition $\mathscr{G} \supset \lambda\mathscr{F}$ holds with a certain $\lambda \in K$.

5.1.1. Remark. The filter $\mathscr{F}$ from the above condition is a subset of the filter $[0]$.

Indeed, the case $X = \{0\}$ is trivial. So, let $X \neq \{0\}$. We take an $x \in X$, $x \neq 0$. $Vx \in \tau(0)$, so there exists a $\lambda \in K$ such that $Vx \supset \lambda\mathscr{F}$. The case $\lambda = 0$ implies that $Vx \supset [0]$. So $\{0\} \supset I_\varepsilon \cdot x$ for a certain $\varepsilon > 0$. But it is impossible. Therefore $\lambda \neq 0$ and $Vx \supset \mathscr{F}$. Moreover, $[0] \supset Vx$. So $[0] \supset \mathscr{F}$.

5.1.2. Remark. In condition (M) we may assume that $\lambda \neq 0$.

Indeed, if $\lambda = 0$, then we obtain $\mathscr{G} \supset [0]$; so $\mathscr{G} = [0]$. But $[0] \supset \mathscr{F} = 1 \cdot \mathscr{F}$ (Remark 5.1.1).

5.2. Let us observe that a generalized modular space is a balanced linear-pseudotopological space satisfying condition (M).

5.3. Let (X, τ) be a balanced linear-pseudotopological space satisfying condition (M). We have

(M) there exists a filter $\mathscr{F} \in \tau(0)$ such that for every $\mathscr{G} \in \tau(0)$ there is a $\lambda \neq 0$ for which $\mathscr{G} \supset \lambda\mathscr{F}$.

5.3.1. Remark. The filter $\mathscr{F}$ need not be balanced.

Example. $X = R$ over R (the set of all real numbers), $\mathscr{F} = [\langle -1, 2\rangle]$ ($\langle a, b\rangle$ denotes a closed segment).

$\mathscr{F}$ need not be balanced but in (M) the filter $I\mathscr{F}$ may be taken instead of $\mathscr{F}$. Indeed, we have $\mathscr{F} \supset I\mathscr{F}$. So, if $\mathscr{G} \supset \lambda\mathscr{F}$, then $\mathscr{G} \supset \lambda I\mathscr{F}$. Moreover, there is a $\mu \neq 0$ such that $I\mathscr{F} \supset \mu\mathscr{F}$. Hence $\mathscr{G} \supset \lambda I\mathscr{F}$ implies that $\mathscr{G} \supset \lambda\mu\mathscr{F}$.

5.3.2. Theorem. *Let (X, τ) be a balanced linear-pseudotopological space satisfying* (M) *with $\mathscr{F}$. Then the following conditions hold*:

(bM1) *there exists a $\mu \neq 0$ such that $I\mathscr{F} + I\mathscr{F} \supset \mu\mathscr{F}$,*

(bM2) *$Vx \supset \mathscr{F}$ for every $x \in X$.*

Proof. The filter $I\mathscr{F} \in \tau(0)$. So $I\mathscr{F} + I\mathscr{F} \in \tau(0)$. Therefore $I\mathscr{F} + I\mathscr{F} \supset \mu\mathscr{F}$, where μ is a certain number different from 0. Now let an $x \in X$ be given. $Vx \in \tau(0)$, so there is a $\lambda \neq 0$ such that $Vx \supset \lambda\mathscr{F}$. It means that $Vx \supset \mathscr{F}$.

5.3.3. THEOREM. *Let $\mathscr{F}$ be a filter in a linear space X. Moreover, let $\mathscr{F}$ satisfy conditions* (bM1) *and* (bM2). *Then the equalities*

$$\tau(0) = \{\mathscr{G} \in F(X)\colon\ \mathscr{G} \supset \lambda\mathscr{F} \text{ for a certain } \lambda \neq 0\},$$

$$\tau(x) = \tau(0) + x \quad (\text{where } x \in X)$$

define a balanced linear pseudotopology in X satisfying condition (M).

