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MILAN R. TASKOVIĆ (Beograd)

Some theorems on fixed point and its applications

Abstract. In this paper fixed point theorems have been established for the mappings which are contractive over two consecutive elements of an orbit, on metric and Banach spaces. With such an extension, a very general fixed point theorem is obtained to include a recent result of the author, which contains, as special cases, some results of S. Banach, F. E. Browder, D. W. Boyd and J. S. Wong, M. Edelstein, J. Daneš, R. Kannan, and many others.

1. Introduction and some results. In recent years a number of generalizations of the well-known Banach contraction principle have appeared in the literature where the authors have introduced mappings of contractive type and studied the existence of their fixed points. A comparative study of these generalization has been made more recently by Rhoades [17]. The well-known Banach contraction principle is the following:

Let $T\colon X \to X$ *be a mapping of a complete metric space* (X, ϱ) *into itself. If* T *is a contraction, i.e. if*

(A) $\varrho[Tx, Ty] \leqslant \alpha\varrho[x, y]$ *for some* $\alpha \in [0, 1)$, *and all* $x, y \in X$,

then:

(a) T *has a unique fixed point* ξ *in* X;

(b) $T^n x \to \xi$ *for all* $x \in X$, *and*

(c) *there exists an open neighbourhood* U *of* ξ *such that for any neighbourhood* V *of* $T\xi$ *there is an* $n(V)$ *which satisfies* $n \geqslant n(V) \Rightarrow T^n(U) \subset V$, *i.e.*

$$T^n x \in K\big(\xi, \alpha^n (1-\alpha)^{-1} \varrho[x, Tx]\big),$$

for every $x \in X$ *and* $n \in N$, *where* K *is a closed ball.*

In other words, if T is a contraction mapping on a complete metric space X, then the equation $Tx = x$ has in X a unique solution. The theorem of Banach and its extensions usualy are proved by the fact that the geometrical series converges. A different proof of the Banach theorem is given by R. Kannan in [11], where he investigates properties of subsets of X, defined as $S_\gamma := \{x \in X\colon \varrho[x, Tx] \leqslant \gamma,\ \gamma > 0\}$. For extension of Banach contraction principle and certain other related results, see References.

Let $T\colon X \to X$ be a mapping of a metric space (X, ϱ) into itself. For $x \in X$, let us denote the subset $\{x, Tx, \ldots, T^k x\}$, $k = 1, 2, \ldots$, of X by $\mathcal{O}(x, k)$ and the diameter of $\mathcal{O}(x, k)$ by $\delta[\mathcal{O}(x, k)]$. For $x, y \in X$ we put

$$\delta[\mathcal{O}(x, \infty)] := \operatorname{diam}\{x, Tx, T^2 x, \ldots\},$$

$$\delta[\mathcal{O}(x, y, \infty)] := \operatorname{diam}\{x, y, Tx, Ty, T^2 x, T^2 y, \ldots\}.$$

A space X is said to be *T-orbitally complete* iff every Cauchy sequence which is contained in $\mathcal{O}(x, \infty)$ for some $x \in X$ converges in X (cf. [20]).

In [20] we have proved the following theorem.

THEOREM A. *Let T be a mapping of a metric space X into itself and let X be T-orbitally complete. Suppose that there exists a self-map φ on $R_+ := [0, +\infty)$ such that φ is $(\forall t \in (0, +\infty))$ $\varphi(t) < t$, $\limsup_{z \to t+0} \varphi(z) < t\,(t \in (0, +\infty))$ and with the property*

$$\varrho[Tx, Ty] \leqslant \varphi(\max\{\varrho[x, y], \varrho[x, Tx], \varrho[y, Ty], \varrho[x, Ty], \varrho[y, Tx]\}),$$

for each $x, y \in X$. Then for each $x \in X$, the sequence $\{T^n x\}$ converges to a fixed point of T. The velocity of this convergence is not necessarily geometrical.

THEOREM B (Tasković [21]). *Let $T\colon X \to X$ be a mapping on X and let X be a T-orbitally complete metric space. If T satisfies the following condition: there exist real numbers α_i, β for every $x, y \in X$ such that $\alpha_1 + \alpha_2 + \alpha_3 > \beta$ and $\beta - \alpha_2 \geqslant 0 \vee \vee\, \beta - \alpha_3 \geqslant 0$, and*

$$\alpha_1 \varrho[Tx, Ty] + \alpha_2 \varrho[x, Tx] + \alpha_3 \varrho[y, Ty] + \\ + \alpha_4 \min\{\varrho[x, Ty], \varrho[y, Tx]\} \leqslant \beta \varrho[x, y],$$

then for each $x \in X$, the sequence $\{T^n x\}$ converges to a fixed point ξ of T.

