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A fixed point property for locally one-to-one mappings

Locally one-to-one mappings $f\colon X \to Y$ are studied in the paper such that $X \subset f(X) \subset Y$. Some sufficient conditions are found under which these mappings have fixed points. In terms of such mappings a characterization of atriodic graphs (i.e. of arcs and simple closed curves) is obtained.

All mappings considered in the paper are continuous. A mapping $f\colon X \to Y$ is called *locally one-to-one* if each point of X has a neighbourhood $U \subset X$ such that the partial mapping $f|U\colon U \to f(U)$ is one-to-one. If X is a subspace of a topological space Y, then we denote by $\mathrm{Fr}\, X$ and $\mathrm{Int}\, X$ the boundary and the interior of X with respect to Y correspondingly.

The following two concepts are well known. An arc X with end points a and b, contained in a space Y, is called free if $X \setminus \{a, b\}$ is an open subset of Y. In other words a free arc X in Y is characterized by the inclusion $\mathrm{Fr}\, X \subset \{a, b\}$. Similarly, a simple closed curve $X \subset Y$ is called free if $\mathrm{Fr}\, X$ consists of at most one point. Note that if X is either an arc or a simple closed curve in a space Y, then X is free if and only if X is a closed domain in Y whose interior is connected. Thus it is natural to accept the following definition.

D1. A subspace X of a space Y is called *free* (or freely embedded into Y) provided that

(1) $$X = \overline{\mathrm{Int}\, X}$$

and

(2) $$\mathrm{Int}\, X \text{ is connected.}$$

We consider now some locally one-to-one mappings whose domains are free subspaces of the range spaces. Namely we admit the following definition.

D2. A mapping $f\colon X \to Y$ is called *stretchy* provided that

(3) $$f \text{ is locally one-to-one,}$$

(4) $$X \text{ is a free subspace of } Y,$$

(5) $$X \subset f(X) \subset Y,$$

(6) $$f^{-1}(\operatorname{Int} X) \subset \operatorname{Int} X.$$

We need the following lemma.

L1. *Let a continuum X be either an arc or a simple closed curve, and let a mapping $f\colon X \to Y$ be stretchy. Then* 1° *there is a component C of $f^{-1}(X)$ such that*

(7) $$C \cap f^{-1}(\operatorname{Int} X) \neq \emptyset,$$

and 2° *for every component C of $f^{-1}(X)$ satisfying* (7) *we have*

(8) $$f(C) = X.$$

Proof. Since X is freely embedded into Y we have (1) which implies $\operatorname{Int} X \neq \emptyset$, and thereby 1° is true. To show 2° let C be any component of $f^{-1}(X)$ satisfying (7). If $C = X$, then $f(C) = f(X) \subset X$ and (8) is a consequence of (5). So let C be a proper subset of X. Take a point $x \in C \cap f^{-1}(\operatorname{Int} X) \subset \operatorname{Int} X$ (see (6)) and observe that, since $f^{-1}(\operatorname{Int} X)$ is an open subset of X and f is locally one-to-one, there is an arc ab in X such that $x \in ab \setminus \{a, b\} \subset ab \subset f^{-1}(\operatorname{Int} X) \subset f^{-1}(X)$ and $f \mid ab$ is one-to-one. Thus $ab \subset C$ by the definition of the component C, so C is non-degenerate. Therefore C is a proper subarc of X. Further, $f(ab)$ is an arc such that $f(x) \in f(ab) \setminus \{f(a), f(b)\} \subset f(ab) \subset f(C)$, whence we conclude that

(9) for every point $x \in C \cap f^{-1}(\operatorname{Int} X)$ its image $f(x)$ is not an end point of $f(C)$.

Obviously, $f(C) \subset X$. Suppose on the contrary that $f(C)$ is a proper subcontinuum of X. Thus it is an arc in X such that at least one of its end points, call it y, lies in $\operatorname{Int} X$. Take a point $x \in C \cap f^{-1}(y)$. Then $x \in C \cap f^{-1}(\operatorname{Int} X)$ and $y = f(x)$ is an end point of the arc $f(C)$, contrary to (9). The proof is complete.

The following lemma is obvious, in which g denotes an arbitrary mapping.

L2. *Let X and Y be arcs with $X \subset Y$. Every mapping $g\colon X \to Y$ of X onto Y has a fixed point.*

Now let a stretchy mapping $f\colon X \to Y$ be defined on an arc X. Then by L1 a component C of $f^{-1}(X)$ exists such that (8) holds. Since $C \subset X$, it is an arc. Applying L2 we see that the mapping $g = f \mid C\colon C \to X$ of C onto X has a fixed point. This point is obviously a fixed point for f. Therefore, the following proposition is established.

P1. *Every stretchy mapping $f\colon X \to Y$ defined on an arc X has a fixed point. If a component C of $f^{-1}(X)$ satisfies* (7), *then there exists a fixed point of f belonging to C.*

Taking a simple closed curve S as Y and an arc $C \subset S$ as X, we get the following corollary to P1.

