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An application of modular spaces to approximation problems, II

Let (Ω, Σ, μ) denote a space with a finite and complete measure μ, defined on Σ, a σ-algebra of subsets of the set $\Omega \neq \emptyset$, $\mu(\Omega) > 0$, $\varrho_n(t, x)$: $\Omega \times \mathfrak{X} \to \langle 0, \infty\rangle$ for $n = 1, 2, \ldots$ and $x \in \mathfrak{X}$ – a space of functions x: $\Omega \to \langle -\infty, \infty\rangle$ which are Σ-measurable and almost everywhere finite, where $x = y$ iff $x(t) = y(t)$ almost everywhere.

Let us assume:

(a) $\varrho_n(t, x)$ is a pseudomodular in $\mathfrak{X}$ for all $t \in \Omega$ and for every $n = 1, 2, \ldots$,

(b) $\varrho_n(t, x)$ is measurable and almost everywhere finite with respect to t for every $x \in \mathfrak{X}$ and every $n = 1, 2, \ldots$,

(c) if for $n = 1, 2, \ldots$, $\varrho_n(t, x) = 0$ for almost all t, then $x = 0$,

(d) if $x, y \in \mathfrak{X}$, $|x(t)| \leqslant |y(t)|$ almost everywhere in Ω, then for $n = 1, 2, \ldots$, $\varrho_n(t, x) \leqslant \varrho_n(t, y)$ almost everywhere in Ω.

Let

$$\varrho_{ns}(x) = \int_{\Omega} \varrho_n(t, x)\, d\mu, \qquad \varrho^s(x) = \sum_{n=1}^{\infty} \frac{1}{2^n} \frac{\varrho_{ns}(x)}{1+\varrho_{ns}(x)}.$$

ϱ^s is a modular in $\mathfrak{X}$. Let

$$X_{\varrho^s} = \{x \in \mathfrak{X}\colon \varrho^s(\lambda x) \to 0 \text{ for } \lambda \to 0\}.$$

We say that a sequence (ϱ_n) preservs constants if $\varrho_n(t, c) = c$ for every $t \in \Omega$ and for every $c \geqslant 0$, $n = 1, 2, \ldots$

We say that a sequence (ϱ_n) preservs constants uniformly approximately if

$$\mathop{\forall}_{\varepsilon>0} \mathop{\exists}_{\Omega_\varepsilon \subset \Omega} \mathop{\exists}_{N=N(\varepsilon)} \mathop{\forall}_{c \geqslant 0} \mathop{\forall}_{n>N} \mathop{\forall}_{t \in \Omega \setminus \Omega_\varepsilon} |\varrho_n(t, c) - c| < \varepsilon,$$

where $\Omega_\varepsilon \in \Sigma$, $\mu(\Omega_\varepsilon) = 0$.

The pseudomodular ϱ_n is called *ψ_n-convex in $\mathfrak{X}$* if there exists $\Omega_n^0 \subset \Omega$,

$\Omega_n^0 \in \Sigma$, $\mu(\Omega_n^0) = 0$, such that for every $x, y \in \mathfrak{X}$ and $t \in \Omega \setminus \Omega_n^0$

$$\varrho_n(t, \alpha x + \beta y) \leqslant \psi_n(\alpha)\varrho_n(t, x) + \psi_n(\beta)\varrho_n(t, y)$$

for $\alpha, \beta \geqslant 0$, $\alpha + \beta = 1$, $\psi_n\colon \langle 0, 1\rangle \to \langle 0, 1\rangle$, $\psi_n(\tau) \geqslant \tau$ for $\tau \in \langle 0, 1\rangle$.

The sequence (ϱ_n) of ψ_n-convex pseudomodulars in $\mathfrak{X}$, $n = 1, 2, \ldots$, is called *singular* at the point $x \in X_{\varrho^s}$, if

$$\forall_{a>0, b\geqslant 1} \ \forall_{m=1,2,\ldots} \int_{\Omega} \varrho_m\big\{t, a\psi_n(1/b)\big[\varrho_n\big(\cdot, b(x_+ - x_+(\cdot))\big) + \varrho_n\big(\cdot, b(x_- - x_-(\cdot))\big)\big]\big\}\, d\mu \to 0$$

with $n \to \infty$.

