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Some characterizations of semi-Fredholm operators

Abstract. A continuous linear operator from one Banach space into another is called semi-Fredholm if its kernel is finite-dimensional and its range is closed. We characterize semi-Fredholm operators in terms of their action on basic sequences. For instance, we prove that if an operator maps normalized basic sequences to such bounded sequences which become basic after deleting a finite number of their terms, then it is semi-Fredholm.

Let X and Y be two Banach spaces. A linear operator $T\colon X \to Y$ is said to be *semi-Fredholm* (see, e.g., [4]) if it is continuous, its kernel $N(T)$ is finite-dimensional, and its range $T(X)$ is a closed subspace of Y. T is an *isomorphism* if it is semi-Fredholm and injective, i.e., $N(T) = \{0\}$.

The properties of semi-Fredholm operators appearing below concern their behaviour on basic sequences and are mostly well known. That they are characteristic just for this class of operators seems to be lees known, and it is the purpose of this paper to establish such characterizations. For other characterizations of semi-Fredholm operators see [2] and [5].

We fix some further terminology and notation. Given two sequences, (x_n) in X and (y_n) in Y, we say that they are *equivalent* if there is an isomorphism between $\operatorname{lin}(x_n)$ and $\operatorname{lin}(y_n)$ which maps each x_n to y_n; we write then $(x_n) \approx (y_n)$. If there is m such that $(x_n)_{n\geqslant m} \approx (y_n)_{n\geqslant m}$, then (x_n) and (y_n) will be called *almost equivalent*, and we shall write $(x_n) \sim (y_n)$. The sequence (x_n) will be called *almost basic* if $(x_n)_{n\geqslant m}$ is basic ([3], Definition 1.a.1), and *almost regular* if $(x_n)_{n\geqslant m}$ is regular (i.e., bounded and bounded away from zero), for some m.

PROPOSITION 1. *If $T\colon X \to Y$ is a continuous [and injective] linear operator, then the following statements are equivalent*:

(a) *T is semi-Fredholm [resp., an isomorphism].*

(b) *There is a closed subspace H of finite codimension in X such that $T|H$ is an isomorphism.*

(c) *Every closed infinite-dimensional subspace E of X contains another such subspace F for which $T|F$ is an isomorphism.*

(d) *There is no closed infinite-dimensional subspace E in X such that $T|E$ is compact.*

(e) *There is no normalized (or regular) basic sequence* (x_n) *in* X *for which* $Tx_n \to 0$.

Remark. The word "closed" in (b), (c) and (d) may be omitted of course.

Proof. We first note that each of the conditions above implies that $N(T)$ has finite dimension.

The implications (b) $\Rightarrow$ (c) $\Rightarrow$ (d) are obvious, and to get (b) from (a) it suffices to take as H any closed complementary subspace to $N(T)$.

(d) $\Rightarrow$ (e): Assuming that (e) fails, we find a normalized basic sequence (x_n) in X such that $\sum_{n=1}^{\infty} \|Tx_n\| < \infty$. Let $K = \sup \|P_n\|$, where P_n is the nth natural projection from $E = \overline{\operatorname{lin}}(x_i : i = 1, 2, \ldots)$ onto $\operatorname{lin}(x_i : 1 \leqslant i \leqslant n)$ ([3], p. 2). Then if $x = \sum_{n=1}^{\infty} a_n x_n \in E$ and $\|x\| \leqslant 1$, we have

$$\|Tx - TP_n x\| = \Big\|\sum_{n+1}^{\infty} a_i\, Tx_i\Big\| \leqslant 2K \sum_{n+1}^{\infty} \|Tx_i\| \to 0,$$

and hence $T|E$ is compact as the norm-limit of the sequence (TP_n) of finite rank operators.