Proof. We shall apply Theorem 3.6. Conditions (α), (2), (4) are obvious. We consider conditions (1) and (β). Let filters $\mathscr{G}_1$, $\mathscr{G}_2 \in \tau(0)$ be given. Then there exist $\lambda_1, \lambda_2 \neq 0$ such that $\mathscr{G}_1 \supset \lambda_1 \mathscr{F}$, $\mathscr{G}_2 \supset \lambda_2 \mathscr{F}$. For $k = 1, 2$ we have $\mathscr{G}_k \supset \lambda_k \mathscr{F} \supset \lambda_k I\mathscr{F} \supset (|\lambda_1|+|\lambda_2|)\cdot I\mathscr{F}$ (see 3.4). So $\mathscr{G}_1+\mathscr{G}_2 \supset (|\lambda_1|+|\lambda_2|)\times \times(I\mathscr{F}+I\mathscr{F}) \supset (|\lambda_1|+|\lambda_2|)\mu\mathscr{F}$. Therefore the filter $\mathscr{G}_1+\mathscr{G}_2$ belongs to $\tau(0)$. Moreover, we have $\mathscr{G}_1 \cap \mathscr{G}_2 \supset (|\lambda_1|+|\lambda_2|)\cdot I\mathscr{F} \supset (|\lambda_1|+|\lambda_2|)\cdot(I\mathscr{F}+I\mathscr{F}) \supset (|\lambda_1| +|\lambda_2|)\mu\mathscr{F}$. So $\mathscr{G}_1 \cap \mathscr{G}_2 \in \tau(0)$.

(3b): Let $F(X) \ni \mathscr{G} \supset \lambda\mathscr{F}$, $\lambda \neq 0$. Since $I\mathscr{G} \supset \lambda I\mathscr{F} \supset \lambda(I\mathscr{F}+I\mathscr{F}) \supset \lambda\mu\mathscr{F}$, the filter $I\mathscr{G}$ belongs to $\tau(0)$. Obviously, the pseudotopology τ satisfies condition (M).

5.3.4. Remark. Condition (bM1) is equivalent to the conjunction of the following ones:

(1bM1) there is a $\lambda \neq 0$ such that $\mathscr{F}+\mathscr{F} \supset \lambda\mathscr{F}$,

(2bM1) there is a $\xi \neq 0$ such that $I\mathscr{F} \supset \xi\mathscr{F}$.

Indeed, from these conditions we have $I\mathscr{F}+I\mathscr{F} \supset \xi\mathscr{F}+\xi\mathscr{F} = \xi(\mathscr{F}+ +\mathscr{F}) \supset \xi\lambda\mathscr{F}$, from (bM1) we get: $\mathscr{F}+\mathscr{F} \supset I\mathscr{F}+I\mathscr{F} \supset \mu\mathscr{F}$, $I\mathscr{F} \supset I\mathscr{F}+I\mathscr{F} \supset \mu\mathscr{F}$.

Moreover, if condition (2bM1) is satisfied, then condition (bM2) is equivalent to the following one:

every set $F \in \mathscr{F}$ is absorbent in X.

First, let condition (bM2) be satisfied and let arbitrary $F \in \mathscr{F}$, $x \in X$ be given. We have $Vx \supset \mathscr{F}$. So there exists an $\varepsilon > 0$ such that $F \supset I_\varepsilon x$. Thus $\varepsilon x \in F$ (condition (2bM1) has not been applied). Now, let each $F \in \mathscr{F}$ be absorbent in X and let the filter $\mathscr{F}$ satisfy condition (2bM1). For every $x \in X$, $F \in \mathscr{F}$ there are $G \in \mathscr{F}$, $\alpha \neq 0$ such that $F \supset \frac{1}{\xi} IG$ and $\alpha x \in G$. So $F \supset \frac{1}{\xi} IG \supset \frac{\alpha}{\xi} Ix \in Vx$.

5.3.5. Remark. In [8] a certain notion called a *premodular base* is considered. Let S be a (real) linear lattice. We say that a filter-base $\mathscr{B} \subset 2^S$ is a *premodular base* if the following condition is satisfied:

(c) there exists a $\beta \neq 0$ such that for every set $U \in \mathscr{B}$ there is a set $U' \in \mathscr{B}$ with $\beta(N(U')+N(U')) \subset U$, where $N(U') = \{y \in S\colon |y| \leqslant |x| \text{ for a certain } x \in U'\}$.

Let us observe that condition (bM1) can be written in the following way (real and complex cases):

there exists a $\beta \neq 0$ such that for each $U \in \mathscr{F}$ there is a $U' \in \mathscr{F}$ with $\beta(IU' + IU') \subset U$.

This condition is like condition (c). Here the set IU' is instead of $N(U')$.