Let for $x \in \mathfrak{X}$

$$F_n(t, x) = \varrho_n(t, x_+) - \varrho_n(t, x_-),$$

where x_+ – the positive part of x, x_- – the negative part of x.

The following theorem is true (see [4]):

THEOREM 1. *If*:

(a) *the sequence* (ϱ_n) *of* ψ_n*-convex pseudomodulars in* $\mathfrak{X}$, $n = 1, 2, \ldots$, *preserves constants uniformly approximately*,

(b) *constant functions belong to* X_{ϱ^s},

(c) *the sequence* (ϱ_n) *is singular at the point* $x \in X_{\varrho^s}$,

then, for every $\lambda > 0$, $\varrho^s\{\lambda[x(\cdot) - F_n(\cdot, x)]\} \to 0$ *with* $n \to \infty$.

Let $\Omega = \langle 0, 1\rangle$, μ – the Lebesgue measure, Σ-σ-algebra of Lebesgue measurable sets in $\langle 0, 1\rangle$. Let $\mathfrak{X}$ denote the set of Σ-measurable and almost everywhere finite functions in $\langle 0, 1\rangle$, extended periodically, with period 1, outside $\langle 0, 1\rangle$. Let K_n $(n = 1, 2, \ldots)$ be functions measurable and positive almost everywhere in $\langle 0, 1\rangle$. φ is a convex φ-function and φ^{-1} is the function inverse to φ for $u \geqslant 0$.

We define the following sequences of functionals:

$$\text{(A)} \qquad \varrho_n(t, x) = \varphi^{-1} \circ F^{-1}\Big\{\int_0^1 K_n(u) F \circ \varphi\big(|x(u+t)|\big)\, du\Big\},$$

where $n = 1, 2, \ldots$, $t \in \langle 0, 1\rangle$, $x \in \mathfrak{X}$, $F(u) = e^u - 1$ for $u \geqslant 0$,

$$\lim_{n\to\infty} \int_0^1 K_n(u)\, du = 1,$$

$$\text{(B)} \qquad \varrho_n(t, x) = \varphi^{-1}\Big\{\int_0^1 K_n(u)\, \varphi\big(|x(u+t)|\big)\, du\Big\},$$

where $n = 1, 2, \ldots$, $t \in \langle 0, 1\rangle$, $x \in \mathfrak{X}$,

$$\int_0^1 K_n(u)\, du = 1 \qquad \text{for } n = 1, 2, \ldots$$

For $n = 1, 2, \ldots$, $\varrho_n(t, x)$, where $\varrho_n(t, x)$ is defined by formula (A) or by formula (B), satisfy conditions (a), (c), (d) and are measurable with respect to t for every $x \in \mathfrak{X}$ and every $n = 1, 2, \ldots$

Sequence (A) preservs constants uniformly approximately ([4]). Sequence (B) preservs constants.

THEOREM 2. *If for a sequence* (ϱ_n) *defined by* (B), *where* φ *satisfies the condition* (Δ_2) *for large* u, *we have*

$$\lim_{n\to\infty} \int_0^1 K_n(v)\Big[\int_0^1 \varphi\big(b\,|x(v+s)-x(s)|\big)ds\Big]dv = 0$$

for every $b > 0$ *and for any* $x \in X_{\varrho s}$, $x \geqslant 0$, *then* (ϱ_n) *is singular at the point* x.

Proof (see [2], [3]).

We prove the next

THEOREM 3. *If for a sequence* (ϱ_n) *defined by* (A), *where* φ *satisfies the condition* (Δ_2) *for large* u, *we have for* $G = F \circ \varphi$

$$\lim_{n\to\infty} \int_0^1 K_n(v)\Big[\int_0^1 G^p\big(b\,|x_\pm(v+s)-x_\pm(s)|\big)ds\Big]^{1/p}dv = 0$$

for every $p \geqslant 1$, $b \geqslant 1$ *and for any* $x \in X_{\varrho s}$, *then* (ϱ_n) *is singular at the point* x.