In other words, in [20] we introduced the concept of a φ-contraction T of a metric space X into itself, i.e., of a mapping $T\colon X \to X$ such that for all $x, y \in X$

$$\text{(B)} \qquad \varrho[Tx, Ty] \leqslant \varphi(\varrho[x, y], \varrho[x, Tx], \varrho[y, Ty], \varrho[y, Tx], \varrho[x, Ty]),$$

where the existing mapping $\varphi\colon R_+^5 \to R_+$ is increasing and has the property $(\forall t \in (0, +\infty))$ $\limsup_{z \to t+0} \varphi(z, \ldots, z) < t$.

In the present paper we introduce the concept of a diametral φ-contraction T of a metric space X into itself, i.e., of a mapping $T\colon X \to X$ such that for every $x, y \in X$,

$$\text{(C)} \qquad \varrho[Tx, Ty] \leqslant \varphi(\delta[\mathcal{O}(x, y, \infty)]), \qquad \delta[\mathcal{O}(x, \infty)] \in R_+,$$

where the existing mapping $\varphi\colon R_+ \to R_+$ with the properties

$$(\forall t \in (0, +\infty))\big(\varphi(t) < t \wedge \limsup_{z \to t+0} \varphi(z) < t\big).$$

It may be noted that T is φ-contraction implies that T is diametral φ-contractive mapping.

And finally, at the next step we prove a very general fixed point theorem which generalizes a great number of known results.

THEOREM 1. *Let T be a diametral φ-contraction on a metric space X and let X be T-orbitally complete. Then for each $x \in X$, the sequence $\{T^n x\}$ converges to a unique fixed point ξ of T. The velocity of this convergence is not necessarily geometrical.*

The proof of this theorem is based upon the proposition, proved in [20].

PROPOSITION 1. *Let the mapping $\varphi\colon (0, +\infty) \to (0, +\infty)$ have the properties $(\forall t \in (0, +\infty))\varphi(t) < t$ and $\limsup_{z \to t+0} \varphi(z) < t$ for $t \in (0, +\infty)$. If the sequence (x_n) of non-negative real numbers satisfy the condition $x_{n+1} \leqslant \varphi(x_n)$, $n = 1, 2, \ldots$, then the sequence (x_n) tends to zero. The velocity of this convergence is not necessarily geometrical.*

Proof of Proposition 1. Since (x_n) is non-increasing sequence in R_+, there is a $t \geqslant 0$ such that $x_n \to t$ $(n \to \infty)$. We claim that $t = 0$. It $t > 0$, then

$$t = \limsup_{n \to \infty} x_{n+1} \leqslant \limsup_{n \to \infty} \varphi(x_n) \leqslant \limsup_{z \to t+0} \varphi(z) < t,$$

which is a contradiction. Consequently $t = 0$, and $\lim x_n = 0$.

Proof of Theorem 1. For $x_0 = x \in X$, let $x_n = T^n x$ $(n = 0, 1, 2, \ldots)$. It is easy to verify that the sequence $\{x_n\}$ satisfies condition $\delta[\mathcal{O}(x_{n+1}, \infty)] \leqslant \varphi(\delta[\mathcal{O}(x_n, \infty)])$, $n = 0, 1, 2, \ldots$, and hence applying Proposition 1 to the sequence $\{\delta[\mathcal{O}(x_n, \infty)]\}$ we obtain $\lim \delta[\mathcal{O}(x_n, \infty)] = 0$. This implies that $\{T^n x\}$ is a Cauchy sequence in X, and hence, by T-orbitally completeness, there is a $\xi \in X$ such that $x_n = T^n x \to \xi$ $(n \to \infty)$. Put $y_n = T^n \xi$ $(n = 0, 1, 2, \ldots)$. Since $\{y_n\}$ is a bounded sequence, $\{\delta[\mathcal{O}(x_n, y_n, \infty)]\}$ is a non-increasing sequence of non-negative reals, for some $\varepsilon_0 \geqslant 0$, $\delta[\mathcal{O}(x_n, y_n, \infty)] \to \varepsilon_0$ $(n \to \infty)$. Similarly we have $\varepsilon_0 = 0$. Thus $\xi = \lim y_n$ and by our Proposition 1 we have $\delta[\mathcal{O}(\xi, \infty)] = 0$ and it means that ξ is a fixed point of T. From (C) we have that $\xi \in X$ is unique.

Special cases of diametral φ-contraction have been discused by

(1) (Rakotch [15]) There exist a monotone decreasing function $\mathfrak{f}\colon (0, +\infty) \to [0, 1)$ such that, for each $x, y \in X$, $x \neq y$, $\varrho[Tx, Ty] \leqslant \mathfrak{f}\varrho[x, y]$.

