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On weak partial differential inequalities of first order with Volterra's operator

In this paper we prove the theorem on strong differential inequalities of first order with Volterra's operator and the theorem on weak differential inequalities of the same type. The proof of the second theorem is similar to P. Besala's proof for partial differential inequalities of first order (see [1], Remark 59.1). I thank Mr K. Zima for help in my work.

I. Introduction. Let E^{n+1} denote the space of points $P(t, x_1, \ldots, x_n)$, $\operatorname{Dim} E^{n+1} = n+1$. Let H be a subset of the space E^{n+1} of the form

$$H = \{t_0 \leqslant t \leqslant t_0 + a,\ a_i + L(t-t_0) \leqslant x_i \leqslant b_i - L(t-t_0),\ i = 1, 2, \ldots, n\},$$

where t, a, a_i, b_i, L are such real numbers that $L > 0$, $a_i < b_i$, $0 < a \leqslant (b_i - a_i)/2L$, $i = 1, 2, \ldots, n$. Let G be a subset of the half-space E^{n+1} $(t \leqslant t_0)$ such that

$$G \cap H = \{t = t_0,\ a_i \leqslant x_i \leqslant b_i,\ i = 1, 2, \ldots, n\}.$$

We introduce the notation:

$$X = (x_1, \ldots, x_n), \quad U := (u^1, \ldots, u^m), \quad Q := (q^1, \ldots, q^n),$$
$$Q(z^\nu) := (\partial z^\nu/\partial x_1, \ldots, \partial z^\nu/\partial x_n).$$

Let $(z^1, \ldots, z^m) \leqslant (v^1, \ldots, v^m)$ denote that $z^i \leqslant v^i$ for all i $(i = 1, 2, \ldots, m)$. Let $f^j(t, X, U, Q)$, $j = 1, \ldots, m$ be functions such that:

(i) The function f^j $(j = 1, \ldots, m)$ is defined on a such set that its projection on the space (t, X) contains the set H.

(ii) The function f^j $(j = 1, \ldots, m)$ is increasing with regard to the variable U (if $U < \bar{U}$, then $f^j(t, X, U, Q) \leqslant f^j(t, X, \bar{U}, Q)$).

(iii) The function f^j $(j = 1, \ldots, m)$ satisfies Lipschitz's condition with regard to the variable Q:

$$|f^j(t, X, U, Q) - f^j(t, X, U, \bar{Q})| \leqslant L \sum_{i=1}^{n} |q_i - \bar{q}_i|,$$

where L is from the definition H.

Let $u^j(t, X)$, $v^j(t, X)$, $j = 1, \ldots, m$, be defined and continuous in the set $G \cup H$, having the continuous derivative in the set H. Let, moreover, the operator $A(t, X, U)$ satisfy Volterra's condition: if $U_1(t, X) = U_2(t, X)$ for $(t, X) \in G \cup H_t$, then $A(t, X, U_1) = A(t, X, U_2)$, where $H_t = \{(\tau, X): t_0 \leqslant \tau \leqslant t\}$.

We assume that A is an operator increasing with regard to U: if $U_1 < U_2$, then $A(t, X, U_1) \leqslant A(t, X, U_2)$.

II. A system of strong partial differential inequalities of first order. Now we prove the theorem on strong partial differential inequalities:

THEOREM 1. *We assume that*:

(i) $u^j(t, X) < v^j(t, X)$ *for* $(t, X) \in G$, $j = 1, 2, \ldots, m$,

(ii) $u^j_t(t, X) \leqslant f^j(t, X, A(t, X, U), Q(u^j))$ *for* $(t, X) \in H$, $j = 1, \ldots, m$,

(iii) $v^j_t(t, X) > f^j(t, X, A(t, X, V), Q(v^j))$ *for* $(t, X) \in H$, $j = 1, \ldots, m$.

Then $u^j(t, X) < v^j(t, X)$, $j = 1, \ldots, m$, *in the whole set* H.