Proof. We shall give a sufficient condition in order that for every $a > 0$, $b \geqslant 1$, $m = 1, 2, \ldots$

$$\begin{aligned} J_n(x) = \int_0^1 G^{-1}\Big\{\int_0^1 K_m(u)\,G\Big[a\psi_n(1/b)\,\times \\ \times\Big(G^{-1}\Big(\int_0^1 K_n(v)\,G\big(b\,|x_+(u+v+t)-x_+(u+t)|\big)dv\Big)+ \\ +G^{-1}\Big(\int_0^1 K_n(v)\,G\big(b\,|x_-(u+v+t)-x_-(u+t)|\big)dv\Big)\Big)\Big]du\Big\}dt \to 0 \end{aligned}$$

with $n \to \infty$.

Because G^{-1} is subadditive, we have

$$\begin{aligned} J_n(x) \leqslant \int_0^1 G^{-1}\Big\{\int_0^1 K_m(u)\,G\Big[2a\psi_n(1/b)\,\times \\ \times\Big(G^{-1}\Big(\int_0^1 K_n(v)\,G\big(b\,|x_+(u+v+t)-x_+(u+t)|\big)dv\Big)\Big)\Big]du\Big\}dt+ \\ +\int_0^1 G^{-1}\Big\{\int_0^1 K_m(u)\,G\Big[2a\psi_n(1/b)\,\times \\ \times\Big(G^{-1}\Big(\int_0^1 K_n(v)\,G\big(b\,|x_-(u+v+t)-x_-(u+t)|\big)dv\Big)\Big)\Big]du\Big\}dt \\ = J_n^+(x)+J_n^-(x). \end{aligned}$$

Since φ satisfies the condition (Δ_2) for large u, so for every $\varepsilon > 0$ there exists $a' = a'(\varepsilon) > 0$ such that $\varphi(2au) \leqslant a'\varphi(u)$ for $u \geqslant \varepsilon$. Thus, if we put $p = [a'] + 1$, where $[a']$ denotes the integer part of a', we have for $z \geqslant e^{\varphi(\varepsilon)} - 1$

$$G\big(2aG^{-1}(z)\big) \leqslant (2^p - 1)\begin{cases} z^p, & \text{when } \quad z \geqslant 1, \\ z, & \text{when } \; 0 < z < 1. \end{cases}$$

Let

$$A_t = \Big\{u \in \langle 0, 1\rangle \colon \int_0^1 K_n(v)\, G\big(b\,|x_+(u+v+t) - x_+(u+t)|\big)\,dv < e^{\varphi(\varepsilon)} - 1\Big\},$$

$$B_t = \langle 0, 1\rangle \setminus A_t.$$

Then for every $t \in \langle 0, 1\rangle$

$$\begin{aligned}
&\int_0^1 K_m(u)\, G\Big[2aG^{-1}\Big(\int_0^1 K_n(v)\, G\big(b\,|x_+(u+v+t) - x_+(u+t)|\big)\,dv\Big)\Big]\,du \\
&\quad = \int_{A_t} + \int_{B_t} \leqslant G\big(2aG^{-1}(e^{\varphi(\varepsilon)} - 1)\big) \int_0^1 K_m(u)\,du + (2^p - 1) \times \\
&\qquad \times \Big\{\int_0^1 K_m(u)\Big[\int_0^1 K_n(v)\, G\big(b\,|x_+(v+u+t) - x_+(u+t)|\big)\,dv\Big]\,du + \\
&\qquad + \int_0^1 K_m(u)\Big[\int_0^1 K_n(v)\, G\big(b\,|x_+(v+u+t) - x_+(u+t)|\big)\,dv\Big]^p\,du\Big\} \\
&\quad \leqslant G\big(2aG^{-1}(e^{\varphi(\varepsilon)} - 1)\big)M + (2^p - 1)\Big\{\int_0^1 K_n(v)\Big[\int_0^1 K_m(u) \times \\
&\qquad \times G\big(b\,|x_+(v+u+t) - x_+(u+t)|\big)\,du\Big]\,dv + \\
&\qquad + \Big[\int_0^1 K_n(v)\Big(\int_0^1 K_m(u)\, G^p\big(b\,|x_+(v+u+t) - x_+(u+t)|\big)\,du\Big)^{1/p}\,dv\Big]^p\Big\},
\end{aligned}$$