(2) (Edelstein [7]) For each $x, y \in X$, $x \neq y$, $\varrho[Tx, Ty] < \varrho[x, y]$.

(3) (Boyd and Wong [2], Browder [3]) There exists a continuous function φ on non-negative reals R_+ satisfying $\varphi(t) < t$ for $t > 0$ such that for all $x, y \in X$

$$\varrho[Tx, Ty] \leqslant \varphi(\varrho[x, y]).$$

(4) (Kannan [11]) There exists a number $\alpha \in (0, 2^{-1})$, such that, for each $x, y \in X$,

$$\varrho[Tx, Ty] \leqslant \alpha\varrho[x, Tx] + \alpha\varrho[y, Ty].$$

(5) (Bianchini [1]) There exists a number $\alpha \in [0, 1)$, such that for each $x, y \in X$,

$$\varrho[Tx, Ty] \leqslant \alpha \max\{\varrho[x, Tx], \varrho[y, Ty]\}.$$

(6) (Reich [16], Rus [18]) There exist non-negative numbers a, b, c satisfying $a+b+c < 1$ such that, for each $x, y \in X$,

$$\varrho[Tx, Ty] \leqslant a\varrho[x, Tx] + b\varrho[y, Ty] + c\varrho[x, y].$$

(7) (Sehgal [19]) For each $x, y \in X$, $x \neq y$,

$$\varrho[Tx, Ty] < \max\{\varrho[x, y], \varrho[x, Tx], \varrho[y, Ty]\}.$$

(8) (Rhoades [17], Chatterjea [4]) There exists a number $h \in [0, 1)$ such that, for each $x, y \in X$,

$$\varrho[Tx, Ty] \leqslant h \max\{\varrho[x, Ty], \varrho[y, Tx]\}.$$

(9) (Hardy and Rogers [8]) There exist non-negative constants a_i satisfying $a_1+a_2+a_3+a_4+a_5 < 1$ such that, for each $x, y \in X$,

$$\varrho[Tx, Ty] \leqslant a_1\varrho[x, y] + a_2\varrho[x, Tx] + a_3\varrho[y, Ty] + a_4\varrho[x, Ty] + a_5\varrho[y, Tx].$$

(10) (S. Massa [13], Ćirić [5]) There exists a constant $q \in [0, 1)$, such that, for each $x, y \in X$

$$\varrho[Tx, Ty] \leqslant q \max\{\varrho[x, y], \varrho[x, Tx], \varrho[y, Ty], \varrho[x, Ty], \varrho[y, Tx]\}.$$

(11) (Daneš [6]) There exists a continuous increasing function $\varphi\colon R_+ \to R_+$, satisfying $\varphi(t) < t$ for $t > 0$, such that for all $x, y \in X$

$$\varrho[Tx, Ty] \leqslant \varphi(\max\{\varrho[x, y], \varrho[x, Tx], \varrho[y, Ty], \varrho[x, Ty], \varrho[y, Tx]\}).$$

Geometrically

$$\begin{tikzcd}
(A) \arrow[r, Rightarrow] & (1) \arrow[r, phantom, "\not\Rightarrow"] \arrow[d, Rightarrow] & (2) \arrow[r, Rightarrow] & (3) & & & & & \\
& (2) \arrow[d, Rightarrow] & (4) \arrow[l, Rightarrow] \arrow[r, Rightarrow] & (5) \arrow[r, Rightarrow] & (7) & (6) \arrow[d, Rightarrow] & & & \\
(4) \arrow[r, Leftrightarrow] & (5) \arrow[r, Rightarrow] & (6) \arrow[r, phantom, "\not\Rightarrow"] & (8) \arrow[r, Rightarrow] & (9) \arrow[r, Leftrightarrow] & (10) \arrow[r, Rightarrow] & (11) \arrow[r, Rightarrow] & (B) \arrow[r, Rightarrow] & (C)
\end{tikzcd}$$

Since conditions (A) and (1)-(11) imply the condition of diametral φ-contractions, our Theorem 1 is a generalizations of theorems of Massa, Ćirić, Kannan, Reich, Rhoades, Daneš, Bianchini, Hardy-Rogers, Kurepa, Rakotch, Boyd and Wong, I. Rus, and others.

The following example shows that a diametral φ-contraction need not satisfy conditions (1)-(11).

