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An application of modular spaces to approximation problems, II

Let \((\Omega, \Sigma, \mu)\) denote a space with a finite and complete measure \(\mu\), defined on \(\Sigma\), a \(\sigma\)-algebra of subsets of the set \(\Omega \neq \emptyset\), \(\mu(\Omega) > 0\), \(q_n(t, x)\): \(\Omega \times \mathcal{X} \to (-\infty, \infty)\) for \(n = 1, 2, \ldots\) and \(x \in \mathcal{X}\) – a space of functions \(x: \Omega \to (-\infty, \infty)\) which are \(\Sigma\)-measurable and almost everywhere finite, where \(x = y\) iff \(x(t) = y(t)\) almost everywhere.

Let us assume:

(a) \(q_n(t, x)\) is a pseudomodular in \(\mathcal{X}\) for all \(t \in \Omega\) and for every \(n = 1, 2, \ldots\),

(b) \(q_n(t, x)\) is measurable and almost everywhere finite with respect to \(t\) for every \(x \in \mathcal{X}\) and every \(n = 1, 2, \ldots\),

(c) if for \(n = 1, 2, \ldots\), \(q_n(t, x) = 0\) for almost all \(t\), then \(x = 0\),

(d) if \(x, y \in \mathcal{X}\), \(|x(t)| \leq |y(t)|\) almost everywhere in \(\Omega\), then for \(n = 1, 2, \ldots\), \(q_n(t, x) \leq q_n(t, y)\) almost everywhere in \(\Omega\).

Let

\[
q^*_{\Sigma}(x) = \int_{\Omega} q^*_n(t, x) \, d\mu, \quad q^*(x) = \sum_{n=1}^{\infty} \frac{q^*_n(x)}{2^n \left(1 + q^*_n(x)\right)}.
\]

\(q^*\) is a modular in \(\mathcal{X}\). Let

\[
X_{q^*} = \{x \in \mathcal{X}: q^*(\lambda x) \to 0 \text{ for } \lambda \to 0\}.
\]

We say that a sequence \((q_n)\) preserves constants if \(q_n(t, c) = c\) for every \(t \in \Omega\) and for every \(c \geq 0\), \(n = 1, 2, \ldots\)

We say that a sequence \((q_n)\) preserves constants uniformly approximately if

\[
\forall \varepsilon > 0 \quad \exists N = N(\varepsilon) \quad \forall n > N \quad \forall t \in \Omega_e \quad \forall c \in \Omega_e \quad |q_n(t, c) - c| < \varepsilon,
\]

where \(\Omega_e \in \Sigma\), \(\mu(\Omega_e) = 0\).

The pseudomodular \(q_n\) is called \(\psi^*\)-convex in \(\mathcal{X}\) if there exists \(\Omega^0_n \subset \Omega\),
\[ \Omega_0^0 \in \Sigma, \mu(\Omega_0^0) = 0, \] such that for every \( x, y \in \mathfrak{X} \) and \( t \in \Omega \setminus \Omega_0^0 \)

\[ \varrho_n(t, \alpha x + \beta y) \leq \psi_n(x) \varrho_n(t, x) + \psi_n(\beta) \varrho_n(t, y) \]

for \( \alpha, \beta \geq 0, \alpha + \beta = 1, \psi_n: \langle 0, 1 \rangle \to \langle 0, 1 \rangle, \psi_n(\tau) \geq \tau \) for \( \tau \in \langle 0, 1 \rangle \).

The sequence \((\varrho_n)\) of \(\psi_n\)-convex pseudomodulars in \(\mathfrak{X}\), \(n = 1, 2, \ldots\), is called singular at the point \(x \in X_{\varrho^*}\), if

\[ \forall a > 0, b \geq 1 \quad \exists m = 1, 2, \ldots \quad \int_{\Omega} \varrho_m \{ t, a \psi_n(1/b) [ \varrho_n(\cdot, b (x_+ - x_+ (\cdot))) + \\
\varrho_n(\cdot, b (x_- - x_- (\cdot)))] \} d\mu \to 0 \]

with \(n \to \infty\).

Let for \(x \in \mathfrak{X}\)

\[ F_n(t, x) = \varrho_n(t, x_+) - \varrho_n(t, x_-), \]

where \(x_+\) is the positive part of \(x\), \(x_-\) is the negative part of \(x\).

The following theorem is true (see [4]):

**Theorem 1.** If:

(a) the sequence \((\varrho_n)\) of \(\psi_n\)-convex pseudomodulars in \(\mathfrak{X}\), \(n = 1, 2, \ldots\), preserves constants uniformly approximately,

(b) constant functions belong to \(X_{\varrho^*}\),

(c) the sequence \((\varrho_n)\) is singular at the point \(x \in X_{\varrho^*}\),

then, for every \(\lambda > 0\),

\[ \varrho^* \{ \lambda [x(\cdot) - F_n(\cdot, x)] \} \to 0 \]

with \(n \to \infty\).