5.4. Let us give some theorems on spaces satisfying condition (M).

5.4.1. THEOREM. *A linear-pseudotopological space satisfying condition* (M) *is linear-topological if and only if it is equable.*

Proof. The necessity is obvious (3.9). Let us verify the sufficiency. Let (X, τ) be the considered equable space and let $\mathscr{F} \in \tau(0)$ be the fixed filter appearing in condition (M). Since the pseudotopology τ is equable, a filter $\mathscr{G}_1 \in \tau(0)$ can be chosen such that $\mathscr{F} \supset V\mathscr{G}_1$. Moreover, there exists a $\nu \neq 0$ such that $V\mathscr{G}_1 \supset \nu\mathscr{F}$. So $\mathscr{F} \supset V\mathscr{G}_1 = \frac{1}{\nu} V\mathscr{G}_1 \supset \mathscr{F}$, and hence $\mathscr{F} = V\mathscr{G}_1$. Thus for each $\lambda \neq 0$ the equality $\lambda\mathscr{F} = \mathscr{F}$ holds. It means that

$$\tau(0) = \{\mathscr{G} \in F(X) \colon \mathscr{G} \supset \mathscr{F}\}.$$

So, by virtue of 2.4, our space is linear-topological.

5.4.2. COROLLARY. *A generlized modular space is linear-topological if and only if it is equable.*

5.4.3. THEOREM. *Let a linear-pseudotopological space (X, τ) satisfy condition* (M) *with a filter $\mathscr{F}$. The space (X, τ) is linear-topological if and only if there exist numbers μ, $\nu \neq 0$ such that $|\mu| < |\nu|$ and $\mu\mathscr{F} \subset \nu\mathscr{F}$.*

Proof. The necessity is obvious because if (X, τ) is linear-topological, then the filter $\mathscr{F}$ is equable, and for every $\lambda \neq 0$ the condition $\lambda\mathscr{F} = \mathscr{F}$ is satisfied. Now we verify the sufficiency. The conditions $\mu\mathscr{F} \subset \nu\mathscr{F}$, μ, $\nu \neq 0$, $|\mu| < |\nu|$ imply the following ones: $\mathscr{F} \subset \frac{\nu}{\mu}\mathscr{F}$, $|\nu/\mu| > 1$. The filter $V\mathscr{F}$ converges to 0, so $V\mathscr{F} \supset \mathscr{F}$. Now let us take an arbitrary set $A \in V\mathscr{F}$. Then $A \supset \varepsilon IF$ for some $\varepsilon > 0$, $F \in \mathscr{F}$. $|\nu/\mu| > 1$, so there is a positive integer n such that $\varepsilon|\nu/\mu|^n > 1$. Since the filter $\mathscr{F}$ is a subset of $\frac{\nu}{\mu}\mathscr{F}$, $F \supset (\nu/\mu)^n F'$ for a certain set $F' \in \mathscr{F}$. Therefore $A \supset \varepsilon IF \supset \varepsilon(\nu/\mu)^n IF' \supset F'$. Hence $A \in \mathscr{F}$. It means that $V\mathscr{F} \subset \mathscr{F}$. Moreover, $\mathscr{F} \subset V\mathscr{F}$, so $\mathscr{F} = V\mathscr{F}$. Therefore our space is linear-topological.

5.4.4. COROLLARY. *Let a linear-pseudotopological space (X, τ) satisfy condition* (M) *with a filter $\mathscr{F}$. Then the space (X, τ) is linear-topological if and only if $\mu\mathscr{F} = \nu\mathscr{F}$ for some μ, $\nu \neq 0$ with $|\mu| \neq |\nu|$.*

5.4.5. THEOREM. *Let a linear-pseudotopological space (X, τ) satisfy condition* (M). *Then there exists a* (*unique*) *linear topology $\tau^\vee \geqslant \tau$ such that for every linear topology $\mathscr{T} \geqslant \tau$ the condition $\tau^\vee \leqslant \mathscr{T}$ holds. Moreover, $\tau^\vee = \tau^\#$.*

Proof. It is obvious that $\tau^{\#} \geqslant \tau$ (3.10). Let $\mathscr{F}$ be the fixed filter of condition (M) and let an arbitrary $\mathscr{G} \in \tau^{\#}(0)$ be given. Then there exists a filter $\mathscr{G}_1 \in \tau(0)$ such that $\mathscr{G} \supset V\mathscr{G}_1$. Moreover, $\mathscr{G}_1 \supset \lambda\mathscr{F}$ for some $\lambda \neq 0$. So $\mathscr{G} \supset V\mathscr{G}_1 \supset \lambda V\mathscr{F} = V\mathscr{F}$. Furthermore, the conditions $\mathscr{H} \in F(X)$, $\mathscr{H} \supset V\mathscr{F}$ imply $\mathscr{H} \in \tau^{\#}(0)$, so

$$\tau^{\#}(0) = \{\mathscr{G} \in F(X):\ \mathscr{G} \supset V\mathscr{F}\}.$$

Therefore $\tau^{\#}$ is a linear topology in X. Now let an arbitrary linear topology $\mathscr{T} \geqslant \tau$ be given. The topology $\mathscr{T}$ is equable (3.9), so we get $\mathscr{T} \geqslant \tau^{\#}$ (3.10). The uniqueness of the topology $\tau^{\vee}$ is obvious.