Example 1. Let $X = [0, +\infty)$ and define $T\colon X \to X$ by $Tx = x(1+x)^{-1}$, and distance function ϱ is the ordinary euclidean distance on the line. The mapping T is a diametral φ-contraction which for mapping $\varphi\colon R_+^5 \to R_+$ is defined as

$$\varphi(t) := t(1+t)^{-1}, \qquad t > 0.$$

Then it is easy to verify that φ satisfies all the conditions of Theorem 1. Furthermore, for any $x, y \in X$

$$\varrho[Tx,\ Ty] = \frac{|x-y|}{1+x+y+xy} \leqslant \frac{|x-y|}{1+|x-y|} \leqslant \varphi(\delta\{x,\ y,\ Tx,\ Ty\}).$$

Thus (C) holds. Since X is T-orbitally complete, it follows by Theorem 1 that T has a unique fixed point – it is a point 0. However, T does not satisfy (A) and (1)–(10) for otherwise there is a $q < 1$ such that for all $x \in X$

$$\varrho[T0,\ Tx] \leqslant \frac{x}{1+x} \leqslant q \max\left\{0, \frac{x^2}{1+x}, x, \frac{x}{1+x}, x\right\}. \tag{12}$$

Since for any $x \in X$, $x^2(1+x)^{-1} \leqslant x$, it follows by (12) that, for each $x > 0$, $x(1+x)^{-1} \leqslant qx$, that is, $(1+x)^{-1} < q$ for each $x > 0$.

This is clearly impossible. Thus, T does not satisfy (A) and (1)–(10) for any value of $q < 1$. On the other hand, let $X = \{-1, 0, 1, 2\}$ and define ϱ by letting

$$\varrho(x, y) = \begin{cases} 0 & \text{if } x = y, \\ \frac{3}{2} & \text{if } (x, y) \in \{(0, 2), (2, 0)\}, \\ 2 & \text{if } (x, y) \in \{(0, 1), (1, 0)\}, \\ 1 & \text{otherwise}. \end{cases}$$

It is clear that (X, ϱ) is a complete metric space. Consider the functions $T\colon X \to X$ and φ defined as follows: $T(-1) = T(0) = 0$, $T(1) = 2$, $T(2) = -1$, $\varphi(t) := \frac{3}{4}t$ for all $t \in R_+$. Inequality (C) holds for every $x \in X$, and inequality (11) does not hold for any non-negative real valued function φ on R_+ satisfying $\varphi(t) < t$ for every $t > 0$, where $x = 1$ and $y = 2$.

Therefore, the results of Massa, Ćirić, Kannan, Reich, Rhoades, Hardy–Rogers, Kurepa, Rakotch, Boyd and Wong, Daneš, Rus and other authors are in fact a special case of Theorem 1.

2. Reflexive Banach space. In this section fixed point theorems are established first for the mappings T which map a closed bounded convex subset K of a reflexive Banach space into itself and satisfy conditions of φ_{RBS}-contractions. The theorems extend and generalize some recent theorems of Kirk, Kannan, Browder, Göhde, Goebel, the author, and many others.

Let X be a reflexive Banach space, let K be a non-empty bounded closed convex subset of X and let $T\colon K \to K$ be a non-expansive mapping, i.e., $\|Tx - Ty\| \leqslant \|x - y\|$ for all $x, y \in K$. Our main concern is with the existence of fixed point of T, i.e., $x \in K$ such that $Tx = x$. In his paper (see [22]), Kirk proved the following theorem: If T is a non-expansive mapping of K into itself and if K

has normal structure (i.e., for each convex subset S of K which contains more than one point, there exists $x \in S$ such that $\sup\{\|x-y\|: y \in S\} < \delta(S)$, $\delta(S)$ being the diameter of S; see [22]), then T has a fixed point in K. This result was also proved in a uniformly convex space X by Browder [3], Göhde and Goebel (see [22]), the reflexivity of the space and the normal structure of K being consequences of the uniform convexity of X.

Kannan [11] considers the existence of fixed points for the mappings $T: K \to K$ which satisfy

$$\|Tx-Ty\| \leqslant 2^{-1}(\|x-Tx\|+\|y-Ty\|), \qquad x, y \in K.$$

In this paper we introduce the concept of a generalized φ_{RBS}-contraction T of K into itself, i.e., of a mapping $T: K \to K$ such that for all $x, y \in K$, $\|Tx-Ty\| \leqslant \varphi(\delta\{x, y, Tx, Ty\})$, where the existing mapping $\varphi: R_+ \to R_+ := [0, +\infty)$ is with the property $\varphi(t) \leqslant t$, $t \in R_+$.

In other words, if $\varphi(t) < t$ ($t \in R_+$), then the mappings T will be referred to as having property of φ_{RBS}-contraction.

Such mappings have been used to study fixed point and other similar problems in [16], [17], [21].