Proof. Since (i) is satisfied and $u^j, v^j, j = 1, \ldots, m$, are continuous in $G \cup H$ the set of elements $\tilde{t}(t_0 \leqslant \tilde{t} \leqslant t_0 + a)$ such that $u^j(t, X) < v^j(t, X)$ in the intersection $H \cap \{(t, X): t_0 \leqslant t < \tilde{t}\}$ is not empty. Let t^* denote its least upper bound. We want to show that $t^* = t_0 + a$. Let us suppose that it is not true. Therefore $t^* < t_0 + a$. Then there exists a k and a point X^* such that

$$(1)\qquad \begin{aligned} &u^\nu(t, X) \leqslant v^\nu(t, X) \quad \text{for } t_0 \leqslant t \leqslant t^*,\ \nu = 1, 2, \ldots, m,\\ &u^k(t^*, X^*) = v^k(t^*, X^*). \end{aligned}$$

We have two possible cases:

Case 1. Let (t^*, X^*) be an interior point of the set H. The function $u^j(t^*, X) - v^j(t^*, X)$ of the variable X attains, by (1), the maximum in X^*. Since (t^*, X^*) is an interior point of H and this function has a derivative with regard to X at this point, we have

$$(2)\qquad Q\big(u^j(t^*, X^*) - v^j(t^*, X^*)\big) = 0 \quad \text{or} \quad Q\big(u^j(t^*, X^*)\big) = Q\big(v^j(t^*, X^*)\big).$$

We consider the function $u^j(t, X^*) - v^j(t, X^*)$ of the variable t. It attains its maximum, in the interval $\langle t_0, t^*\rangle$, at the point t^*. Therefore,

$$(3)\qquad u^j_t(t^*, X^*) - v^j_t(t^*, X^*) \geqslant 0.$$

On the other hand, from (ii), (iii), and (2)

$$(4)\qquad \begin{aligned} u^j_t(t^*, X^*) - v^j_t(t^*, X^*) &< f^j\big(t^*, X^*, A(t^*, X^*, U), Q(u^j(t^*, X^*))\big) - \\ &\quad - f^j\big(t^*, X^*, A(t^*, X^*, V), Q(v^j(t^*, X^*))\big) \\ &= f^j\big(t^*, X^*, A(t^*, X^*, U), Q(u^j(t^*, X^*))\big) - \\ &\quad - f^j\big(t^*, X^*, A(t^*, X^*, V), Q(u^j(t^*, X^*))\big). \end{aligned}$$

Since A is the Volterra operator, $A(t^*, X^*, U)$ depends only on a behaviour of

U in the set $G \cup H_{t^*}$. Moreover, A is increasing with regard to U. Since for any $(t, X) \in G \cup H_{t^*}$

$$u^j(t, X) \leqslant v^j(t, X), \quad j = 1, \ldots, m,$$

therefore,

$$A(t^*, X^*, U) \leqslant A(t^*, X^*, V). \tag{5}$$

From (4), (5), and (ii) we have

$$u_t^j(t^*, X^*) - v_t^j(t^*, X^*) < 0 \tag{6}$$

which contradicts (3).

Case 2. Let (t^*, X^*) be a point on a side of the set H. Then we can assume that

$$\begin{aligned} & x_p^* = b_p - L(t^* - t_0), \quad p = 1, 2, \ldots, s, \\ & x_q^* = a_q + L(t^* - t_0), \quad q = s+1, s+2, \ldots, s+r, \\ & a_l + L(t^* - t_0) < x_l^* < b_l - L(r^* - t_0), \quad l = s+r+1, \ldots, n. \end{aligned} \tag{7}$$

(In the case where we change indexes a proof is analogous.)

We consider the function

$$\begin{aligned} & u^j(t^*, x_1^*, x_2^*, \ldots, x_{p-1}^*, x_p, x_{p+1}^*, \ldots, x_n^*) - \\ & \qquad - v^j(t^*, x_1^*, x_2^*, \ldots, x_{p-1}^*, x_p, x_{p+1}^*, \ldots, x_n^*) \end{aligned}$$

of the variable x_p, $a_p + L(t - t_0) \leqslant x_p \leqslant b_p - L(t - t_0)$.