where

$$\int_0^1 K_m(u)\,du \leqslant M \qquad \text{for } m = 1, 2, \ldots$$

Let us denote

$$v_\varepsilon = G\big(2aG^{-1}(e^{\varphi(\varepsilon)} - 1)\big)\cdot M, \qquad \delta_\varepsilon = v_\varepsilon \sup_{u \geqslant v_\varepsilon} \frac{G^{-1}(u)}{u}, \qquad c_\varepsilon = \delta_\varepsilon / v_\varepsilon.$$

Then $G^{-1}(u) \leqslant c_\varepsilon u$ for $u \geqslant v_\varepsilon$ and

$$J_n^+(x) \leqslant \delta_\varepsilon + (2^p - 1)\, c_\varepsilon \int_0^1 \int_0^1 K_m(u)\, K_n(v) \times$$

$$\times\Big(\int_0^1 G\big(b|x_+(v+u+t)-x_+(u+t)|\big)dt\Big)du\,dv+(2^p-1)c_\varepsilon\int_0^1\Big\{\Big[\int_0^1 K_n(v)\times$$

$$\times\Big(\int_0^1 K_m(u)G^p\big(b|x_+(v+u+t)-x_+(u+t)|\big)du\Big)^{1/p}dv\Big]^p\Big\}dt$$

$$\leqslant \delta_\varepsilon+(2^p-1)c_\varepsilon M\int_0^1 K_n(v)\Big(\int_0^1 G\big(b|x_+(v+s)-x_+(s)|\big)ds\Big)dv+$$

$$+(2^p-1)c_\varepsilon M\Big[\int_0^1 K_n(v)\Big(\int_0^1 G^p\big(b|x_+(v+s)-x_+(s)|\big)ds\Big)^{1/p}dv\Big]^p.$$

The expression $J_n^-(x)$ is estimated in an analogous manner.

Since $\delta_\varepsilon\to 0$ with $\varepsilon\to 0$, so $J_n(x)\to 0$ for $n\to\infty$. Thus the sequence (ϱ_n) is singular at the point $x\in X_{\varrho^s}$.

We say that (K_n) is a *singular kernel*, if for every $\delta\in(0, 1\rangle$

$$\lim_{n\to\infty}\int_\delta^1 K_n(u)\,du=0.$$

Let us denote

$$\varrho_{G^p}(x)=\int_0^1 G^p(|x(t)|)\,dt,$$

$$X_{\varrho_{G^p}}=\{x\in\mathfrak{X}\colon \varrho_{G^p}(\lambda x)\to 0 \text{ with } \lambda\to 0\},$$

$$E_{\varrho_{G^p}}=\{x\in X_{\varrho_{G^p}}\colon \varrho_{G^p}(\lambda x)<\infty \text{ for every } \lambda>0\},$$

$$\omega_{G^p}(\delta,\lambda x)=\sup_{0\leqslant v\leqslant\delta}\Big[\int_0^1 G^p\big(\lambda|x(v+s)-x(s)|\big)ds\Big]^{1/p},\qquad p\geqslant 1,\ \lambda>0.$$

If $x\in E_{\varrho_{G^p}}$, then $\omega_{G^p}(\delta,\lambda x)\to 0$ with $\delta\to 0$ for every $p\geqslant 1$, $\lambda>0$ (see [1]).