5.4.6. Remark. There exist linear pseudotopologies τ for which there is no linear topology $\tau^{\vee} \geqslant \tau$ satisfying the following condition:

if a linear topology $\mathscr{T} \geqslant \tau$, then $\mathscr{T} \geqslant \tau^{\vee}$.

Example. X is an infinite-dimensional linear space,

$$\tau(0) = \{\mathscr{F} \in F(X):\ \mathscr{F} \supset Vx_1 + \ldots + Vx_n \text{ for certain } x_1, \ldots, x_n \in X\},$$
$$\tau(x) = \tau(0) + x \quad \text{for } x \in X.$$

Here there is no linear topology which would be $\geqslant \tau$. Let us observe that $\tau^{\#} = \tau$ and $\tau^{\#}$ is not a linear topology.

5.4.7. Remark. Let a balanced linear pseudotopology τ (in a certain linear space over the field K) satisfy condition (M) with a filter $\mathscr{F}$. Then $\sup\{\varepsilon\mathscr{F}:\ \varepsilon > 0\}$ is the neighbourhood filter of 0 for the topology $\tau^{\vee} = \tau^{\#}$, i.e.

$$V\mathscr{F} = \sup\{\varepsilon\mathscr{F}:\ \varepsilon > 0\}.$$

Moreover, if $\varepsilon_n \neq 0$, $\varepsilon_n \to 0$ as $n \to \infty$, then

$$V\mathscr{F} = \sup\{\varepsilon_n\mathscr{F}:\ n \in N\}.$$

Indeed, we have

$$V\mathscr{F} = [\{\varepsilon IF:\ \varepsilon > 0,\ F \in \mathscr{F}\}] = \sup\{\varepsilon I\mathscr{F}:\ \varepsilon > 0\}.$$

Since there exists a $\mu > 0$ such that $I\mathscr{F} \subset \mathscr{F} \subset \mu I\mathscr{F}$, we get

$$\sup\{\varepsilon I\mathscr{F}:\ \varepsilon > 0\} \subset \sup\{\varepsilon\mathscr{F}:\ \varepsilon > 0\} \subset \sup\{\varepsilon\mu I\mathscr{F}:\ \varepsilon > 0\}$$
$$= \sup\{\varepsilon I\mathscr{F}:\ \varepsilon > 0\}.$$

So $\sup\{\varepsilon\mathscr{F}:\ \varepsilon > 0\} = V\mathscr{F}$ (see also 3.1 in [6]).

Now let an $\varepsilon > 0$ be given. For an ε_{n_0} with $|\varepsilon_{n_0}| < \varepsilon\mu$ we have $\varepsilon\mathscr{F} \subset \varepsilon\mu I\mathscr{F} \subset \varepsilon_{n_0}\mathscr{F} \subset \sup\{\varepsilon_n\mathscr{F}:\ n \in N\}$. Therefore $\sup\{\varepsilon\mathscr{F}:\ \varepsilon > 0\} \subset \sup\{\varepsilon_n\mathscr{F}:\ n \in N\}$. Moreover, from $\varepsilon_n\mathscr{F} \subset \varepsilon_n\mu I\mathscr{F} \subset |\varepsilon_n\mu|\,\mathscr{F}$ $(n \in N)$ we get $\sup\{\varepsilon_n\mathscr{F}:\ n \in N\} \subset \sup\{\varepsilon\mathscr{F}:\ \varepsilon > 0\}$. Therefore $\sup\{\varepsilon\mathscr{F}:\ \varepsilon > 0\} = \sup\{\varepsilon_n\mathscr{F}:\ n \in N\}$.

5.4.8. THEOREM. *Let (X, τ) be a linear-pseudotopological space. Then there exists a (unique) linear topology $\tau^{\wedge} \leqslant \tau$ such that for every linear topology $\mathcal{T} \leqslant \tau$ the condition $\mathcal{T} \leqslant \tau^{\wedge}$ holds.*

Proof. From [4] we get that

$$\sup\{\mathcal{G} \in F(X)\colon\ \mathcal{G} \subset \mathcal{U} \cap (\mathcal{G}+\mathcal{G}) \cap V\mathcal{G}\},$$

where $\mathcal{U} = \inf \tau(0)$, is the neighbourhood filter of 0 for the topology $\tau^{\wedge}$. Here we shall give another construction of this filter (see [6]). It is known that $\mathcal{V} \in F(X)$ is the neighbourhood filter of 0 for a linear topology in X if and only if the following conditions are satisfied:

(LT1) $\mathcal{V} \subset \mathcal{V}+\mathcal{V}$,

(LT2) $\mathcal{V} \subset I\mathcal{V}$,

(LT3) $\mathcal{V} \subset Vx$ for every $x \in X$.