Let \(\Omega = \langle 0, 1 \rangle\), \(\mu\) is the Lebesgue measure, \(\Sigma\)-\(\sigma\)-algebra of Lebesgue measurable sets in \(\langle 0, 1 \rangle\). Let \(\mathfrak{X}\) denote the set of \(\Sigma\)-measurable and almost everywhere finite functions in \(\langle 0, 1 \rangle\), extended periodically, with period 1, outside \(\langle 0, 1 \rangle\). Let \(K_n(n = 1, 2, \ldots)\) be functions measurable and positive almost everywhere in \(\langle 0, 1 \rangle\). \(\varphi\) is a convex \(\varphi\)-function and \(\varphi^{-1}\) is the function inverse to \(\varphi\) for \(u \geq 0\).

We define the following sequences of functionals:

**A**

\[ \varrho_n(t, x) = \varphi^{-1} \circ F^{-1} \quad \{ \frac{1}{0} K_n(u) F \circ \varphi([x(u + t)]) du \}, \]

where \(n = 1, 2, \ldots, t \in \langle 0, 1 \rangle, x \in \mathfrak{X}, F(u) = e^u - 1\) for \(u \geq 0\),

\[ \lim_{n \to \infty} \frac{1}{0} K_n(u) du = 1, \]

**B**

\[ \varrho_n(t, x) = \varphi^{-1} \quad \{ \frac{1}{0} K_n(u) \varphi([x(u + t)]) du \}, \]

where \(n = 1, 2, \ldots, t \in \langle 0, 1 \rangle, x \in \mathfrak{X}, \)

\[ \frac{1}{0} K_n(u) du = 1 \quad \text{for} \quad n = 1, 2, \ldots \]
For \( n = 1, 2, \ldots \), \( \varphi_n(t, x) \), where \( \varphi_n(t, x) \) is defined by formula (A) or by formula (B), satisfy conditions (a), (c), (d) and are measurable with respect to \( t \) for every \( x \in \mathcal{X} \) and every \( n = 1, 2, \ldots \).

Sequence (A) preserves constants uniformly approximately ([4]). Sequence (B) preserves constants.

**Theorem 2.** If for a sequence \( (\varphi_n) \) defined by (B), where \( \varphi \) satisfies the condition (A2) for large \( u \), we have

\[
\lim_{n \to \infty} \int_0^1 K_n(v) \left[ \int_0^1 \varphi(b |x(v+s)−x(s)|) ds \right] dv = 0
\]

for every \( b > 0 \) and for any \( x \in \mathcal{X} \), \( x \geq 0 \), then \( (\varphi_n) \) is singular at the point \( x \).

**Proof.** (see [2], [3]).

We prove the next

**Theorem 3.** If for a sequence \( (\varphi_n) \) defined by (A), where \( \varphi \) satisfies the condition (A2) for large \( u \), we have for \( G = F \circ \varphi \)

\[
\lim_{n \to \infty} \int_0^1 K_n(v) \left[ \int_0^1 G^p(b |x_\pm (v+s)−x_\pm (s)|) ds \right]^{1/p} dv = 0
\]

for every \( p \geq 1 \), \( b \geq 1 \) and for any \( x \in \mathcal{X} \), then \( (\varphi_n) \) is singular at the point \( x \).

**Proof.** We shall give a sufficient condition in order that for every \( a > 0 \), \( b \geq 1 \), \( m = 1, 2, \ldots \)

\[
J_n(x) = \int_0^1 G^{-1} \left\{ \int_0^1 K_m(u) G \left[ a\psi_n(1/b) \times \right. \right.
\]

\[
\left. \times \left( G^{-1} \left( \int_0^1 K_n(v) G(b |x_+ (u+v+t)−x_+ (u+t)|) dv \right) \right) + \right.
\]

\[
\left. + G^{-1} \left( \int_0^1 K_n(v) G(b |x_- (u+v+t)−x_- (u+t)|) dv \right) \right] du \right\} dt \to 0
\]

with \( n \to \infty \).

