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The maximal topological extension of a locally solid Riesz space with the Fatou property

Abstract. Using certain Abramovič's ideas presented in [1] we give a construction of such topological extension (called "*maximal*") of a locally solid Riesz space with a Fatou topology (L, τ), which contains the other topological extensions of the space (L, τ).

Throughout the note, in what concerns topological Riesz spaces, we use the terminology of [2].

τ, π will always denote *Hausdorff* topologies on a Riesz space.

Let (L, τ) and (K, π) be two locally solid Riesz spaces. An operator $T: L \to K$ will be called an *order-topological isomorphism* if it is a continuous Riesz isomorphism (into) and $T^{-1}: T(L) \to L$ is also continuous. A locally solid Riesz space (K, π) is a *topological extension* of a locally solid Riesz space (L, τ), if there exists an order-topological isomorphism which carries L onto an order dense Riesz subspace in K.

If τ is a locally solid topology on L, then L is Archimedean, so L has the *universal completion* L^u.

$\mathscr{F}(A)$ will denote the family of all finite subsets of the set A ordered by inclusion.

THEOREM 1. *Let (L, τ) be a locally solid Riesz space and let τ be a Fatou topology. There exists a topological extension $(L^\vee, \tau^\vee)$ of (L, τ) which has the following properties*:

(i) *$\tau^\vee$ is a Fatou topology.*

(ii) *If (K, π) is a topological extension of (L, τ), then there exists a continuous Riesz isomorphism carrying K onto an order dense Riesz subspace in $L^\vee$. Furthermore, this isomorphism can be chosen to be an order-topological isomorphism if and only if π is a Fatou topology.*

Proof. Let $\mathscr{V}$ be a basis of τ-neighbourhoods of zero consisting of solid and order closed sets. Let $Q(z) = [0, |z|] \cap L_+$ for $z \in L^u$. Given $U \in \mathscr{V}$, define the subset of L^u

$$U^\vee = \{z \in L^u: Q(z) \subset U\}.$$

The family $\mathscr{V}^\vee = \{U^\vee: U \in \mathscr{V}\}$ forms a filter base and each $U^\vee$ is solid.

Now, we will show that for each $U^\vee \in \mathscr{V}^\vee$ there exists a set $V^\vee \in \mathscr{V}^\vee$

with the property $V^\vee + V^\vee \subset U^\vee$. Let $V \in \mathscr{V}$ be such a set that $V+V \subset U$ and let $z_i \in V^\vee$ $(i = 1, 2)$.

L is order dense in L^u, so we can find families (x_α), $(x_\beta) \subset L_+$ which increase to $|z_1|$ and $|z_2|$, respectively. Let $x \in Q(z_1+z_2)$. We have $x \wedge (x_\alpha + x_\beta) \uparrow_{\alpha,\beta} x$ and $x \wedge (x_\alpha + x_\beta) \leqslant x_\alpha + x_\beta \in Q(z_1) + Q(z_2) \subset V+V \subset U$. Therefore $x \in U$ because the neighbourhood U is order closed. This means that $Q(z_1+z_2) \subset U$, so by the definition of the set $U^\vee$, we get $z_1+z_2 \in U^\vee$. Finally, $V^\vee + V^\vee \subset U^\vee$. Let

$$L^\vee = \{z \in L^u \colon \forall U^\vee \in V^\vee\ \exists t_0 > 0\ \forall t \in [-t_0, t_0]\ tz \in U^\vee\}.$$

The space $L^\vee$ is an ideal in L^u and the family

$$\mathscr{V}_L^\vee = \{U^\vee \cap L^\vee \colon U \in \mathscr{V}\}$$

forms a basis of neighbourhoods of zero of some Fatou topology $\tau^\vee$ on $L^\vee$. $\mathscr{V}_L^\vee$ is of course a basis of neighbourhoods of zero of some locally solid topology. We must only verify that $\tau^\vee$ has the Fatou property. Let $z_\alpha \in U^\vee$ and $0 \leqslant z_\alpha \uparrow z$. If $Q = \bigcup_\alpha Q(z_\alpha)$, then $z = \sup Q$. We have $x \wedge y_\Delta \uparrow x$ for each $x \in Q(z)$, where $\Delta \in \mathscr{F}(Q)$ and $y_\Delta = \sup \Delta$. Moreover, $0 \leqslant x \wedge y_\Delta \leqslant y_\Delta \in Q = \bigcup_\alpha Q(z_\alpha) \subset U$. Hence $x \in U$ because U is order closed. In other words $Q(z) \subset U$, so $z \in U^\vee$.