(e) $\Rightarrow$ (a): Let M be a closed complement to $N(T)$ in X. If T is not semi-Fredholm, then $T|M$ is not an isomorphism (though it is continuous and injective). Then from Theorem 1 in [1] (with $E = M$ and $\varrho =$ the topology on M defined by the norm $x \to \|Tx\|$) it follows easily that there exists a normalized basic sequence (x_n) in M such that $Tx_n \to 0$. ∎

PROPOSITION 2. *If* $T: X \to Y$ *is a continuous injective linear operator, then the following are equivalent.*

(i) T *is an isomorphism.*

(ii) *For every basic sequence* (x_n) *in* X,

$$(x_n) \approx (Tx_n).$$

(iii) T *maps basic sequences in* X *to basic sequences in* Y.

(iv) T *maps almost basic sequences in* X *to almost basic sequences in* Y.

Remark. Note that (iii) implies T injective.

Proof. Only the implication (iv) $\Rightarrow$ (i) needs a proof. Suppose T is not an isomorphism. Then, using Proposition 1 (non(a) $\Rightarrow$ non(e)), we find a normalized basic sequence (x_n) in X such that $2^n Tx_n \to 0$. Fix some $u \in X \setminus \{0\}$. Then by the Krein–Milman–Rutman theorem ([3], Proposition 1.a.9) the sequence $(x_n + 2^{-n} u)$ is almost basic, and so is clearly $(2^n x_n + u)$. By (iv), the sequence $(2^n Tx_n + Tu)$ is almost basic in Y; moreover, it is almost regular. Since

$2^n Tx_n + Tu \to Tu$, we have a contradiction with the easily verified fact that an almost regular almost basic sequence must not have any limit point. ∎

PROPOSITION 3. *If $T: X \to Y$ is a continuous linear operator, then the following are equivalent.*

(j) *T is semi-Fredholm.*

(jj) *For every almost basic sequence (x_n) in X,*

$$(x_n) \sim (Tx_n).$$

(jjj) *T maps almost basic sequences in X to almost basic sequences in Y.*

Proof. We first note that $N(T)$ is finite-dimensional in each of cases (j)–(jjj).

(j) $\Rightarrow$ (jj): Let (x_n) be an almost basic sequence in X. Choose m large enough to have $G \cap N(T) = \{0\}$ for $G = \overline{\text{lin}}(x_n)_{n \geqslant m}$. Then (j) implies $T|G$ is an isomorphism, whence $(x_n) \sim (Tx_n)$.

(jj) $\Rightarrow$ (jjj) is obvious.

(jjj) $\Rightarrow$ (j): It suffices to apply Proposition 2 ((iv) $\Rightarrow$ (i)) to $T|M$, where M is any closed complement of $N(T)$ in X, and then use Proposition 1 ((b) $\Rightarrow$ (a)). ∎

LEMMA. *If a linear operator $T: X \to Y$ maps normalized basic sequences (and thus also almost regular almost basic sequences) to bounded sequences, then T is continuous.*

Proof. Suppose T is not continuous. Then there exists a sequence (u_n) in X such that $\sum_{n=1}^{\infty} \|u_n\| < \infty$ and (Tu_n) is not bounded. Fix a normalized basic sequence (x_n) in X. Then the sequence $(x_n + u_n)$ is almost regular and almost basic (by [3], 1.a.9). By assumption, both (Tx_n) and $(Tx_n + Tu_n)$ are bounded, which implies (Tu_n) is bounded. A contradiction. ∎

An immediate consequence of the lemma is the following

COROLLARY 1. *A linear operator $T: X \to Y$ is continuous if (and only if) its restriction to any closed subspace with a basis is continuous.*

The next result follows easily from the lemma and Propositions 2 and 3.

COROLLARY 2. *If $T: X \to Y$ is a linear operator, then:*

(a) *T is an isomorphism if and only if T maps normalized (or regular) basic sequences to bounded basic sequences.*

(b) *T is semi-Fredholm if and only if T maps normalized (or regular) basic sequences to bounded almost basic sequences.*

Remark. We do not know whether condition (c) in Proposition 1 or condition (iii) in Proposition 2 imply that T is continuous.

References

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