(C) *Let an arc C be contained in a simple closed curve S, and let a one-to-one mapping g: $C \to S$ of C onto $S = g(C)$ satisfy the condition*

$$g^{-1}(\mathrm{Int}\ C) \subset \mathrm{Int}\ C. \tag{10}$$

Then g has a fixed point.

We need Corollary (C) to apply it only in a proof of the following proposition.

P2. *Every stretchy mapping f: $X \to Y$ from a simple closed curve X into a connected space $Y \neq X$ has a fixed point. If a component C of $f^{-1}(X)$ satisfies (7), then there exists a fixed point of f belonging to C.*

Proof. Since X is a free proper subset of a connected space Y, its boundary is a singleton. Denote this boundary point of X by t. Take a component C of $f^{-1}(X)$ such that (7) (and thus (8)) holds, and consider two cases. If $t \in C$, then $f(t) \in X$ by (8), whence $f(t) \in X \setminus \mathrm{Int}\ X$ by (6), so $f(t) = t$ and we are done. If t is not in $C \subset f^{-1}(X) \subset X$, then C is an arc. Put $g = f \mid C$. Thus g maps C onto the simple closed curve X by (8), g is locally one-to-one and, since C is a component of $f^{-1}(X)$, its end points are mapped onto t under g, whence (10) is satisfied. So (C) can be applied, from which we conclude that g (and thus f) has a fixed point belonging to C.

The reader can verify by easy examples that all hypotheses assumed in propositions P1 and P2 are essential to attain the conclusions.

Consider now a class Δ of continua defined as follows.

D3. A continuum X is in Δ provided that whenever embedded into a continuum Y as a free proper subspace, every stretchy mapping f: $X \to Y$ has a fixed point.

Propositions P1 and P2 show that an arc and a simple closed curve are members of Δ. The next proposition assures us that, if we restrict our considerations to graphs only, then the mentioned curves are the only two members of Δ.

A *graph* means a one-dimensional polyhedron. By a *triod* we understand the union of three arcs emanating from a point and disjoint out of it. A space that contains no triod is called *atriodic*. Thus an arc and a simple closed curve are the only two atriodic graphs. The reader is referred to [1] for definitions of some concepts concerning graphs we need below. Given a graph X, we denote by $E(X)$ the set of end points of X.

We have the following proposition.

P3. *For every graph X containing a triod there exist a graph Y, in which X is freely embedded as a proper subset, and a stretchy mapping f: $X \to Y$ such that no point of X is fixed under f.*

Proof. Given a graph X which contains a triod, we define an auxiliary graph G containing X. Namely, if $E(X) = \emptyset$, we put $G = X$. If $E(X) = \{e_1, e_2, \ldots, e_k\}$, then G is defined as the union of X and of k mutually disjoint simple closed curves $S_1, S_2, \ldots, S_k$ such that $X \cap S_i = \{e_i\}$ for every $i \in \{1, 2, \ldots, k\}$. Note that each point disconnecting X disconnects also G. Since G is not a simple closed curve, there exist two points a and b of G such that the set $G \setminus \{a, b\}$ is connected (see [3], Theorem, p. 58), whence

(11) $$X \setminus \{a, b\} \text{ is connected.}$$

Let G' denote a copy of G (both G and G' are placed in the 3-space) which is disjoint with G; similarly $a', b' \in G'$ denote copies of points $a, b \in G$ respectively. We define a graph Y as the union of G and G' and of two disjoint arcs aa' and bb' which have their end points only in common with $G \cup G'$. Thus these arcs are free in Y, and by the definition of Y we have

(12) $$E(Y) = \emptyset.$$

Note that X is freely embedded into Y as a proper subset. In fact,

(13) $$\mathrm{Fr}\, X \subset \{a, b\} \cup E(X)$$

by construction. Thus condition (1) holds. Further, since $X \setminus \mathrm{Fr}\, X = \mathrm{Int}\, X$, it follows from (13) by (11) and Theorem 4 of [2], § 51, V, p. 293, that (2) is satisfied.

By construction we can easily find both a triod in X with the top at a point v and two points x_1 and x_2 in X which are ramification points of Y and such that some arcs vx_1 and vx_2, which lie entirely in X, are free in Y. Let v', x_1' and x_2' be copies of v, x_1 and x_2 lying in G', and for $j = 1$ and 2 let C_j denote the component of the set $D_j = Y \setminus \mathrm{Int}\, vx_j$ containing the free arc $v'x_j'$. Since v, x_1 and x_2 are ramification points of Y, we conclude from (12) that

(14) $$E(C_1) = E(C_2) = \emptyset.$$

Further, considering the three possibilities: 1° both D_1 and D_2 are connected, 2° exactly one of them is connected, and 3° no one of D_1 and D_2 is connected, the reader can easily verify that

(15) $$C_1 \cup C_2 = Y.$$

The reader is referred to § 2 of [1] for definitions of some concepts used in the rest of the proof.

