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IVAN DOBRAKOV (Bratislava, Czechoslovakia)

Uniform boundedness principle for exhaustive set functions

Abstract. In this paper the well-known theorem of Nikodým on uniform boundedness of measures is generalized to the case of arbitrary exhaustive set functions with values in $[0, +\infty)$. We also prove a version of Vitali–Hahn–Saks theorem for such set functions.

The theorem of Nikodým on uniform boundedness of scalar measures, cf. [7], IV.9.8, was generalized in many ways, see the non-complete list of references at the end of the paper. In all these generalizations the considered set functions are either countably additive or only additive, or satisfy some kind of subadditivity requirements (K-subadditivity, N-triangularity). Novelty in our generalization is, see Theorem 1 below, that we consider arbitrary exhaustive ($=$ strongly bounded) set functions with values in $[0, +\infty)$. Since a set in a locally convex linear topological space is bounded if and only if the range of each continuous pseudonorm on it is bounded, we can immediately formulate the corresponding result for exhaustive set functions with values in such spaces.

In the following T will be a non-empty set, $\mathscr{R} \subset 2^T$ a ring, $\mathscr{S} \subset 2^T$ a σ-ring, and $N = \{1, 2, \ldots\}$.

Let $\mu\colon \mathscr{R} \to [0, +\infty)$ be a set function. For $A \in \mathscr{R}$ we put

$$\bar{\mu}(A) = \sup\{\mu(B);\ B \in \mathscr{R},\ B \subset A\},$$

and

$$\mu^{+0}(A) = \lim_{n\to\infty} \sup\{\mu(A \cup B);\ B \in \mathscr{R},\ \bar{\mu}(B) < 1/n\}.$$

We shall say that μ is *subadditively continuous from the right* (*at* A) if $\mu^{+0} = \mu$ ($\mu^{+0}(A) = \mu(A)$). Clearly, any subadditive, or K-subadditive ($\mu(A \cup B) \leqslant \mu(A) + K\mu(B)$ for some $K \geqslant 1$ and each $A, B \in \mathscr{R}$) set function is subadditively continuous from the right. If μ is K-subadditive, then clearly $K\mu(A-B) + K\mu(B-A) \geqslant |\mu(A) - \mu(B)|$ for each $A, B \in \mathscr{R}$. Conditions (i) in Lemma 1 and Theorem 1 below, which play a crucial role in our considerations, are in a certain sense generalizations of this inequality. What is more important, they are so general that they hold for any uniformly bounded family of set functions.

A set function $\mu\colon \mathscr{R} \to [0, +\infty)$ is said to be *exhaustive* (or *strongly* *bounded*) if $\mu(A_n) \to 0$ for each infinite sequence $A_n \in \mathcal{R}$, $n = 1, 2, \ldots$, of pairwise disjoint sets. If μ is exhaustive, then clearly $\bar{\mu}$ is also exhaustive.

LEMMA 1. *Let M be a family of set functions $\mu: \mathcal{R} \to [0, +\infty)$. Then $\sup_{\mu \in M, E \in \mathcal{R}} \mu(E) < +\infty$ if and only if the following two conditions hold:*

(i) *for each $N \in \boldsymbol{N}$ there is a $K(N) \in \boldsymbol{N}$ such that $\bar{\mu}(A-B)+\bar{\mu}(B-A) > N$ whenever $|\mu(A)-\mu(B)| > K(N)$, $\mu \in M$, $A, B \in \mathcal{R}$, and*

(ii) $\sup_{\mu \in M} \sup_n \mu(D_n) < +\infty$ *for every sequence $\{D_n\}$ of pairwise disjoint sets from $\mathcal{R}$.*

Proof. The necessity of (i) and (ii) is immediate ((i) for any $N \in \boldsymbol{N}$ take $K(N) = [\sup_{\mu \in M, E \in \mathcal{R}} \mu(E)]+1)$.