Because \( G^{-1} \) is subadditive, we have

\[
J_n(x) \leq \int_0^1 G^{-1} \left\{ \int_0^1 K_m(u) G \left[ 2a\psi_n(1/b) \times \right. \right.
\]

\[
\left. \times \left( G^{-1} \left( \int_0^1 K_n(v) G(b |x_+ (u+v+t)−x_+ (u+t)|) dv \right) \right) \right\} dt +
\]

\[
+ \int_0^1 G^{-1} \left\{ \int_0^1 K_m(u) G \left[ 2a\psi_n(1/b) \times \right. \right.
\]

\[
\left. \times \left( G^{-1} \left( \int_0^1 K_n(v) G(b |x_- (u+v+t)−x_- (u+t)|) dv \right) \right) \right\} dt
\]

\[= J_n^+(x)+J_n^-(x).\]
Since \( \varphi \) satisfies the condition \((A_2)\) for large \( u \), so for every \( \varepsilon > 0 \) there exists \( a' = a'(\varepsilon) > 0 \) such that \( \varphi(2au) \leq a' \varphi(u) \) for \( u \geq \varepsilon \). Thus, if we put \( p = [a'] + 1 \), where \([a']\) denotes the integer part of \( a'\), we have for \( z \geq e^{\varphi(\varepsilon)} - 1 \)

\[
G(2aG^{-1}(z)) \leq (2^p - 1) \begin{cases} z^p, & \text{when } z \geq 1, \\ z, & \text{when } 0 < z < 1. \end{cases}
\]

Let

\[
A_t = \{ u \in \langle 0, 1 \rangle : \int_0^1 K_n(v) G(b |x_+(u+v+t) - x_+(u+t)|) dv < e^{\varphi(\varepsilon)} - 1 \},
\]

and

\[
B_t = \langle 0, 1 \rangle \setminus A_t.
\]

Then for every \( t \in \langle 0, 1 \rangle \)

\[
\int_0^1 K_m(u) G\left[ 2aG^{-1}\left( \int_0^1 K_n(v) G(b |x_+(u+v+t) - x_+(u+t)|) dv \right) \right] du
\]

\[= \int_{A_t} + \int_{B_t} \leq G(2aG^{-1}(e^{\varphi(\varepsilon)} - 1)) \int_0^1 K_m(u) du + (2^p - 1) \times
\]

\[
\times \left\{ \int_0^1 K_m(u) \left[ \int_0^1 K_n(v) G(b |x_+(v+u+t) - x_+(u+t)|) dv \right] du + \right. \\
+ \left. \int_0^1 K_m(u) \left[ \int_0^1 K_n(v) G(b |x_+(v+u+t) - x_+(u+t)|) dv \right]^p dv \right\}
\]

\[\leq G(2aG^{-1}(e^{\varphi(\varepsilon)} - 1)) M + (2^p - 1) \left\{ \int_0^1 K_n(v) \left[ \int_0^1 K_m(u) \times \\
\times G(b |x_+(v+u+t) - x_+(u+t)|) du \right] dv + \right. \\
+ \left. \left[ \int_0^1 K_n(v) \left[ \int_0^1 K_m(u) G^p(b |x_+(v+u+t) - x_+(u+t)|) du \right]^{1/p} dv \right] \right\},
\]

where

\[
\int_0^1 K_m(u) du \leq M \quad \text{for } m = 1, 2, \ldots
\]

Let us denote

\[
v_\varepsilon = G(2aG^{-1}(e^{\varphi(\varepsilon)} - 1)) \cdot M, \quad \delta_\varepsilon = v_\varepsilon \sup_{u \geq v_\varepsilon} \frac{G^{-1}(u)}{u}, \quad c_\varepsilon = \delta_\varepsilon/v_\varepsilon.
\]

Then \( G^{-1}(u) \leq c_\varepsilon u \) for \( u \geq v_\varepsilon \) and

\[
J_n^+(x) \leq \delta_\varepsilon + (2^p - 1) c_\varepsilon \int_0^1 \int_0^1 K_m(u) K_n(v) \times
\]
The expression \( J_n^- (x) \) is estimated in an analogous manner.

Since \( \delta \to 0 \) with \( \varepsilon \to 0 \), so \( J_n^- (x) \to 0 \) for \( n \to \infty \). Thus the sequence \( (g_n) \)
is singular at the point \( x \in X_{\varepsilon^d} \).

We say that \( (K_n) \) is a singular kernel, if for every \( \delta \in (0, 1) \)

\[
\lim_{n \to \infty} \int_0^1 K_n(u) \, du = 0.
\]

Let us denote

\[
\varrho_{G_p}(x) = \int_0^1 G_p(|x(t)|) \, dt,
\]

\( X_{\varrho_{G_p}} = \{ x \in \mathcal{X}: \varrho_{G_p}(\lambda x) \to 0 \text{ with } \lambda \to 0 \} \),

\( E_{\varrho_{G_p}} = \{ x \in X_{\varrho_{G_p}}: \varrho_{G_p}(\lambda x) < \infty \text{ for every } \lambda > 0 \} \),

\[
\omega_{G_p}(\delta, \lambda x) = \sup_{0 \leq v \leq \delta} \left[ \int_0^1 G_p(\lambda |x(v+s) - x(s)|) \, ds \right]^{1/p}, \quad p \geq 1, \lambda > 0.
\]

If \( x \in E_{\varrho_{G_p}} \), then \( \omega_{G_p}(\delta, \lambda x) \to 0 \) with \( \delta \to 0 \) for every \( p \geq 1, \lambda > 0 \) (see [1]).

