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On characterizations of semi-simple and semi-local rings by properties of their modules

Various characterizations of the class of semi-simple Artinian rings and the class of semi-local rings are known. Several of these involve injectivity and projectivity conditions for the corresponding module classes.

In a recent article, J. Ahsan proved that a ring R with identity is semi-simple and Artinian if and only if every finitely generated R-module is quasi-injective, and that R is semi-local if and only if every finitely generated Jacobson semi-simple R-module is quasi-projective [1], p. 6, Propositions 1, 5.

The purpose of this note is to show that Ahsan's characterizations remain valid if the term *finitely generated* is replaced by *2-generated*. Examples will be given showing that further reduction, from *finitely generated* to *1-generated*, is not possible. In addition, we will establish the validity of the dual criteria obtained by interchanging quasi-injectivity and quasi-projectivity.

Throughout, R is an associative ring with identity and all modules are unital left *R-modules*. The Jacobson radical of an R-module M is denoted by $J(M)$. In order to avoid confusion, the *semi-simple* modules of Ahsan [1] will be called *Jacobson semi-simple* here. Thus, M is Jacobson semi-simple if $J(M) = 0$. Most of our terminology will follow Anderson and Fuller [3] and Faith [4], [5]. In particular, a module M is called *semi-simple* if M is a (direct) sum of simple submodules [3], p. 117, 9.6.

We collect various characterizations of the semi-simple Artinian rings contained in the literature. The equivalences of (1), (2*i*) and (2*p*) in the following theorem are well known (cf. [4], p. 357, 7.33 A.2). By Ahsan [1], [2], either condition (6*i*) or (6*p*) implies (1). Using a result of Osofsky's [8], p. 649, Theorem, Michler and Villamayor have proved that (3*i*) implies (1) [7], p. 189, 3.2; Satyanarayana [9] has shown the equivalence of (1) and (4*p*). The remaining implications are trivial.

THEOREM [1], [2], [4], [7], [8], [9]. *The following conditions of the ring R are equivalent.*

(1) *R is semi-simple and Artinian.*

(2*i*) *Every R-module is injective.*

(2*p*) *Every R-module is projective.*

(3*i*) *Every cyclic Jacobson semi-simple R-module is injective.*

(3*p*) *Every cyclic Jacobson semi-simple R-module is projective.*

(4*p*) *Every simple R-module is projective.*

(5*i*) *Every R-module is quasi-injective.*

(5*p*) *Every R-module is quasi-projective.*

(6*i*) *Every finitely generated R-module is quasi-injective.*

(6*p*) *Every finitely generated R-module is quasi-projective.*

Remark. The dual of (4*p*), involving injectivity instead of projectivity, does not imply (1); see [4], p. 356.

We shall prove the following results. A module M is said to be *n-generated*, n a cardinal, if M contains a generating set containing at most n elements. The ring R is *semi-local* if $R/J(R)$ is semi-simple and Artinian.

Proposition 1. *The following properties of the ring R are equivalent.*

(1) *R is semi-simple and Artinian.*

(7*i*) *Every 2-generated R-module is quasi-injective.*

(7*p*) *Every 2-generated R-module is quasi-projective.*

Proposition 2. *The following properties of the ring R are equivalent.*

(1) *R is semi-local.*

(2) *Every Jacobson semi-simple R-module is semi-simple.*

(3*i*) *Every Jacobson semi-simple R-module is quasi-injective.*

(3*p*) *Every Jacobson semi-simple R-module is quasi-projective.*

(4*i*) *Every 2-generated Jacobson semi-simple R-module is quasi-injective.*

(4*p*) *Every 2-generated Jacobson semi-simple R-module is quasi-projective.*

Remarks. 1. In (7*i*) and (7*p*) of Proposition 1, the term *2-generated* cannot be weakened to *1-generated*: every cyclic Z/p^2Z-module, p a prime, is both quasi-injective and quasi-projective [3], p. 191, 17 (1); [5], p. 207, 24.10.

2. In (4*p*) of Proposition 2, *2-generated* cannot be replaced by *1-generated*: as was shown by Fuchs and Rangaswamy [6], p. 7, Theorem, every cyclic Z-module is quasi-projective. Whether the same holds true for (4*i*), we were unable to decide.

Problem. *If every Jacobson semi-simple cyclic R-module is quasi-injective, must R be semi-local?*

We note that, if R is a left duo ring (i.e., every left ideal of R is two-sided), the answer to this problem is affirmative.

For the proofs of Propositions 1 and 2 we shall need a few auxiliary results.

Lemma 3. *Let M be an R-module.*

(i) *If the R-module $M \oplus_R R$ is quasi-injective, then M is injective.*

(ii) *If the R-module $M \oplus_R R$ is quasi-projective and M is cyclic, then M is projective.*

Proof. (i) [1], p. 6, Proof of Proposition 1; (ii) [1], p. 6, Lemma 4.

LEMMA 4. *Let I be a two-sided ideal of R and let n be a cardinal. If every Jacobson semi-simple n-generated R-module is quasi-injective (quasi-projective), then every Jacobson semi-simple n-generated R/I-module is quasi-injective (quasi-projective).*

Proof. Straightforward (cf. [1], p. 6, Lemma 2).

We are ready to prove our results.

Proof of Proposition 1. In view of the above theorem, it suffices to derive (1) from both ($7i$) and ($7p$). Let M be a cyclic R-module. Then $M \oplus_R R$ is 2-generated so that, by Lemma 3, ($7i$) implies M injective and ($7p$) implies M projective. The theorem above completes the proof.

Proof of Proposition 2. In order to derive (2) from (1), let M be an R-module such that $J(M) = 0$. Let $J = J(R)$. Clearly $M = \sum_{x \in M} Rx$, and $J(Rx) = 0$ for each $x \in M$. Let, for $x \in M$, I denote the left annihilator of x in R. Then $Rx \cong R/I$ and $0 = J(Rx) \cong J(R/I)$ implies $J \subseteq I$. Hence

$$R/I \cong (R/J)/(I/J)$$

and, since R/J is a semi-simple R-module, so is $R/I \cong Rx$. By [3], p. 117, 9.6, M is semi-simple. Thus, (1) implies (2). Since semi-simple modules are both quasi-injective and quasi-projective [3], p. 191, 17 (1), (2) implies both ($3i$) and ($3p$). Trivially, ($3i$) and ($3p$) imply ($4i$) and ($4p$), respectively. For the final implications, let $\bar{R} = R/J$ and let $\bar{M}$ be cyclic Jacobson semi-simple $\bar{R}$-module. Then $\bar{M} \oplus_{\bar{R}} R$ is a 2-generated Jacobson semi-simple $\bar{R}$-module. Using Lemmas 3 and 4, ($4i$) implies $\bar{M}$ injective and ($4p$) implies $\bar{M}$ projective. The theorem above shows $\bar{R}$ semi-simple Artinian, completing the proof.

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