Suppose conversely that (i) and (ii) hold and $\sup_{\mu \in M, E \in \mathcal{R}} \mu(E) = +\infty$. Then there is a sequence $\{v_i\}_1^\infty \subset M$ and a sequence $\{A_i\}_1^\infty \subset \mathcal{R}$ such that $v_i(A_i) > i$ for each $i = 1, 2, \ldots$ Put $\mu_1 = \lambda_1 = v_1$ and $B_1 = D_1 = A_1$. Since $|v_i(A_1)-v_i(A_i)| \to +\infty$, according to (i) either $\bar{v}_i(A_i - A_1) \to +\infty$, case $(+)$, or $\bar{v}_i(A_1 - A_i) \to +\infty$, case $(-)$. In case $(+)$ we take a subsequence $\{i_j\}_2^\infty \subset \boldsymbol{N}$ and sets $A_j^1 \subset A_{i_j}$, $A_j^1 \in \mathcal{R}$ such that $v_{i_j}(A_j^1 - A_1) > j$ for each $j = 2, 3, \ldots$ Next we put $\mu_2 = v_{i_2}$, $D_2 = A_2^1 - A_1$ and repeat the consideration for the sequences $\{v_{i_j}\}_1^\infty$ and $\{A_j^1 - A_1\}_2^\infty$. In case $(-)$ we take a subsequence $\{i_j\}_2^\infty \subset \boldsymbol{N}$ and sets $A_j^1 \supset A_{i_j}$, $A_j^1 \in \mathcal{R}$ such that $v_{i_j}(A_1 - A_j^1) > j$ for each $j = 2, 3, \ldots$ Next we put $\lambda_2 = v_{i_2}$, $B_2 = A_1 - A_2^1$ and repeat the consideration for the sequences $\{v_{i_j}\}_2^\infty$ and $\{A_1 - A_j^1\}_2^\infty$. Continuing in this way, we either obtain an infinite sequence $\{\mu_n\}_1^\infty \subset M$ and pairwise disjoint sets $D_n \in \mathcal{R}$, $n = 1, 2, \ldots$ such that $\mu_n(D_n) > n$ for each $n = 1, 2, \ldots$, a contradiction with (ii), or we obtain an infinite sequence $\{\lambda_n\}_1^\infty \subset M$ and a non-increasing sequence of sets $B_n \in \mathcal{R}$, $n = 1, 2, \ldots$ such that $\lambda_n(B_n) > n$ for each $n = 1, 2, \ldots$ But then by (i) there is a subsequence $\{n_k\}_0^\infty$, $n_0 = 1$ such that $\bar{\lambda}_{n_k}(B_{n_{k-1}} - B_{n_k}) > k$ for each $k = 1, 2, \ldots$, what again contradicts (ii) since $D_k = B_{n_{k-1}} - B_{n_k}$, $k = 1, 2, \ldots$, are pairwise disjoint. The lemma is proved.

COROLLARY. *Let M be a family of uniformly exhaustive set functions $\mu: \mathcal{R} \to [0, +\infty)$. Then $\sup_{\mu \in M, E \in \mathcal{R}} \mu(E) < +\infty$ if and only if* (i) *holds and $\sup_{\mu \in M} \mu(E) < +\infty$ for each $E \in \mathcal{R}$. In particular, an exhaustive set function $\mu: \mathcal{R} \to [0, +\infty)$ is bounded if and only if* (i) *holds.*

We shall need the following

LEMMA 2. *Let $\mu: \mathcal{S} \to [0, +\infty)$ ($\mu: \mathcal{R} \to [0, +\infty)$) be exhaustive, and let $A_n \in \mathcal{S}$ ($A_n \in \mathcal{R}$), $n = 1, 2, \ldots$, be pairwise disjoint. Then for each $\varepsilon > 0$ there is a* *subsequence* $\{A_{n_i}\}_1^\infty \subset \{A_n\}_1^\infty$ *such that* $\bar{\mu}(\bigcup_{i\in I} A_{n_i}) < \varepsilon$ *for any (finite)* $I \subset \boldsymbol{N}$.

Proof. Let us decompose N into infinite number of infinite pairwise disjoint sets $\boldsymbol{N}_1, \boldsymbol{N}_2, \ldots$ Then the exhaustivity of $\bar{\mu}$ implies that $\bar{\mu}(\bigcup_{i\in \boldsymbol{N}_n} A_i) > \varepsilon$ for at most a finite number of $\boldsymbol{N}_n$-s.

THEOREM 1. *Let M be a family of exhaustive set functions $\mu\colon \mathscr{S} \to [0, +\infty)$. Then* $\sup_{\mu\in M, E\in\mathscr{S}} \mu(E) < +\infty$ *if and only if the following two conditions hold:*

(i) *for each* $N \in \boldsymbol{N}$ *there is a* $K(N) \in \boldsymbol{N}$ *such that* $\mu(A-B)+\mu(B-A) > N$ *whenever* $|\mu(A)-\mu(B)| > K(N)$, *and*

(ii) $\sup_{\mu\in M} \mu^{+0}(A) < +\infty$ *for each* $A \in \mathscr{S}$.