The function \( x \), where

\[
x(t) = \begin{cases} 
  n & \text{for } t \in A_n = \left( \frac{1}{n + e^{-n \phi(n)}}, \frac{1}{n} \right), \quad n = 1, 2, \ldots, \\
  0 & \text{for } t \in \langle 0, 1 \rangle \setminus \bigcup_{n=1}^\infty A_n,
\end{cases}
\]

\( \psi \) is a \( \varphi \)-function, is not essentially bounded, because for every \( K > 0 \) there exists \( n_0 \) such that \( x(t) > K \) for \( t \in A_{n_0} \). For every \( p \geq 1, \lambda > 0 \) we have

\[
\varrho_{G_p}(\lambda x) = \sum_{n=1}^\infty \left[ e^{\psi(\lambda n)} - 1 \right]^p \frac{e^{-n \phi(n)}}{n \left[ n + e^{-n \phi(n)} \right]} < \infty,
\]

when

\[
\frac{p \left[ \psi(\lambda (n+1)) - \psi(\lambda n) \right]}{\psi(n+1) - \psi(n)} < n+1 \quad \text{for } n > N.
\]
Condition (1) holds for example for $\psi(t) = t^q$, $q > 0$, and $n > p\lambda^d - 1$. Thus $x \in E_{\psi, p}$ for every $p \geq 1$.

We say an element $x \in X_{\psi}$ is strictly regular, when $x \in E_{\psi, p}$ for every $p \geq 1$.

**Theorem 4.** If for a sequence $(\varphi_n)$ defined by (A), where $\varphi$ satisfies the condition (A$_2$) for large $u$, $(K_n)$ is a singular kernel, then $(\varphi_n)$ is singular at the every strictly regular element $x \in X_{\psi}$.

**Proof.** For $\delta \in (0, 1)$ we have

$$\int_0^\delta K_n(v) \left[ \int_0^1 G^p(b |x_\pm| v + s - x_\pm(s)) ds \right]^{1/p} dv \leq M\omega_{G^p}(\delta, bx_\pm),$$

where $G = F \circ \varphi$,

$$\int_0^1 K_n(v) dv \leq M \quad \text{for } n = 1, 2, \ldots,$$

and

$$\int_0^\delta K_n(v) \left[ \int_0^1 G^p(b |x_\pm| v + s - x_\pm(s)) ds \right]^{1/p} dv$$

$$\leq \left\{ \int_0^1 \left[ G(2b |x_\pm(s)|) \right]^{p} ds \right\}^{1/p} \int_0^1 K_n(v) dv.$$}

Thus

$$\int_0^\delta K_n(v) \left[ \int_0^1 G^p(b |x_\pm| v + s - x_\pm(s)) ds \right]^{1/p} dv$$

$$\leq M\omega_{G^p}(\delta, bx_\pm) + \left\{ \int_0^1 \left[ G(2b |x_\pm(s)|) \right]^{p} ds \right\}^{1/p} \int_0^1 K_n(v) dv.$$}

If $\delta > 0$ is so small that $\omega_{G^p}(\delta, bx_\pm) < \varepsilon/2M$, $\varepsilon > 0$, and $n$ is so large that

$$\int_0^1 K_n(v) dv < \frac{1}{2} \varepsilon \left[ \omega_{G^p}(2bx_\pm) \right]^{-1/p},$$

then

$$\int_0^1 K_n(v) \left[ \int_0^1 G^p(b |x_\pm| v + s - x_\pm(s)) ds \right]^{1/p} dv < \varepsilon.$$

Thus $(\varphi_n)$ is singular at the point $x$.

From Theorems 1 and 4 it follows

**Theorem 5.** If for a sequence $(\varphi_n)$ defined by (A), where $\varphi$ satisfies the
condition \((A2)\) for large \(u\), \((K_n)\) is a singular kernel, then for every \(\lambda > 0\)
\[
\varrho_n \{ \lambda [x(\cdot) - F_n(\cdot, x)] \} \to 0 \quad \text{with } n \to \infty.
\]
at every strictly regular element \(x \in X_s\), such that \(\varrho_n(t, x)\) is almost everywhere finite with respect to \(t\) for \(n = 1, 2, \ldots\).

References