The equality $U^\vee \cap L = U$ holds for all $U \in \mathscr{V}$, thus $(L^\vee, \tau^\vee)$ is a topological extension of (L, τ).

Let (K, π) be an arbitrary topological extension of (L, τ) and let $T_0\colon L \to K$ be an order-topological isomorphism which carries L onto an order dense Riesz subspace in K. T_0 has the unique extension to a Riesz isomorphism T from L^u onto K^u ([2], Theorem 23.18). We have $L = T^{-1}(T_0(L)) \subset T^{-1}(K) \subset L^\vee$. Indeed, the set $A_y = \{x \in L\colon T_0 x \in [0, |y|]\}$ is τ-bounded for each $y \in K$ because for every $U \in \mathscr{V}$ there exists a solid π-neighbourhood of zero W such that $T_0^{-1}(W \cap T_0(L)) \subset U$ (T_0^{-1} is continuous) and $t \cdot [0, |y|] \subset W$ for sufficiently small t. Therefore $Q(tT^{-1}(y)) = |t|\, Q(T^{-1}(y)) = |t|\, A_y \subset U$ for sufficiently small t. This implies that $T^{-1}(y) \in L^\vee$.

The operator $T^{-1}|K$ is continuous. Indeed, continuity of the operator T_0^{-1} implies the existence, for arbitrary $U^\vee \in \mathscr{V}^\vee$, of a solid π-neighbourhood of zero W such that $T_0^{-1}(W \cap T_0(L)) \subset U$. If $z \in T^{-1}(W)$ and $x \in Q(z)$, then $Tx \in W \cap T_0(L)$, so $x \in U$. Hence $Q(z) \subset U$ and $z \in U^\vee$. Thus $T^{-1}(W) \subset U^\vee$.

Assume π is a Fatou topology. Let W be a solid order closed π-neighbourhood of zero. Since T_0 is continuous, there exists an $U \in \mathscr{V}$ such that $T_0(U) \subset W$. For each $z \in U^\vee \cap L^\vee \cap T^{-1}(K)$ we can find a family $(x_\alpha) \subset U$ which increases to $|z|$. T is a Riesz isomorphism onto, so $T_0 x_\alpha = Tx_\alpha \uparrow |Tz|$. Therefore, from solidity and order closedness of W, $Tz \in W$, but this gives the inclusion $T(U^\vee \cap L^\vee \cap T^{-1}(K)) \subset W$. Thus we have showed continuity of the operator $T|T^{-1}(K)$.

Conversely, if $T\colon K \to L^\vee$ is an order-topological isomorphism carrying K onto an order dense Riesz subspace in $L^\vee$, then the family $\mathscr{B} = \{T^{-1}(W)\colon W \in \mathscr{V}_L^\vee\}$ forms a basis of π-neighbourhoods of zero consisting of solid and order closed sets (it is sufficient to observe that T is a normal Riesz homomorphism).

The locally solid Riesz space $(L^\vee, \tau^\vee)$ is called the *maximal topological extension* of the space (L, τ).

THEOREM 2. *Let (L, τ) be a locally solid Riesz space and let τ be a Fatou topology. The following conditions are equivalent:*

(i) $L = L^\vee$.

(ii) *L is Dedekind complete and every family of positive disjoint elements $(x_\alpha)_{\alpha\in A}$ such that the set $\{x_\Delta\colon x_\Delta = \sup_{\alpha\in\Delta} x_\alpha,\ \Delta \in \mathscr{F}(A)\}$ is τ-bounded, has the least upper bound in L.*

Proof. (i)$\Rightarrow$(ii). Since $L^\vee$ is an ideal in L^u, L is Dedekind complete. Let $(x_\alpha)_{\alpha\in A}$ be a family of positive disjoint elements such that the set $\{x_\Delta\colon x_\Delta = \sup_{\alpha\in\Delta} x_\alpha,\ \Delta \in \mathscr{F}(A)\}$ is τ-bounded. The family $(x_\alpha)_{\alpha\in A}$ has the least upper bound x^u in L^u. Moreover, $x^u = \sup_\Delta x_\Delta$.

