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The fundamental formula for the operator $(\Delta \pm c^2)^p$ and its application

1. In this paper we shall prove the fundamental formula for the operator $(\Delta \pm c^2)^p$. Such a problem for the Laplace operator was investigated in [2].

These formulas we can applied for the construction of the effective solution of some boundary value problems for the equation of the type

$$\Delta^m u + a_1 \Delta^{m-1} u + \ldots + a_m u = 0,$$

where $a_1, a_2, \ldots, a_m$ are constant.

2. Let $x = (x_1, x_2, \ldots, x_n)$, $y = (y_1, y_2, \ldots, y_n)$, $r^2 = \sum_{i=1}^{n} (x_i - y_i)^2$. We need some lemmas.

LEMMA 1. *If q is an arbitrary real number, p is an arbitrary non-negative integer and c is an arbitrary positive real, then*

$$\Delta_y^p \big((cr)^q K_q(cr)\big) = c^{2p} \sum_{k=0}^{p} (-1)^k \binom{p}{k} a_k (cr)^{q-k} K_{q-k}(cr), \tag{1}$$

where $a_0 = 1$, $a_k = \prod_{j=1}^{k} (n+2q-2j)$, $k = 1, 2, \ldots, p$, and K_q is a MacDonald function.

The simple induction proof of Lemma 1 follows from the formula

$$\Delta_y U(r) = U''(r) + \frac{n-1}{r} U'(r)$$

and the formulas ([3], p. 79)

$$K_{q-1}(z) - K_{q+1}(z) = -\frac{2q}{z} K_q(z), \qquad \frac{d}{dz}\big(z^q K_q(z)\big) = -z^q K_{q-1}(z).$$

Similarly, we obtain

LEMMA 2. *If q is an arbitrary real number, p is an arbitrary non-negative integer, and c is an arbitrary positive real, then*

$$\Delta_y^p\big((cr)^{-q} N_q(cr)\big) = c^{2p} \sum_{k=0}^{p} (-1)^{p+k} \binom{p}{k} b_k (cr)^{-(q+k)} N_{q+k}(cr),$$

where $b_0 = 1$, $b_k = \prod_{j=1}^{p} (2q-n+2j)$, $k = 1, 2, \ldots, p$, and N_q is a Neumann function.

By induction we have

LEMMA 3. *The function $(cr)^s K_s(cr)$ $[(cr)^s N_{-s}(cr)]$ is the fundamental solution of the equation $(\Delta - c^2)^p u(x) = 0$ $[(\Delta + c^2)^p u(x) = 0]$, where $s = p - \frac{1}{2}n$.*

We shall prove

LEMMA 4. *Let D be a bounded domain whose boundary we denote by S. Let S consist of a finite number of piecewise-smooth hypersurfaces. If the functions $u, v \in C^{2p}(D)$ and are continuous with the derivatives up to order $2p-1$ in $D \cup S$, then*

$$(2) \quad \int_D \big(v(x)(\Delta - c^2)^p u(x) - u(x)(\Delta - c^2)^p v(x)\big)\, dx$$

$$= \sum_{k=0}^{p-1} \binom{p}{k-1} (-c^2)^{p-k-1} \sum_{j=0}^{k} \int_S \Big(\Delta^j u(x) \frac{d}{dn} \Delta^{k-j} v(x) - \Delta^j v(x) \frac{d}{dn} \Delta^{k-j} u(x)\Big)\, ds,$$

where n is the invard normal to S.

Proof. By the formula ([2], p. 182)

$$\int_D \big(v(x) \Delta^p u(x) - u(x) \Delta^p v(x)\big)\, dx$$

$$= \sum_{i=0}^{p-1} \int_S \left(\Delta^i u(x) \frac{d}{dn} \Delta^{p-1-i} v(x) - \Delta^i v(x) \frac{d}{dn} \Delta^{p-1-i} u(x) \right) ds$$

we have

$$\int_D \big(v(x)(\Delta - c^2)^p u(x) - u(x)(\Delta - c^2)^p v(x)\big)\, dx$$

$$= \sum_{k=0}^{p-1} \binom{p}{k+1} (-c^2)^{p-k-1} \int_D \big(v(x) \Delta^{k+1} u(x) - u(x) \Delta^{k+1} v(x)\big)\, dx$$

$$= \sum_{k=0}^{p-1} \binom{p}{k+1} (-c^2)^{p-k-1} \sum_{j=0}^{k} \int_S \left(\Delta^j u(x) \frac{d}{dn} \Delta^{k-j} v(x) - \Delta^j v(x) \frac{d}{dn} \Delta^{k-j} u(x) \right) ds,$$

which ends the proof of formula (2).