Before going to the theorems, we first recall the following definitions. A mapping T of a bounded subset K of a normed space X into itself is said to have *property* B_k on K if for every closed convex subset F of K, mapped into itself by T and containing more than one element, there exist $x \in F$ and a positive integer k ($\in N$) such that $\|x-T^k x\| < \sup\{\|x-T^k y\|: y \in F\}$.

If T is a mapping of K into itself such that for each $x \in K$, $\lim_n \delta[\mathcal{O}(T^n x)] < \delta[\mathcal{O}(x)]$ when $\delta[\mathcal{O}(x)] > 0$, where $\mathcal{O}(T^r x) := \{T^r x, T^{r+1} x, \ldots\}$, $r \geqslant 0$, $T^0 x = x$, then T is said to have *diminishing orbital diameters* over K (see [22]).

It has been shown in [22] that if K has normal structure, then a mapping T, having property of φ_{RBS}-contraction on K, of K into itself must have property B_k on K but not conversely.

Here we obtain some fixed point theorems for mappings having property of φ_{RBS}-contraction by using certain additional hypotheses. Then we compare the notions of diminishing orbital diameters, normal structure, and property B_k.

We are now in a position to formulate our theorem.

THEOREM 2. *Let X be a normed space and let T be a mapping of X into itself having the property of φ_{RBS}-contraction over X. Then if T has diminishing orbital diameters over X, T has the property B_k over X.*

Proof. Let F be a closed subset of X, mapped into itself by T, containing more than one element. If possible, let for every element $x \in F$, $\|x-T^k x\| = \sup\{\|y-T^k y\|: y \in F\} = \alpha$ ($=$ const). α is evidently non-zero, for if $\alpha = 0$, then F would contain more than one fixed point of T, which is not possible.

Now we use the following lemma, proved in our paper [22].

LEMMA ([22], p. 243). *Let $T: X \to X$ be a φ_{RBS}-contraction or generalized*

φ_{RBS}-contraction on X and let n be any positive integer. Then for each $x \in X$ and all positive integers i and j: $i, j \in \{1, \ldots, n\}$ we have $\|T^i x - T^j x\| \leqslant \delta[\mathcal{O}(x, n)]$, and for every positive integer n there exists a positive integer $k \leqslant n$ such that

$$\|x - T^k x\| = \delta[\mathcal{O}(x, n)], \qquad \textit{where } \mathcal{O}(x, n) = \{x, Tx, \ldots, T^n x\}.$$

Now, for $x \in F$ and from lemma we have $\|T^r x - T^s x\| \leqslant \alpha$ $(r, s \geqslant 1)$. Hence, for $r \geqslant 1$, $\delta[\mathcal{O}(T^r x)] = \delta\{T^r x, T^{r+1} x, \ldots\}$ (because $\|T^r x - T^s x\| \leqslant \alpha$ and $\|T^r x - T^{r+1} x\| = \alpha$). Hence, at $Tx \in F$, T does not have a diminishing orbital diameter. This contradiction completes the proof.

Remark. From the lemma we are now in a position to prove our result: Let X be a normed space and let T be a mapping of X into itself having the property of φ_{RBS}-contraction over X. If T has diminishing orbital diameters over X, T mapping a subset K of a normed space X into itself, then for every closed subset F of K mapped into itself by T and containing more than one element there exist $x \in F$ and positive integer $k \leqslant n$ such that $\|x - T^k x\| < \delta[\mathcal{O}(x, n)]$.

2.1. *Main result.* Throughout this section, unless otherwise mentioned, X is a reflexive Banach space and K a non-empty bounded closed convex subset of X.

THEOREM 3. *Let T be a mapping of a non-empty bounded, closed and convex subset K of a reflexive Banach space X into itself and let T have the property of generalized φ_{RBS}-contraction over K. Then if there exists a positive integer k such that* $\sup\{\|y - T^k y\|\colon y \in F\} < \delta(F)$ *for every non-empty bounded closed convex subset F of K (containing more than one element and mapped into itself by T), the set $I(K, T) := \{x \in K\colon Tx = x\}$ is non-empty.*

In the proof of the theorem we shall make use of the following theorem.