Proof. The necessity of conditions (i) and (ii) is obvious. Suppose conversely that (i) and (ii) hold and the family M is not uniformly bounded on $\mathscr{S}$. Then by Lemma 1 there is a sequence $\mu_n \in M$, $n = 1, 2, \ldots$, and a sequence of pairwise disjoint sets $D_n \in \mathscr{S}$, $n = 1, 2, \ldots$, such that $\mu_n(D_n) > n$ for each $n = 1, 2, \ldots$ Put $N_1 = [\sup_n \mu_n^{+0}(D_1)]+1$. Then $N_1 < +\infty$ by (ii). Take $K(N_1)$ according to (i), and $n_1 > K(N_1)+1$. Since μ_{n_1} is exhaustive, by Lemma 2 and the definition of μ^{+0} there is a subsequence $\{D_1^1, D_2^1, \ldots\} \subset \{D_{n_1+1}, D_{n_1+2}, \ldots\}$ such that $\mu_{n_1}(D_1 \cup \bigcup_{i\in I} D_i^1) < N_1$ for any $I \subset \boldsymbol{N}$. Put $N_2 = [\sup_n \mu_n^{+0}(D_1 \cup D_{n_1})]+1$. Then $N_2 < +\infty$ by (ii). Take $K(N_2)$ according to (i), and $n_2 > K(N_2)+2$ such that $D_{n_2} = D_i^1$ for some $i = i(n_2)$. Since μ_{n_2} is exhaustive, by Lemma 2 and the definition of μ^{+0} there is a subsequence $\{D_1^2, D_2^2, \ldots\} \subset \{D_{i(n_2)+1}, D_{i(n_2)+2}, \ldots\}$ such that $\mu_{n_2}(D_1 \cup D_{n_1} \cup \bigcup_{i\in I} D_i^2) < N_2$ for each $I \subset \boldsymbol{N}$. Continuing in this manner we obtain two sequences $\{N_k\}_1^\infty$ and $\{n_k\}_0^\infty$, $n_0 = 1$ such that

$$N_k = [\sup_n \mu_n^{+0}(\bigcup_{i=0}^{k-1} D_{n_i})]+1, \qquad \mu_{n_k}(D_{n_k}) > n_k > K(N_k)+k,$$

and

$$\mu_{n_k}(\bigcup_{i=0}^{\infty} D_{n_i} - D_{n_k}) < N_k \qquad \text{for each } k = 1, 2, \ldots$$

Take any $k > [\sup_j \mu_{n_j}(\bigcup_{i=0}^{\infty} D_{n_i})]$. Then

$$|\mu_{n_k}(\bigcup_{i=0}^{\infty} D_{n_i}) - \mu_{n_k}(D_{n_k})| > K(N_k),$$

hence by (i)

$$\mu_{n_k}(\bigcup_{i=0}^{\infty} D_{n_i} - D_{n_k}) > N_k, \qquad \text{a contradiction.}$$

The theorem is proved.

By small modifications of the just given proof we obtain a uniform boundedness result for exhaustive set functions defined on a ring, see also Lemma 1. Namely, we have:

THEOREM 1R. *A family M of exhaustive set functions $\mu\colon \mathscr{R} \to [0, +\infty)$ is uniformly bounded on $\mathscr{R}$ if and only if* (i) *and* (ii) *of Theorem* 1 *hold, and* $\sup_{\mu\in M, n\in N} \mu(A_n) < +\infty$ *for any non-decreasing sequence* $A_n \in \mathscr{R}$, $n = 1, 2, \ldots$

The modifications in the proof of Theorem 1 are: we consider finite $I \subset N$, for each $k = 1, 2, \ldots$ we have the inequality $\mu_{n_k}\left(\bigcup_{i=0}^{n_{k+1}} D_{n_i} - D_{n_k}\right) < N_k$, and we take any $k > \left[\sup_{j,r} \mu_{n_j}\left(\bigcup_{i=0}^{r} D_{n_i}\right)\right]$.

The following theorem is a version of the Vitali–Hahn–Saks theorem for exhaustive set functions. We note that conditions (a) and (b) below are automatically fulfilled for subadditive or K-subadditive set functions.