THEOREM 1. *Let D be a bounded domain whose boundary we denote by S. We assume that S consists of a finite number of piecewise-smooth hypersurfaces. If the* *function* $u \in C^{2p}(D)$ *is continuous with the derivatives up to the order* $2p-1$ *in* $D \cup S$ *and* $(\Delta - c^2)^p u(x) = 0$ *in* D, *then*

$$\sum_{k=0}^{p-1} \sum_{j=0}^{k} \alpha_k^p \int_S \left(\Delta^j u(y) \frac{d}{dn} \Delta_y^{k-j} \big((cr)^s K_s(cr)\big) - \Delta_y^j \big((cr)^s K_s(cr)\big) \frac{d}{dn} \Delta^{k-j} u(y) \right) d\sigma_y$$

$$= \begin{cases} 0 & \text{for } x \in R^n \setminus (D \cup S), \\ \pi^{n/2} (-1)^{p-1} 2^{p+n/2-1} (p-1)!\, c^{2p-n} u(x) & \text{for } x \in D, \end{cases}$$

where $\alpha_k^p = \binom{p}{k+1} (-c^2)^{p-k-1}$, $s = p - \frac{1}{2} n$.

Proof. We write $V_s(r) = (cr)^s K_s(cr)$.

(i) If $x \in R^n \setminus (D \cup S)$, then $r > 0$ for $y \in D \cup S$ and $V_s(r) \in C^{2p}(D \cup S)$. Hence Lemma 3 and formula (2) imply the first part of the theorem.

(ii) Let $x \in D$. We consider the ball $K_R \subset D$ with center x and radius R. Let S_R denote the boundary K_R. From Lemma 3 we get

$$\int_{D \setminus K_R} \big(u(y)(\Delta_y - c^2)^p V_s(r) - V_s(r)(\Delta_y - c^2)^p u(y) \big) dy = 0. \tag{3}$$

In view of (2) and (3) we can write

$$\sum_{k=0}^{p-1} \sum_{j=0}^{k} \alpha_k^p \int_{S \cup S_R} \left(\Delta^j u(y) \frac{d}{dn} \Delta_y^{k-j} V_s(y) - \Delta_y^j V_s(y) \frac{d}{dn} \Delta^{k-j} u(y) \right) d\sigma_y = 0;$$

hence

$$\sum_{k=0}^{p-1} \sum_{j=0}^{k} \alpha_k^p \int_S \left(\Delta^j u(y) \frac{d}{dn} \Delta_y^{k-j} V_s(r) - \Delta_y^j V_s(r) \frac{d}{dn} \Delta^{k-j} u(y) \right) d\sigma_y$$

$$= \sum_{k=0}^{p-1} \sum_{j=0}^{k} \alpha_k^p \int_{S_R} \left(\Delta_y^j V_s(r) \frac{d}{dr} \Delta^{k-j} u(y) - \Delta^j u(y) \frac{d}{dr} \Delta_y^{k-j} V_s(r) \right) d\sigma_y.$$

According to Lemma 1 we obtain

$$\sum_{k=0}^{p-1} \sum_{j=0}^{k} \alpha_k^p \int_{S_R} \left(\Delta_y^j V_s(r) \frac{d}{dr} \Delta^{k-j} u(y) - \Delta^j u(y) \frac{d}{dr} \Delta_y^{k-j} V_s(r) \right) d\sigma_y$$

$$= \sum_{k=0}^{p-1} \sum_{j=0}^{k} \alpha_k^p \left(c^{2j} \sum_{i=0}^{j} (-1)^i \binom{j}{i} a_i \int_{S_R} V_{s-i}(r) \frac{d}{dr} \Delta^{k-j} u(y)\, d\sigma_y - \right.$$

$$\left. - c^{2(k-j)} \sum_{i=0}^{k-j} (-1)^i \binom{k-j}{i} a_i \int_{S_R} \frac{d}{dr} V_{s-i}(r) \Delta^j u(y)\, d\sigma_y \right) = A_1 + A_2,$$

where

$$A_1 = \sum_{k=0}^{p-1} \sum_{j=0}^{k} \sum_{i=0}^{j} \alpha_k^p c^{2j} (-1)^i \binom{j}{i} a_i \int_{S_R} V_{s-i}(r) \Delta^{k-j} u(y)\, d\sigma_y,$$

$$A_2 = \sum_{k=0}^{p-1} \sum_{j=0}^{k} \sum_{i=0}^{k-j} \alpha_k^p c^{2(k-j)+1} (-1)^i \binom{k-j}{i} a_i \int_{S_R} (cr)^{s-i} K_{s-i-1}(cr) \Delta^j u(y)\, d\sigma_y.$$