THEOREM 4 (Smulian). *A necessary and sufficient condition that a Banach space X be reflexive is that: Every bounded descending sequence (transfinite) of non-empty closed convex subsets of X has a non-empty intersection.*

Proof of Theorem 3. Let $\mathcal{F}$ be the family of all closed convex bounded subsets of K, mapped into itself by T. Obviously, $\mathcal{F}$ is non-empty. By the result of Smulian (see [22]) and applying Zorn's lemma, we get a minimal element S in $\mathcal{F}$, S being minimal with respect to being non-empty, bounded, closed and convex and invariant under T. If S contains only one element, then that element is a fixed point of T. If not, let S contain more than one element. Now for $x, y \in S$ (from the lemma)

$$\begin{aligned}\|Tx - Ty\| &\leqslant \varphi(\delta\{x, y, Tx, Ty\}) \leqslant \delta\{x, y, Tx, Ty\} \leqslant \delta[\mathcal{O}(x, n)]\\ &\leqslant \sup\{\|x - T^k x\|\colon x \in S\}, \qquad k \in N.\end{aligned}$$

Hence, $T(S)$ is contained in the closed ball M with T as a center and $\sup\{\|x - T^k x\|\colon x \in S\}$ as a radius. Also $S \cap M$ is invariant under T, therefore, by the minimality of S it follows that $S \subset M$, i.e., $\|Ty-x\| \leqslant \sup\{\|x-T^k x\|: x\in S\}$ for every $x\in S$. Hence, for any arbitrary but fixed $y\in S$, we have

$$(13) \qquad \sup\{\|Ty-x\|: x\in S\} \leqslant \sup\{\|x-T^k x\|: x\in S\}.$$

Let

$$S_0 = \{z\in S: \sup\{\|z-x\|: x\in S\} \leqslant \sup\{\|x-T^k x\|: x\in S\}\}.$$

Obviously, S_0 is closed, convex and non-empty ($Ty\in S_0$). Again if $z\in S_0$, then $z\in S$, and hence $Tz\in S_0$ by (13). Hence S_0 is invariant under T. Also $\delta(S_0) \leqslant \sup\{\|x-T^k x\|: x\in S\} < \delta(S)$, by hypothesis. Hence S_0 is a proper subset of S, which contradicts the minimality of S. Therefore, S has only one element which is a fixed point of T, and the set $I(K, T)$ is non-empty.

3. On a family of contractive maps on a Banach space. Kakutani [10] has shown that if a commutative family of continuous linear transformations of a linear topological space into itself leaves some non-empty compact convex subset invariant, then the family has a common fixed point in this invariant subset. The question naturally arises as to whether this is true if one considers a commutative family of continuous not necessarily linear transformations. We shall show that it is true in a rather special, but non-trivial, case, thus giving some hope that further investigation of the general question will yield positive results. The main result of this section is the following Theorem 5.

In this section we introduce the concept of a diametral contraction T of a Banach space X into itself, i.e., of a mapping $T: X \to X$ such that for every $x, y\in X$,

$$\|Tx-Ty\| \leqslant \varphi(\sup\{\|x-y\|: y\in X\}),$$

where the existing mapping $\varphi: R_+ \to R_+$ with the property $\varphi(t) \leqslant t$ for $t\in(0, +\infty)$.

THEOREM 5. *Let B be a Banach space and let X be a non-empty compact convex subset of B. If $\mathcal{F}$ is a non-empty commutative family of diametral contractive mappings of X into itself, then the family $\mathcal{F}$ has a common fixed point in X.*

Some remarks. If the norm for B is strictly convex, then the above theorem is almost trivial since in this case each contraction mapping has a fixed-point set which is non-empty, compact, and convex. In the general case, however, the fixed-point set of a diametral contraction mapping is not convex. An example illustrating this fact is constructed as follows. Let B be the space of all ordered pairs (a, b) of real numbers, where if $x=(a, b)$, then $\|x\| = \max\{|a|, |b|\}$. Define $X = \{x: \|x\| \leqslant 1\}$ and $T: X \to X$ as follows: if $x=(a, b)$, then $T(x) = (|b|, b)$. It is easily shown that T is a diametral contraction mapping and that $x=(1, 1)$ and $y=(1, -1)$ are fixed points for T. However, $\frac{1}{2}(x+y) = (1, 0)$ is not a fixed point for T.

Proof of Theorem 5. One may show by using Zorn's lemma that there exists a minimal non-empty compact convex set $X_0 \subset X$ such that X_0 is invariant under each $T \in \mathscr{F}$. If X_0 consists of a single point, then the theorem is proved. We shall now show that if X_0 consists of more than one point, then we obtain a contradiction.

We may use Zorn's lemma again to show that there exists a minimal non-empty compact but not necessarily convex set $M \subset X_0$ such that M is invariant under each $T \in \mathscr{F}$. We will now show that $M = \{T(x)\colon x \in M\}$ for each $T \in \mathscr{F}$. Since each T is continuous and M is compact, $T(M)$ must also be compact. For all $T \in \mathscr{F}$ we have $T(M) \subset M$. Let us assume that for some $g \in \mathscr{F}$ we have $g(M) = N \neq M$. Now, for any $x \in N$ there exists $y \in M$ such that $x = g(y)$. Since all functions in $\mathscr{F}$ commute, we have, for all $T \in \mathscr{F}$, $T(x) = T(g(y)) = g(T(y)) \in N$, because $T(y) \in M$. Thus, we have $T(N) \subset N \subset M$ for all $T \in \mathscr{F}$. But since N is a non-empty compact subset of X_0 which is invariant under each $T \in \mathscr{F}$ and since $N \subset M$ and $N \neq M$, we have contradicted the minimality of M. Consequently, our assumption that $M \neq N$ is false. We may assume that M has at least two points; otherwise, the theorem is proved.