Let $\mathcal{U} = \inf \tau(0)$ and $\mathcal{V} = [\{\bigcup_{n=1}^{\infty} (A_1 + \ldots + A_n)\colon A_1, A_2, \ldots \in I\mathcal{U}\}]$. It is easy to see that $\mathcal{V} \subset I\mathcal{U}$. We shall show that $\mathcal{V}$ is the neighbourhood filter of 0 for a linear topology in X. Let an arbitrary $x \in X$ be given. We have $Vx \supset \mathcal{U} \supset I\mathcal{U} \subset \mathcal{V}$, so condition (LT3) is satisfied. Now let a set $V \in \mathcal{V}$ be given. Then there exist $U_1, U_2, \ldots \in \mathcal{U}$ such that $V \supset \bigcup_{n=1}^{\infty} (IU_1 + \ldots + IU_n)$. By virtue of the equality

$$\bigcup_{n=1}^{\infty} (IU_1 + \ldots IU_n) = I \cdot \bigcup_{n=1}^{\infty} (IU_1 + \ldots + IU_n)$$

we infer that $V \in I\mathcal{V}$. So condition (LT2) is satisfied for the considered filter. On the other hand, for the same sets $V, U_1, U_2, \ldots$ we get

$$V \supset \bigcup_{n=1}^{\infty} (IU_1 + \ldots + IU_n) \supset \bigcup_{k=1}^{\infty} \big(I(U_1 \cap U_2) + \ldots + I(U_{2k-1} \cap U_{2k})\big) + \\ + \bigcup_{k=1}^{\infty} \big(I(U_1 \cap U_2) + \ldots + I(U_{2k-1} \cap U_{2k})\big) \in \mathcal{V}+\mathcal{V},$$

so condition (LT1) is satisfied. Let us denote by $\mathcal{T}_1$ the linear topology for which $\mathcal{V}$ is the neighbourhood filter of 0. Since $\mathcal{V} \subset I\mathcal{U} \subset \mathcal{U}$, $\mathcal{T}_1 \leqslant \tau$. Now let $\mathcal{W}$ be the neighbourhood filter of 0 for a linear topology $\mathcal{T}$ in X and let $\mathcal{T} \leqslant \tau$. Obviously, we have $\mathcal{W} \subset \mathcal{U}$ and $I\mathcal{W} = \mathcal{W}$. We shall consider the filter

$$\mathcal{Y} = [\{\bigcup_{n=1}^{\infty} (W_1 + \ldots + W_n)\colon W_1, W_2, \ldots \in \mathcal{W}\}].$$

The condition $\mathcal{W} \subset I\mathcal{U}$ implies $\mathcal{Y} \subset \mathcal{V}$. Let a set $W \in \mathcal{W}$ be given. Since $\mathcal{W} \subset \mathcal{W}+\mathcal{W}$, there exist sets $W_1, W_2, \ldots \in \mathcal{W}$ satisfying the conditions $W \supset W_1 + W_1$, $W_1 \supset W_2 + W_2, \ldots, W_{n-1} \supset W_n + W_n, \ldots$ Therefore we have

$W_1 + \ldots + W_{n-1} + W_n \subset W_1 + \ldots + W_{n-1} + W_n + W_n \subset W_1 + \ldots + W_{n-1} + W_{n-1} \subset \ldots \subset W_1 + W_1 \subset W$. Thus $W \supset \bigcup_{n=1}^{\infty} (W_1 + \ldots + W_n) \in \mathcal{Y}$. Therefore $W \in \mathcal{Y}$. Hence $\mathcal{W} \subset \mathcal{Y}$ (moreover, $\mathcal{Y} \subset \mathcal{W}$, so $\mathcal{W} = \mathcal{Y}$). Now we can see that $\mathcal{W} \subset \mathcal{V}$. It means that $\mathcal{T} \leqslant \mathcal{T}_1$. So $\mathcal{T}_1 = \tau^{\wedge}$.

5.4.9. Let a linear pseudotopology τ satisfy condition (M). Then we say that $\tau^{\vee}$ is the Orlicz topology for the pseudotopology τ and that $\tau^{\wedge}$ is the Wiweger topology for τ (see [8]).

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