Let $A_1 = A_1^+ + A_1^0 + A_1^-$, where

A_1^+ denotes the sum of these ingredients of the sum A_1 for which $s-i > 0$,

A_1^0 denotes the sum of these ingredients of the sum A_1 for which $s-i = 0$,

A_1^- denotes the sum of these ingredients of the sum A_1 for which $s-i < 0$.

Now we shall prove that $A_1^+, A_1^0, A_1^- \to 0$ as $R \to 0$. We need the following formulas ([3], p. 79):

$$K_q(z) \underset{\substack{z\to 0^+\\ q>0}}{\approx} \frac{2^{q-1}\Gamma(q)}{z^q}, \qquad K_0(z) \underset{z\to 0^+}{\approx} \ln\frac{2}{z}, \qquad K_q(z) = K_{-q}(z).$$

For $s-i > 0$ we get

$$\left|\int_{S_R} (cr)^{s-i} K_{s-i}(cr)\frac{d}{dr}\Delta^{k-j}u(y)\,d\sigma_y\right| \leqslant M_1 (cR)^{s-i}K_{s-i}(cR)R^{n-1}$$
$$\leqslant M_2 R^{n-1} \to 0 \quad \text{as } R\to 0,$$

thus $A_1^+ \to 0$ as $R \to 0$.

If $s-i = 0$, then

$$\left|\int_{S_R} (cr)^{s-i} K_{s-i}(cr)\frac{d}{dr}\Delta^{k-j}u(y)\,d\sigma_y\right| \leqslant M_1 K_0(cR)R^{n-1} \leqslant M_3(1+|\ln R|)R^{n-1}.$$

Hence $A_1^0 \to 0$ as $R \to 0$.

For $s-i < 0$ we have

$$\left|\int_{S_R} (cr)^{s-i} K_{s-i}(cr)\frac{d}{dr}\Delta^{k-j}u(y)\,d\sigma_y\right| \leqslant M_1 (cR)^{s-i}K_{i-s}(cR)R^{n-1}$$
$$\leqslant M_4 R^{2p-2i-1}.$$

Since $i < p-1$, we have $2p-2i-1 > 0$. Hence $A_1^- \to 0$ as $R \to 0$.

The numbers M_i, $i = 1, 2, 3, 4$ denote the positive constants.

Let $A_2 = A_2^+ + A_2^0 + A_2^-$, where A_2^+, A_2^0, A_2^- denote the sum of these ingredients of the sum A_2 for which $s-i-1 > 0$, $s-i-1 = 0$, $s-i-1 < 0$, respectively.

Similarly, we can verify that $A_2^+, A_2^0 \to 0$ as $R \to 0$.

Let $A_2^- = A_{21}^- + A_{22}^-$, where A_{21}^-, A_{22}^- denote sums of these ingredients in the sum A_2^- for which $i \neq p-1$, $i = p-1$, respectively.

If $s-i-1 < 0$ and $i \neq p-1$, then

$$\left|\int_{S_R} (cr)^{s-i} K_{s-i-1}(cr)\Delta^j u(y)\,d\sigma_y\right| \leqslant M_1^* (cR)^{s-i}K_{-s+i+1}(cR)R^{n-1}$$
$$\leqslant M_2^* R^{2p-2j-2},$$

where M_1^*, M_2^* are positive constants.

Since $i = 0, 1, \ldots, k-j$, $j = 0, 1, \ldots, k$, $k = 0, 1, \ldots, p-1$, the condition

$i \neq p-1$ is equivalent to the condition $j < p-1$. In this case $2p-2i-2 > 0$ and $A_{21}^{-} \to 0$ as $R \to 0$.