THEOREM 2. *Let $\mu_n\colon \mathscr{S} \to [0, +\infty]$, $n = 1, 2, \ldots$, be exhaustive set functions and suppose that:*

(a) *for each $\varepsilon > 0$ there is a $\delta > 0$ such that $\mu_n(A - B) > \varepsilon$ whenever $\mu_n(A)$, $\mu_n(B) < \delta$ and $B \subset A$, $A, B \in \mathscr{S}$, $n = 1, 2, \ldots$,*

(b) *for each $\delta > 0$ there is an $\eta > 0$ such that $\mu_n^{+0}(A) < \delta$ whenever $\mu_n(A) < \eta$, $n = 1, 2, \ldots$, $A \in \mathscr{S}$, and*

(c) $\lim_{n\to\infty} \mu_n(E) = \mu_0(E) \in [0, +\infty]$ *exists for each* $E \in \mathscr{S}$.

Then μ_0 is exhaustive if and only if μ_n, $n = 1, 2, \ldots$, are uniformly exhaustive. If $\mu_0 = 0$, then condition (b) *may be weakened to*

(b_0) *for each $A \in \mathscr{S}$ and $\delta > 0$ there is an $\eta > 0$ such that $\mu_n^{+0}(A) < \delta$ whenever $\mu_n(A) < \eta$, $n = 1, 2, \ldots$*

Proof. Uniform exhaustivity of μ_n, $n = 1, 2, \ldots$ and (c) immediately imply the exhaustivity of μ_0. Suppose conversely that μ_0 is exhaustive, but μ_n, $n = 1, 2, \ldots$, are not uniformly exhaustive. Then there are an $\varepsilon > 0$, a sequence of pairwise disjoint sets $E_k \in \mathscr{S}$, $k = 1, 2, \ldots$, and a subsequence $n_k \in N$, $k = 1, 2, \ldots$ such that $\mu_{n_k}(E_k) > \varepsilon$ for each $k = 1, 2, \ldots$ Let $\delta > 0$ correspond to ε according to (a), and $\eta < \delta$ to δ according to (b). Since μ_0 is exhaustive, by Lemma 2 there is an infinite subsequence $\{E_{k_i}\} \subset \{E_k\}$ such that $\mu_0\left(\bigcup_{i\in I} E_{k_i}\right) < \eta$ for any $I \subset N$. Put $\nu_i = \mu_{n_{k_i}}$, $F_i = E_{k_i}$, $i = 1, 2, \ldots$, and $i_1 = 1$. Then by (c) there is an i_2 such that $\nu_{i_2}(F_{i_1}) < \eta$. Since ν_{i_2} is exhaustive, by (b) and Lemma 2 there is an infinite subsequence $\{F_i^2\}_1^\infty \subset \{F_i\}_{i_1+1}^\infty$ such that $\nu_{i_2}\left(F_{i_1} \bigcup_{i\in I} F_i^2\right) < \delta$ for any $I \subset N$. Owing to (c) there is an i_3 such that $\nu_{i_3}(F_{i_1} \cup F_{i_2}) < \eta$ and $F_{i_3} = F_i^2$ for some i. Since ν_{i_3} is exhaustive, by (b) and Lemma 2 there is an infinite subsequence $\{F_i^3\}_1^\infty \subset \{F_i^2\}_{i_2+1}^\infty$ such that $\nu_{i_3}(F_{i_1} \cup F_{i_2} \cup \bigcup_{i \in I} F_i^3) < \delta$ for any $I \subset N$. Continuing in this way, we construct subsequently $i_4, i_5, \ldots$ Put $F = \bigcup_{j=1}^{\infty} F_{i_j}$, and according to (c) take j_0 so that $\nu_{i_{j_0}}(F) < \eta < \delta$. By construction $\nu_{i_{j_0}}(F - F_{i_{j_0}}) < \delta$; hence $\nu_{i_{j_0}}(F - (F - F_{i_{j_0}})) = \nu_{i_{j_0}}(F_{i_{j_0}}) < \varepsilon$ by (a), a contradiction. The second part of the theorem may be proved similarly.

From this proof it is clear that the theorem remains valid if $\mathscr{S}$ is replaced by $\mathscr{R}$ provided condition (c) is strengthened to uniform convergence on non-decreasing sequences.

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MATEMATICKÝ ÚSTAV SAV
BRATISLAVA, CZECHOSLOVAKIA