Now we use the following proposition, proved in [21].

PROPOSITION 2. (a) *Let B be a Banach space and let M be a non-empty compact subset of B and let K be the closed convex hull of M. Let d be the diameter of M. If $d > 0$, then there exists an element $u \in K$ such that* $\sup\{\|x-u\|\colon x \in M\} < d$.

(b) *Let X_0 be a non-empty convex subset of a Banach space and let T be a diametral contraction mapping of X_0 into itself. If there is a compact set $M \subset X_0$ such that $M = \{T(x)\colon x \in M\}$ and M has at least two points, then there exists a non-empty closed convex set K_1 such that $T(x) \in K_1 \cap X_0$ for all $x \in K_1 \cap X_0$ and $M \cap CK_1 \neq \emptyset$. CK_1 is the complement of K_1.*

We may now apply Proposition 2 to each $T \in \mathscr{F}$. Referring to the notation of Proposition 2, we see that the set $K_1 \cap X_0$ is invariant under each $T \in \mathscr{F}$. Since K_1 is closed, we see that $K_1 \cap X_0$ is a non-empty compact convex subset of X_0. Since $X_0 \cap CK_1 \supset M \cap CK_1 \neq \emptyset$, we see that $K_1 \cap X_0 \neq X_0$. Thus, we see that if X_0 has more than one point, then we obtain a contradiction to the minimality of X_0.

4. Some applications. Non-linear functional equation of order 1. We will prove in this chapter that we can use obtained theorems, when we are concerned with the integrable solutions of functional equation in a single variable. The functional equations which appear in this paper have been thoroughly investigated in many classes of functions, such as continuous, differentiable, analytic functions, etc. Concerning the integrable solutions of functional equations, the situation is different. There are two papers on this subject. Therefore, we consider the Lebesgue integrable solutions, in turn for non-linear equations of the single order.

Let R be the set of real numbers and $R_\infty := R \cup \{-\infty, +\infty\}$, and let

(X, S, μ) be a measure space. For a $p > 0$ we denote by $\mathscr{L}^p(X, S, \mu)$ the set of all S-measurable functions $\beta \colon X \to R$ such that $\int\limits_X |\beta|^p \, d\mu < \infty$. The relation "$\sim$" in $\mathbb{L}^p(X, S, \mu)$ defined as follows: $\beta_1 \sim \beta_2$ iff $\beta_1 = \beta_2$ a.e. in X, is an equivalence. We denote by $\mathscr{L}^p(X, S, \mu)$ the set $\mathbb{L}^p(X, S, \mu)/\sim$ and by $[\beta]$ the class of equivalence of a $\beta \in \mathbb{L}^p(X, S, \mu)$. It is known that for every $p \in (0, 1)$, the space $\mathscr{L}(X, S, \mu)$ with the metric $\varrho([\beta_1], [\beta_2]) := \int\limits_X |\beta_1 - \beta_2|^p \, d\mu$ is a complete metric space, and for $p \geqslant 1$, $\mathscr{L}^p(X, S, \mu)$ with the norm $\|[\beta]\| := (\int\limits_X |\beta|^p \, d\mu)^{1/p}$ is a Banach space. Put $\alpha(p) = 1$ for $p \in (0, 1)$, and $1/p$ for $p \geqslant 1$. For every $p > 0$, $(\int\limits_X |\beta|^p \, d\mu)^{\alpha(p)}$, $[\beta] \in \mathscr{L}^p(X, S, \mu)$ is a paranorm. The convergence of β_n to β in the sense of this paranorm means the convergence in measure.

To simplify the formulation of the results, in the sequel we assume the following convention. The expression "$\beta \in \mathbb{L}^p(X, S, \mu)$ is a solution of some functional equation" means, in particular, that after inserting β into this equation its both sides are identically equal in X, whereas the statement "$[\beta] \in \mathscr{L}^p(X, S, \mu)$ is a solution of some functional equation" means that for every $g \in [\beta]$, g satisfies this equation a.e. in X. Besides these conventions, we treat the elements of $\mathscr{L}^p(X, S, \mu)$ as functions.