Let $s-i-1 < 0$ and $i = p-1$. This case is true if and only if $k = p-1$ and $j = 0$. Hence

$$A_{22}^{-} = \alpha_{p-1}^{p} c^{2p-1} (-1)^{p-1} a_{p-1} (cR)^{1-n/2} K_{-n/2}(cR) \int_{S_R} u(y)\, d\sigma_y.$$

We remark that

$$(cR)^{n/2} K_{n/2}(cR) \to 2^{n/2-1} \Gamma(n/2) \quad \text{as } R \to 0.$$

Moreover,

$$\frac{1}{\omega_n R^{n-1}} \int_{S_R} u(y)\, d\sigma_y \to u(x) \quad \text{as } R \to 0,$$

where ω_n denotes the surface area of the unit sphere in R^n. Hence

$$A_{22}^{-} \to c^{2p-n}(-1)^{p-1}(p-1)!\, 2^{p+n/2-2} \Gamma(n/2)\, \omega_n u(x) \quad \text{as } R \to 0.$$

We find $\Gamma(n/2)\,\omega_n = 2\pi^{n/2}$, thus

$$A_{22}^{-} \to c^{2p-n}(-1)^{p-1}(p-1)!\, 2^{p+n/2-1} \pi^{n/2} u(x) \quad \text{as } R \to 0.$$

Finally

$$A_1 + A_2 \to c^{2p-n}(-1)^{p-1}(p-1)!\, 2^{p+n/2-1} \pi^{n/2} u(x) \quad \text{as } R \to 0,$$

which ends the proof of Theorem 1.

Similarly we can prove

THEOREM 2. *Let D be a bounded domain whose boundary we denote by S. We assume that S consists of a finite number of piecewise-smooth hypersurfaces. If the function $u \in C^{2p}(D)$ is continuous with the derivatives up to the order $2p-1$ in $D \cup S$ and $(\Delta + c^2)^p u(x) = 0$ in D, then*

$$\sum_{k=0}^{p-1} \sum_{j=0}^{k} \beta_k^p \int_S \left(\Delta^j u(y) \frac{d}{dn} \Delta_y^{k-j} \big((cr)^s N_{-s}(cr)\big) - \Delta_y^j \big((cr)^s N_{-s}(cr)\big) \frac{d}{dn} \Delta^{k-j} u(y) \right) d\sigma_y$$
$$= \begin{cases} 0 & \text{for } x \in \mathbf{R}^n \setminus (D \cup S), \\ (-1)^p (p-1)!\, c^{2p-n}\, 2^{p+n/2} \pi^{n/2-1} u(x) & \text{for } x \in D, \end{cases}$$

where $\beta_k^p = \binom{p}{k+1} c^{2p-2k-2}$.

3. We consider the following boundary value problem

$$(4) \qquad \begin{aligned} (\Delta - c^2)^p u(x) &= 0 && \text{for } x \in D, \\ \Delta^j u(x) &= f_j(x) && \text{for } x \in S,\ j = 0, 1, \ldots, p-1, \end{aligned}$$

where f_j are given functions defined on S.

Let $G(x, y)$ be Green function for problem (4) (we assume that it exists). Under some assumptions on the function f_j we can prove that the function

$$u(x) = A^{-1} \sum_{k=0}^{p-1} \sum_{j=0}^{k} \alpha_k^p \int_S f_j(y) \frac{d}{dn} \Delta_y^{k-j} G(x, y)\, d\sigma_y$$

is a solution of problem (4), where

$$A = \pi^{n/2}(-1)^{p-1} 2^{p+n/2-1} (p-1)!\, c^{2p-n}.$$

We can obtain a similar formula for the equation

$$(\Delta + c^2)^p u(x) = 0.$$

Moreover, using the Vekua results [1], [2], we can obtain a solution of the following problem

$$\Delta^{p_0}(\Delta \mp c_1^2)^{p_1}(\Delta \mp c_2^2)^{p_2} \ldots (\Delta \mp c_k^2)^{p_k} u(x) = 0 \quad \text{for } x \in D,$$

$$\Delta^j u(x) = h_j(x) \quad \text{for } x \in S,\ j = 0, 1, \ldots, \sum_{i=0}^{k} p_i - 1.$$

The analogous result can be obtained for the problem with boundary condition of the type

$$\frac{d}{dn} \Delta^j u(x) = h_j(x) \quad \text{for } x \in S,\ j = 0, 1, \ldots, \sum_{i=0}^{k} p_i - 1.$$

Hence, a solution of these problems can be reduced to the construction of respective Green's functions.

References

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[3] G. N. Watson, *A treatise on the theory of Bessel function*, Cambridge 1962.