For an arbitrary $U^\vee \in \mathscr{V}_L^\vee$ there exists a positive number t_0 such that $tx_\Delta \in U$ for all $\Delta \in \mathscr{F}(A)$ and all $t \in [-t_0, t_0]$. If $x \in Q(x^u)$, then $x \wedge x_\Delta \uparrow x$, so $tx \in U$ for $t \in [-t_0, t_0]$ because U is order closed. In other words $Q(tx^u) = |t|\,Q(x^u) \subset U$ for $t \in [-t_0, t_0]$. Therefore $tx^u \in U^\vee$ for $t \in [-t_0, t_0]$. It means that $x^u \in L^\vee = L$.

(ii)$\Rightarrow$(i). Let $x^\vee \in L_+^\vee$. Since L is Dedekind complete and an order dense Riesz subspace in $L^\vee$ then $x^\vee = \sup_\alpha x_\alpha$, where $(x_\alpha)_{\alpha\in A} \subset L_+$ is a family of disjoint elements ([2], Lemma 23.15). The set $H = \{x_\Delta\colon x_\Delta = \sup_{\alpha\in\Delta} x_\alpha,\ \Delta \in \mathscr{F}(A)\}$ is order bounded in $L^\vee$, so H is τ-bounded. By the assumption there exists an element $x \in L$ such that $x = \sup_\Delta x_\Delta = \sup_\alpha x_\alpha$. L is a regular Riesz subspace in $L^\vee$ ([2], Theorem 1.10). Hence $x = x^\vee$. Therefore we have showed the equality $L = L^\vee$.

In particular, Theorem 2 states that $L^\vee = (L^\vee)^\vee$.

THEOREM 3. *Let (L, τ) be a locally solid Riesz space and let τ be a Fatou topology. The maximal topological extension $(L^\vee, \tau^\vee)$ of (L, τ) is $\tau^\vee$-complete.*

Proof. Let $(L^\wedge, \tau^\wedge)$ be a topological completion of $(L^\vee, \tau^\vee)$. $L^\vee$ is an order dense Riesz subspace in $L^\wedge$ because $\tau^\vee$ is a Fatou topology ([2], Theorem 13.4). Therefore $(L^\wedge, \tau^\wedge)$ is a topological extension of $(L^\vee, \tau^\vee)$. We have $(L^\wedge)^u = L^u$.

The identity operator $I\colon L^{\vee} \to L^{\wedge}$ has the unique extension to a Riesz isomorphism "*onto*" $\bar{I}\colon L^{u} \to L^{u}$ ([2], Theorem 23.18), so $\bar{I}$ must be identity operator. Since $L^{\wedge} = \bar{I}^{-1}(L^{\wedge}) \subset (L^{\vee})^{\vee} = L^{\vee}$, we have $L^{\wedge} = L^{\vee}$ (the inclusion is a consequence of the fact that $L^{\wedge}$ is a topological extension of $L^{\vee}$).

THEOREM 4. *Let (L, τ) be a locally solid Riesz space which satisfies the Lebesgue property. If $(L^{\vee}, \tau^{\vee})$ is the maximal topological extension of (L, τ), then the following conditions are equivalent*:

(i) *$(L^{\vee}, \tau^{\vee})$ is the topological completion of (L, τ).*

(ii) *$\tau^{\vee}$ has the Lebesgue property.*

Proof. Since τ has the Lebesgue property, τ is a Fatou topology ([2], Theorem 11.6), so there exists the maximal topological extension of (L, τ). The implication (i)$\Rightarrow$(ii) is a consequence of Theorem 10.6 from [2].

(ii)$\Rightarrow$(i). L is an order dense Riesz subspace in $L^{\vee}$ and $\tau^{\vee}$ has the Lebesgue property, so L is topologically dense in $L^{\vee}$. Moreover, $\tau^{\vee} \mid L = \tau$ and $(L^{\vee}, \tau^{\vee})$ is complete by the preceding theorem. Thus $(L^{\vee}, \tau^{\vee})$ is the topological completion of (L, τ).

Remarks. 1. It is obvious that if (L, τ) is a locally convex-solid Riesz space (a metrizable locally solid Riesz space), then $(L^{\vee}, \tau^{\vee})$ is also a locally convex-solid Riesz space (a metrizable locally solid Riesz space).