The general functional equation of order 1 has the form $F(x, \beta(x), \beta[g(x)]) = 0$, where F and g are given and β is unknown. We confine ourselves to the less general equations, namely, to the equation $\beta(x) = h(x, \beta[g(x)])$, when we are interested in the uniqueness of solutions, or $\beta[g(x)] = i(x, \beta(x))$, when the problem of the dependence of solutions on an arbitrary function is considered.

In this section we formulate the general assumptions on given functions for the equation $\beta(x) = h(x, \beta[g(x)])$ and we prove a uniqueness theorem. We assume:

(a) g is strictly increasing in an interval $I = (0, x_0)$, $0 < x_0 \leqslant +\infty$ and g, g^{-1} are absolutely continuous in I and $g(I)$, respectively; and $0 < g(x) < x$ $(x \in I)$.

(b) For every $y \in R$, function $h(x, y) \colon I \times R \to R$ is measurable in I; and for almost every $x \in I$, $h(x, y) \colon R \to R$ is continuous, such that

$$|h(x, y) - h(x, z)| \leqslant \varphi(|y - z|) \qquad (x \in I;\ y, z \in R),$$

where $\varphi \colon R_+ \to R_+$ is concave and fulfils the conditions $(\forall t \in R_+)\varphi(t) < t$ and $\limsup\limits_{z \to t+0} \varphi(z) < t$ $(t \in R_+)$.

At the end of this section we give one more result, whose proof is based on fixed point Theorem A (Theorem 1). We shall use the following lemma which contains Jensen's inequality for concave functions (cf. W. Feller, *An introduction to probability theory and its applications*, Vol. II, Chapter V).

PROPOSITION 3. *If* $\varphi\colon (a,b)\to R$, $-\infty \leqslant a < b \leqslant +\infty$, *is concave, then for every function* $\beta\in L(X,S,\mu)$, $\mu(X)=1$, *such that* $\beta\colon X\to(a,b)$ *we have*

$$\int_X \varphi\circ\beta d\mu \leqslant \varphi\big(\int_X \beta d\mu\big).$$

THEOREM 6. *Let* $I=(0,1)$ *and let* (a) *and* (b) *be fulfilled, where* $h(x,0)\in L(0,1)$. *Then the functional equation* $\beta(x)=h(x,\beta[g(x)])$ *has exactly one solution* $[\beta]\in\mathscr{L}(0,1)$. *Moreover, for every* $\beta_0\in L(0,1)$ *the sequence of successive approximations* $\beta_{n+1}(x)=h(x,\beta_n[g(x)])$, $n\in N$, *converges to* β *in measure.*

Proof. We shall prove that the transformation T defined by $T([\beta]) = |h(x,\beta[g(x)])|$ maps $L(0,1)$ into itself. Take a $\beta\in L(0,1)$. Then, by (b) we have

$$|h(x,\beta[g(x)])| \leqslant \varphi(|\beta[g(x)]|)+|h(x,0)|,$$

and, consequently, in view of (a) and (b), we get

$$\int_I |T([\beta])| \leqslant \int_I |\beta\circ g| + \int_I |h(x,0)| \leqslant \int_{g(I)} |\beta| + \int_I |h(x,0)| < +\infty.$$

Let $\beta_1,\beta_2\in L(0,1)$. It follows from (a) and (b) that

$$\begin{aligned}\varrho(T([\beta_1]),T([\beta_2])) &= \int_I |h(x,\beta_1[g(x)])-h(x,\beta_2[g(x)])|\,dx\\ &\leqslant \int_I \varphi(|\beta_1[g(x)]-\beta_2[g(x)]|)\,dx = \int_{g(I)} \varphi(|\beta_1(x)-\beta_2(x)|)\,dx\\ &\leqslant \int_I \varphi(|\beta_1(x)-\beta_2(x)|)\,dx.\end{aligned}$$

Hence, be the lemma of Feller, we have

$$\varrho(T([\beta_1]),T([\beta_2])) \leqslant \varphi\big(\int_I |\beta_1(x)-\beta_2(x)|\,dx\big) = \varphi(\varrho[\beta_1,\beta_2]).$$

Now the result follows from Theorem A (Theorem 1).

EXAMPLE 2. Apply Theorem 6 to the functional equation

$$\beta(x) = 2(x-x^2)\frac{\beta(x^2)}{1+|\beta(x^2)|}+\frac{1}{\sqrt{x}},\qquad x\in(0,1),$$

assuming $p=1$. Since $h(x,y)$ satisfies conditions of Theorem 6, there exists exactly one solution $[\beta]\in\mathscr{L}^p(0,1)$.

Remark. This idea is due to Matkowski [14].

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PRIRODNO-MATEMATIČKI FAKULTET
BEOGRAD, YUGOSLAVIA