The function x, where

$$x(t)=\begin{cases} n & \text{for } t\in A_n=\Big\langle\dfrac{1}{n+e^{-n\psi(n)}},\dfrac{1}{n}\Big\rangle,\qquad n=1,2,\ldots,\\[2ex] 0 & \text{for } t\in\langle 0,1\rangle\setminus\bigcup\limits_{n=1}^{\infty}A_n,\end{cases}$$

ψ is a φ-function, is not essentially bounded, because for every $K>0$ there exists n_0 such that $x(t)>K$ for $t\in A_{n_0}$. For every $p\geqslant 1$, $\lambda>0$ we have

$$\varrho_{G^p}(\lambda x)=\sum_{n=1}^{\infty}\big[e^{\psi(\lambda n)}-1\big]^p\frac{e^{-m\psi(n)}}{n\big[n+e^{-n\psi(n)}\big]}<\infty,$$

when

$$(1)\qquad \frac{p\big[\psi(\lambda(n+1))-\psi(\lambda n)\big]}{\psi(n+1)-\psi(n)}<n+1\qquad\text{for } n>N.$$

Condition (1) holds for example for $\psi(t) = t^q$, $q > 0$, and $n > p\lambda^q - 1$. Thus $x \in E_{\varrho_{G^p}}$ for every $p \geqslant 1$.

We say an element $x \in X_{\varrho^s}$ is *strictly regular*, when $x \in E_{\varrho_{G^p}}$ for every $p \geqslant 1$.

THEOREM 4. *If for a sequence* (ϱ_n) *defined by* (A), *where* φ *satisfies the condition* (Δ_2) *for large* u, (K_n) *is a singular kernel, then* (ϱ_n) *is singular at the every strictly regular element* $x \in X_{\varrho^s}$.

Proof. For $\delta \in (0, 1\rangle$ we have

$$\int_0^\delta K_n(v)\Big[\int_0^1 G^p\big(b\,|x_\pm(v+s) - x_\pm(s)|\big)\,ds\Big]^{1/p} dv \leqslant M\omega_{G^p}(\delta, bx_\pm),$$

where $G = F \circ \varphi$,

$$\int_0^1 K_n(v)\,dv \leqslant M \qquad \text{for } n = 1, 2, \ldots,$$

and

$$\int_\delta^1 K_n(v)\Big[\int_0^1 G^p\big(b\,|x_\pm(v+s) - x_\pm(s)|\big)\,ds\Big]^{1/p} dv$$
$$\leqslant \Big\{\int_0^1 \big[G(2b\,|x_\pm(s)|)\big]^p ds\Big\}^{1/p} \cdot \int_\delta^1 K_n(v)\,dv.$$

Thus

$$\int_0^1 K_n(v)\Big[\int_0^1 G^p\big(b\,|x_\pm(v+s) - x_\pm(s)|\big)\,ds\Big]^{1/p} dv$$
$$\leqslant M\omega_{G^p}(\delta, bx_\pm) + \Big\{\int_0^1 \big[G(2b\,|x_\pm(s)|)\big]^p ds\Big\}^{1/p} \int_\delta^1 K_n(v)\,dv.$$

If $\delta > 0$ is so small that $\omega_{G^p}(\delta, bx_\pm) < \varepsilon/2M$, $\varepsilon > 0$, and n is so large that

$$\int_\delta^1 K_n(v)\,dv < \tfrac{1}{2}\varepsilon\,[\varrho_{G^p}(2bx_\pm)]^{-1/p},$$

then

$$\int_0^1 K_n(v)\Big[\int_0^1 G^p\big(b\,|x_\pm(v+s) - x_\pm(s)|\big)\,ds\Big]^{1/p} dv < \varepsilon.$$

Thus (ϱ_n) is singular at the point x.

From Theorems 1 and 4 it follows

THEOREM 5. *If for a sequence* (ϱ_n) *defined by* (A), *where* φ *satisfies the* *condition* (Δ_2) *for large* u, (K_n) *is a singular kernel, then for every* $\lambda > 0$

$$\varrho^s\{\lambda[x(\cdot)-F_n(\cdot, x)]\} \to 0 \quad \textit{with } n \to \infty$$

at the every strictly regular element $x \in X_{\varrho^s}$ *such that* $\varrho_n(t, x)$ *is almost everywhere finite with respect to* t *for* $n = 1, 2, \ldots$

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