By (1) and (7) this function attains maximum at $x_p^* = b_p - L(t^* - t_0)$. Therefore

$$u_{x_p}^j(t^*, X^*) - v_{x_p}^j(t^*, X^*) \geqslant 0, \quad p = 1, 2, \ldots, s. \tag{8}$$

We have also:

$$\begin{aligned} & u_{x_q}^j(t^*, X^*) - v_{x_q}^j(t^*, X^*) \leqslant 0, \quad q = s+1, \ldots, r, \\ & u_{x_l}^j(t^*, X^*) - v_{x_l}^j(t^*, X^*) = 0, \quad l = s+r+1, \ldots, n. \end{aligned} \tag{9}$$

For $t_0 \leqslant t \leqslant t^*$ we consider the function $K(t) := u^j(t, b_1 - L(t - t_0), \ldots$ $\ldots, b_s - L(t - t_0), a_{s+1} + L(t - t_0), \ldots, a_{s+r} + L(t - t_0), x_{s+r+1}^*, \ldots, x_n^*) - v^j(t, b_1 -$ $- L(t - t_0), \ldots, b_s - L(t - t_0), a_{s+1} + L(t - t_0), \ldots, a_{s+p} + L(t - t_0), x_{s+r+1}^*, \ldots, x_n^*)$ of the variable t. This function attains, by (1) and (7), maximum at t^*. Therefore,

$$\left. \frac{dK(t)}{dt} \right|_{t=t^*} \geqslant 0,$$

$$\left.\frac{dK(t)}{dt}\right|_{t=t^*} = u_t^j(t^*, X^*) - v_t^j(t^*, X^*) + (-L)\sum_{p=1}^{s}\left(u_{x_p}^j(t^*, X^*) - v_{x_p}^j(t^*, X^*)\right) + $$
$$+L\sum_{q=s+1}^{s+r}\left(u_{x_q}^j(t^*, X^*) - v_{x_q}^j(t^*, X^*)\right)$$
$$= u_t^j(t^*, X^*) - v_t^j(t^*, X^*) - L\Big[\sum_{p=1}^{s}\left(u_{x_p}^j(t^*, X^*) - v_{x_p}^j(t^*, X^*)\right) - $$
$$- \sum_{q=s+1}^{s+r}\left(u_{x_q}^j(t^*, X^*) - v_{x_q}^j(t^*, X^*)\right)\Big].$$

Therefore

$$(10)\qquad u_t^j(t^*, X^*) - v_t^j(t^*, X^*)$$
$$\geqslant L\Big[\sum_{p=1}^{s}\left(u_{x_p}^j(t^*, X^*) - v_{x_p}^j(t^*, X^*)\right) - \sum_{q=s+1}^{s+r}\left(u_{x_q}^j(t^*, X^*) - v_{x_q}^j(t^*, X^*)\right)\Big].$$

On the other hand, we have from (ii) and (iii):

$$u_t^j(t^*, X^*) - v_t^j(t^*, X^*) < f^j\big(t^*, X^*, A(t^*, X^*, U), Q(u^j(t^*, X^*))\big) - $$
$$-f^j\big(t^*, X^*, A(t^*, X^*, V), Q(v^j(t^*, X^*))\big)$$
$$= \Big[f^j\big(t^*, X^*, A(t^*, X^*, U), Q(u^j(t^*, X^*))\big) - $$
$$-f^j\big(t^*, X^*, A(t^*, X^*, V), Q(u^j(t^*, X^*))\big)\Big] + $$
$$+\Big[f^j\big(t^*, X^*, A(t^*, X^*, V), Q(u^j(t^*, X^*))\big) - $$
$$-f^j\big(t^*, X^*, A(t^*, X^*, V), Q(v^j(t^*, X^*))\big)\Big].$$

The part in the first brackets is non-positive by properties of the operator A and the function f^j.

From (iii) and (8), (9) we obtain

$$u_t^j(t^*, X^*) - v_t^j(t^*, X^*) < L\Big[\sum_{p=1}^{s}\left(u_{x_p}^j(t^*, X^*) - v_{x_p}^j(t^*, X^*)\right) - $$
$$- \sum_{q=s+i}^{s+r}\left(u_{x_q}^j(t^*, X^*) - v_{x_q}^j(t^*, X^*)\right)\Big]$$

which contradicts (10).

The theorem is proved.