It can be assumed without loss of generality that the graph Y is simple and directed, and it has vx_1 and vx_2 as its edges. By (14) and Lemma 2 of

[1], p. 238, there exists, for $j = 1$ and $j = 2$, a directed path P_j from v' to x'_j such that

(16) $$v'x'_j \ (x'_j v') \text{ is the first (the last) edge of } P_j,$$

and

(17) $$P_j^* = C_j$$

(here P_j^* denotes the union of all edges of P_j).

To define the needed mapping $f\colon X \to Y$ we shall use the concept of a standard mapping (see § 3 of [1], p. 241) and we shall define f separately on each edge of X. First, for $j \in \{1, 2\}$ we define the restriction $f|vx_j\colon vx_j \to C_j$ as the standard mapping associated with the sequence $\sigma(P_j)$. Thus in particular we have for $j \in \{1, 2\}$

(18) $$f(v) = v' \quad \text{and} \quad f(x_j) = x'_j$$

by (16), and

(19) $$f(vx_j) = C_j$$

by (17). Second, for an arbitrary edge $xy \neq vx_j$ of X we take $f|xy\colon xy \to x'y'$ to be a homeomorphism of xy onto $x'y'$ with $f(x) = x'$ and $f(y) = y'$, where $x'y'$ denotes the copy (in G') of the edge xy. Observe that the mapping f is well defined.

It is evident just from the definition that f is locally one-to-one. Further, f maps X onto Y by (19) and (15). To see that f satisfies (6) note that each point of Fr X (see (13)) is an end point of an edge of X, so it is mapped onto its copy in G', i.e., $f(\mathrm{Fr}\, X) \subset G'$, whence we have $X \cap f(\mathrm{Fr}\, X) = \emptyset$, which gives (6).

Finally, we show that f is fixed point free. In fact, for $j \in \{1, 2\}$ we have $f(vx_j) = C_j \subset Y \setminus \mathrm{Int}\, vx_j$ by (19), whence it follows that $vx_j \cap f(vx_j) \subset \{v, x_j\}$. But (18) implies $f(v) \neq v$ and $f(x_j) \neq x_j$, so $f(x) \neq x$ whenever $x \in vx_j$. If an edge xy of X is different from vx_j, then $xy \cap f(xy) = xy \cap x'y' \subset G \cap G' = \emptyset$. The proof is finished.

Using the concept of the class Δ of continua one can reformulate P3 saying that if a graph is in Δ, then it is atriodic. Thus propositions P1, P2 and P3, as well as D3, lead to the following theorem

THEOREM. *A graph is atriodic if and only if it is a member of Δ.*

Recall that a *regular curve* (in the sense of the theory of order) means a continuum each point of which has a local basis of open sets with finite boundaries ([2], p. 275; [3], p. 82). Obviously, each graph is a regular curve. Since every atriodic regular curve is either an arc or a simple closed curve, it is natural to ask if the term *a graph* can be replaced in P3 (and thus in the Theorem) by a *regular curve*. In other words a question is whether the characterization of an arc and a simple closed curve via stretchy mappings is valid for graphs only, i.e., whether it cannot be extended for some wider classes of continua. The example below (due to W. J. Charatonik) shows an affirmative answer to this question, whence we conclude that, in this sense, the result obtained in the Theorem is the best possible.

EXAMPLE. A bouquet of circles is a regular triodic curve which is a member of Δ.

Let C_n denote the circle $(x-1/n)^2+y^2 = 1/n^2$ in the euclidean plane. The union $X = \bigcup \{C_n\colon n \in \{1, 2, \ldots\}\}$ is called the *bouquet* of the circles C_n. Denote by $p = (0, 0)$ the common point of all circles. If X is freely embedded into a continuum Y, then $p \in \operatorname{Int} X$ by (2). By (1) and (2) we see that for each $n \in \{1, 2, \ldots\}$ the circle C_n contains at most one point of $\operatorname{Fr} X$, whence we conclude that $\operatorname{Fr} X$, being closed, is finite. Let $f\colon X \to Y$ be a stretchy mapping. The point $f(p)$ cannot be in $\operatorname{Int} X \setminus \{p\}$ by (3), therefore, if p is not a fixed point of f, then $f(p)$ lies out of $\operatorname{Int} X$. Consider a small ball B about p which is mapped homeomorphically under f (by (3)) into $f(X)$, and note that its image $f(B)$ either lies entirely out of X (if $f(p)$ lies out of X), or has only an arc in common with X (if $p \in \operatorname{Fr} X$). In any case the set $X \setminus B$, being the union of finitely many arcs, is mapped under f onto a (compact) set $f(X \setminus B)$ containing the whole X by (5); in particular, a neighbourhood of p in X is contained in $f(X \setminus B)$ which is impossible by (3). Thus p is fixed under f, and so $X \in \Delta$.

Using very similar arguments one can verify that the one-point union of countably many straight line segments of lengths tending to zero is also a member of Δ (and a regular curve, of course). Out of the class of regular curves, the sin $1/x$-curve is a member of Δ, as it can be shown with standard arguments and some ideas taken from the proof of P1. Thus the following two problems seem to be natural.

Q1. Characterize all regular curves being members of Δ.

Q2. Characterize all continua belonging to the class Δ.

References

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