2. In the metrizable case we can say more about the maximal topological extension.

Let $(L, \|\cdot\|)$ be an F^*-lattice with a Fatou F-norm $\|\cdot\|$, i.e. L is a vector lattice and $\|\cdot\|$ is an F-norm with the following properties:

(i) $|x| \leqslant |y|$ implies $\|x\| \leqslant \|y\|$ (monotonicity),

(ii) $0 \leqslant x_\alpha \uparrow x$ implies $\sup_\alpha \|x_\alpha\| = \|x\|$.

The balls determined by $\|\cdot\|$ form a basis of a Fatou topology τ.

It is not difficult to see that $L^{\vee} = \{z \in L^{u}\colon |||tz||| \to 0 \text{ if } t \to 0\}$, where $|||z||| = \sup_{x \in Q(z)} \|x\|$. The functional $|||\cdot||| \mid L^{\vee}$ is a Fatou F-norm and, moreover, the topology $\tau^{\vee}$ equals the topology determined by this F-norm. Since $|||\cdot||| \mid L = \|\cdot\|$, the identity operator $I\colon L \to L^{\vee}$ is an isometry. The following theorem was proved in [3] (Theorem 5.2.1):

If $(K, \|\cdot\|_K)$ is an F^*-lattice (i.e. $\|\cdot\|_K$ is a monotone F-norm) containing an order dense Riesz subspace Riesz isomorphic and isometric to $(L, \|\cdot\|)$, then there exists a Riesz isomorphism $T\colon K \to L^{\vee}$ carrying K onto an order dense Riesz subspace in $L^{\vee}$ such that $|||Ty||| \leqslant \|y\|_K$ for all $y \in K$.

Moreover, $(K, \|\cdot\|_K)$ can be carried onto an order dense Riesz subspace in $L^{\vee}$ by a Riesz isomorphism being an isometry if and only if $\|\cdot\|_K$ is a Fatou F-norm.

The proof of this statement is similar to the proof of Theorem 1, so we will omit it.

3. Let L be a Dedekind complete Riesz space and let τ be a locally solid

Hausdorff topology on L. There exists a topological extension $(L^{\vee}, \tau^{\vee})$ of (L, τ) such that if (K, π) is another topological extension of (L, τ), then it is possible to carry the space K onto an order dense Riesz subspace in $L^{\vee}$ by a continuous Riesz isomorphism. Indeed, to prove this fact it is sufficient to repeat the argument used in the proof of Theorem 1 and observe that now L is an ideal in L^u, so addition in $L^{\vee}$ is continuous.

Investigations of maximal topological extensions were begun by Ju. A. Abramovič. His paper [1] is just devoted to the maximal normed extensions of Dedekind complete normed Riesz spaces.

Let (S, Σ, μ) be a measure space and let a function $\psi\colon [0, \infty)\times S \to [0, \infty)$ satisfy the following conditions:

(ψ1) the function $\psi(r, \cdot)\colon S \to [0, \infty)$ is Σ-measurable for each r;

(ψ2) the function $\psi(\cdot, s)\colon [0, \infty) \to [0, \infty)$ is non-decreasing, left continuous, continuous at zero for each $s \in S$ and $\psi(r, s) = 0$ iff $r = 0$.

The space of Σ-measurable real functions finite μ-almost everywhere (functions equal μ-almost everywhere are identified) will be denoted by $L^0(S, \Sigma, \mu)$. $L^0(S, \Sigma, \mu)$ is a Riesz space with respect to the standard order and it contains the so-called *Musielak–Orlicz space* $L^{\psi}(S, \Sigma, \mu)$,

$$L^{\psi}(S, \Sigma, \mu) = \{f \in L^0(S, \Sigma, \mu)\colon \exists k > 0 \;\; M_{\psi}(kf) = \int_S \psi(k|f(s)|, s)\, d\mu < \infty\}.$$

The functional $|\cdot|_{\psi}\colon L^{\psi}(S, \Sigma, \mu) \to [0, \infty)$ defined by the equality

$$|f|_{\psi} = \inf\{\varepsilon > 0\colon M_{\psi}(\varepsilon^{-1}f) \leqslant \varepsilon\}$$

is an F-norm. A Musielak–Orlicz space $\big(L^{\psi}(S, \Sigma, \mu), |\cdot|_{\psi}\big)$ is a complete locally solid Riesz space with the σ-Levi property. Moreover, each order bounded family of positive disjoint elements is at most countable, so $L^{\psi}(S, \Sigma, \mu)$ is super Dedekind complete and the topology on $L^{\psi}(S, \Sigma, \mu)$ is a Fatou topology.