III. A system of weak partial differential inequalities of first order. We prove the theorem on weak partial differential inequalities of first order:

THEOREM 2. *We assume that*

(i′) $u^j(t, X) \leqslant v^j(t, X)$, $j = 1, \ldots, m$ *for* $(t, X) \in G$,

(ii) $u_t^j(t, X) \leqslant f^j(t, X, A(t, X, U), Q(u^j))$, $j = 1, \ldots, m$, $(t, X) \in H$, (iii′) $v^j_t(t, X) \geqslant f^j(t, X, A(t, X, V), Q(v^j))$, $j = 1, \ldots, m$, $(t, X) \in H$.

Suppose that there exists a sequence $\{W^\nu(t, X)\}$, $\nu = 1, 2, \ldots$ ($W^\nu = \{w^{\nu 1}, w^{\nu 2}, \ldots, w^{\nu n}\}$), *which satisfies the conditions*:

(iv) *Functions* $w^\nu(t, X)$, $j = 1, \ldots, m$, *are continuous and positive in* $G \cup H$.

(v) $\lim\limits_{\nu \to \infty} w^\nu(t, X) = 0$ *for* $(t, X) \in G \cup H$.

(vi) $v^j_t(t, X) + w^{\nu j}_t(t, X) > f^j(t, X, A(t, X, V) + W^\nu(t, X), Q(v^j + w^{\nu j}))$, $j = 1, \ldots, m$, $(t, X) \in H$.

Let, moreover,

(vii) $A(t, X, V) + W^\nu(t, X) \geqslant A(t, X, V + W^\nu)$, $\nu = 1, 2, \ldots$

Then

$$u^j(t, X) \leqslant v^j(t, X), \quad j = 1, \ldots, m, \text{ in the set } H.$$

Proof. Let $\tilde{V}(t, X) := V(t, X) + W^\nu(t, X)$. Then

$$f^j(t, X, A(t, X, V) + W^\nu(t, X), Q(v^j + w^{\nu j})) \geqslant f^j(t, X, A(t, X, V + W^\nu), Q(v^j + w^{\nu j})).$$

This follows from (7) and (ii). Therefore, from (vi) and this inequality, we infer

$$v^j_t(t, X) + w^{\nu j}_t(t, X) > f^j(t, X, A(t, X, W^\nu), Q(v^j + w^{\nu j}))$$

or

$$\tilde{v}^j_t(t, X) > f^j(t, X, A(t, X, \tilde{V}), Q(\tilde{v}^j)).$$

From the theorem on strong inequalities:

$$u^j(t, X) < \tilde{v}^j(t, X) \quad \text{for } (t, X) \in H \cup G,\ j = 1, 2, \ldots, m,$$

or

$$u^j(t, X) < v^j(t, X) + w^{\nu j}(t, X) \quad \text{for } (t, X) \in G \cup H,\ j = 1, 2, \ldots, m.$$

Passing in this inequality, to the limit as $\nu \to \infty$, we get

$$u^j(t, X) \leqslant v^j(t, X), \quad j = 1, \ldots, m,\ (t, X) \in G \cup H.$$

IV. Examples of the operator $A(t, X, U)$ which is increasing with regard to the variable U and satisfies Volterra's condition and property

(*) $$A(t, X, V + W) \leqslant A(t, X, V) + W(t, X).$$

(1) $$A(t, X, U) := t \cdot U(t^{1+\beta}, X), \quad \beta > 0,\ t \in (0, 1),$$

(2) $$A(t, X, U) := U(t - \alpha, X), \quad \alpha > 0.$$

The operator A fulfils condition (*) for functions $W(t, X)$ non-decreasing with regard to t,

$$A(t, X, U) := \max_{t \in \langle t_0, t_0 + a\rangle} U(t, X). \tag{3}$$

V. Certain properties of the function f implying the existence of the sequence $\{W^\nu(t, X)\}$ fulfilling the assumptions of Theorem 2. We assume that the operator A satisfies condition $(*)$ for functions W non-decreasing with regard to t.

EXAMPLE 1. Assume that $t_0 := 0$. In other cases considerations are analogous. Let

$$f^j(t, X, U, Q(u^j)) - g^j(t, X, \tilde{U}, Q(\tilde{u}^j)) \leqslant \sigma^j(t, U - \bar{U}), \quad j = 1, 2, \ldots, m,$$

where $\sigma^j(t, U)$ is a comparison function of the first type (see [1]). Let $z^{\nu j}(t)$ be a solution of the equation

$$z_t^j(t) = \sigma^j(t, z) + 1/\nu, \quad t \in \langle 0, a\rangle, \ z = (z^1, z^2, \ldots, z^m)$$

with the condition $z^j(0) = 1/\nu$.

We define the sequence $\{W^\nu(t, X)\}$, $\nu = 1, 2, \ldots$, in the following way:

$$w^{\nu j}(t, X) = \begin{cases} z^{\nu j}(t) & \text{for } (t, X) \in H; \\ 1/\nu & \text{for } (t, X) \in G. \end{cases}$$

Since σ^j is a comparison function of the first type the sequence $\{W^\nu(t, X)\}$ satisfies assumptions (iv)–(vi).

EXAMPLE 2. Assume that $t_0 = 0$. Let

$$f^j(t, X, U, Q(u^j)) - f^j(t, X, \tilde{U}, Q(\tilde{u}^j)) < \sigma^j(t, U - \bar{U}),$$

where $\sigma^j(t, U)$ is a comparison function of the second type (see [1]). Let $z^{\nu j}(t)$ be a solution of the equation

$$z_t^j(t) = \sigma^j(t, z), \quad t \in \langle 0, a\rangle$$

with the condition $z^j(a) = 1/\nu$, $\nu = 1, 2, \ldots$

We define the sequence $\{W^\nu(t, X)\}$, $\nu = 1, 2, \ldots$, in the following way:

$$w^{\nu j}(t, X) = \begin{cases} z^{\nu j}(t), & (t, X) \in H - \{(t, X)\colon t = 0\}, \\ \alpha^\nu, & (t, X) \in G, \end{cases}$$

where $\alpha^\nu = \lim_{t \to 0^+} w^\nu(t)$. The sequence $\{W^\nu(t, X)\}$ satisfies assumptions (iv)–(vi), as is easy to verify.

EXAMPLE 3. Assume that $t_0 = 0$. Let

$$f^j(t, X, U, Q(u^j)) - f^j(t, X, \tilde{U}, Q(\tilde{u}^j)) < \min\left(c \cdot |u^l - \tilde{u}^l|^\lambda, \ \frac{k}{t}|u^l - \tilde{u}^l|\right),$$

where l are constans, $c > 0$, $k > 0$, $\lambda \in (0, 1)$, $k(1-\lambda) < 1$.

We introduce the notation

$$z^l = |u^l - \tilde{u}^l|.$$

Then

$$f^j(t, X, U, Q(u^j)) - f^j(t, X, \bar{U}, Q(\tilde{u}^j)) < \min\left(c\cdot(z^l)^\lambda, \frac{k}{t} z^l\right).$$

We consider solutions of the equations:

$$[z_1^l(t)]' = c(z_1^l)^\lambda, \qquad z_1^l(0) = \frac{1}{v}, \qquad t \in \left\langle 0, \frac{1}{v}\right\rangle,$$

$$[z_2^l(t)]' = \frac{k}{t} z_2^l, \qquad z_2^l\left(\frac{1}{v}\right) = z_1^l\left(\frac{1}{v}\right), \qquad t \in \left\langle \frac{1}{v}, a\right\rangle.$$

We form the sequence $\{W^v(t, X)\}$:

$$w^{vj}(t, X) = \begin{cases} z_1^l(t) = [(-\lambda+1)ct]^{1/(-\lambda+1)} + 1/v, & t \in \langle 0, 1/v\rangle, \\ z_2^l(t) = \left\{\left[c\dfrac{1-\lambda}{v}\right]^{1/(-\lambda+1)} \left(\dfrac{1}{v}\right)^{-k} + \left(\dfrac{1}{v}\right)^{-k+1}\right\}^{tk}, & t \in \left\langle \dfrac{1}{v}, a\right\rangle. \end{cases}$$

This sequence satisfies conditions (iv)–(vi).

References

[1] J. Szarski, *Differential inequalites*, Monografie Mat. T. 43, PWN, Warszawa 1967.

[2] K. Zima, *On differential inequality with a lagging argument*, Ann. Polon. Math. 18 (1966), 227–233.