Let $L^{\psi}_f(S, \Sigma, \mu) = \{f \in L^{\psi}(S, \Sigma, \mu)\colon \forall k > 0 \;\; M_{\psi}(kf) < \infty\}$. The space $L^{\psi}_f(S, \Sigma, \mu)$ is a closed and super order dense ideal in $L^{\psi}(S, \Sigma, \mu)$ ([3], Theorem 2.13).

THEOREM 5. *The Musielak–Orlicz space $L^{\psi}(S, \Sigma, \mu)$ is the maximal topological extension of $L^{\psi}_f(S, \Sigma, \mu)$ and $L^{\psi}(S, \Sigma, \mu)$.*

Proof. Let $L^{\vee}$ be the maximal topological extension of $L^{\psi}_f(S, \Sigma, \mu)$. We have $L^{\psi}(S, \Sigma, \mu) \subset L^{\vee}$. Each order bounded family $(x^{\vee}_{\alpha})_{\alpha\in A} \subset L^{\vee}$ of positive disjoint elements is at most countable. Indeed, let $x^{\vee}_{\alpha} \leqslant x^{\vee}$ for all $\alpha \in A$ and put $M^{\vee}(z^{\vee}) = \sup\{M_{\psi}(x)\colon x \in Q(z^{\vee})\}$ for all $z^{\vee} \in L^{\vee}$. It is not difficult to verify that

1° $M^{\vee}(z^{\vee}) = 0$ iff $z^{\vee} = 0$,

2° $|z^{\vee}| \leqslant |y^{\vee}|$ implies $M^{\vee}(z^{\vee}) \leqslant M^{\vee}(y^{\vee})$,

3° $M^{\vee}(tz^{\vee}) \to 0$ if $t \to 0$,

4° $M^{\vee}(z^{\vee} + y^{\vee}) = M^{\vee}(z^{\vee}) + M^{\vee}(y^{\vee})$ if $z^{\vee} \wedge y^{\vee} = 0$.

Let $t > 0$ be a number such that $M^{\vee}(tx^{\vee}) < \infty$. We have $\{x_\alpha^{\vee}: x_\alpha^{\vee} \neq 0\} = \bigcup_{n=1}^{\infty} \{x_\alpha^{\vee}: M^{\vee}(tx_\alpha^{\vee}) > n^{-1}\}$. Each set $A_n = \{x_\alpha^{\vee}: M^{\vee}(tx_\alpha^{\vee}) > n^{-1}\}$ must be finite. Otherwise, there exist a number n_0 and a sequence $(x_{\alpha_n}^{\vee}) \subset A_{n_0}$ with the property $M^{\vee}(tx_{\alpha_n}^{\vee}) > n_0^{-1}$ for all n. The elements x_{α_n} are disjoint, so

$$jn_0^{-1} < \sum_{n=1}^{j} M^{\vee}(tx_{\alpha_n}^{\vee}) = M^{\vee}(t \sup_{1 \leqslant n \leqslant j} x_{\alpha_n}^{\vee}) \leqslant M^{\vee}(tx^{\vee})$$

for all natural numbers j — a contradiction.

Since $L_f^{\psi}(S, \Sigma, \mu)$ is Dedekind complete and order dense in $L^{\vee}$ then each $x^{\vee} \in L_+^{\vee}$ is the least upper bound of some family $(x_\alpha)_{\alpha \in A} \subset L_f^{\psi}(S, \Sigma, \mu)$ of positive disjoint elements ([2], Lemma 23.15). The family $(x_\alpha)_{\alpha \in A}$ is at most countable and the set $\{x_\Delta: x_\Delta = \sup_{\alpha \in \Delta} x_\alpha, \Delta \in \mathscr{F}(A)\}$ is topologically bounded in $L^{\psi}(S, \Sigma, \mu)$. The space $L^{\psi}(S, \Sigma, \mu)$ has the σ-Levi property, so $x = \sup_\alpha x_\alpha$ exists in $L^{\psi}(S, \Sigma, \mu)$. The inclusion $L^{\psi}(S, \Sigma, \mu) \subset L^{\vee}$ implies the inequality $x^{\vee} \leqslant x$. Since $L^{\psi}(S, \Sigma, \mu)$ is Dedekind complete and order dense in $L^{\vee}$, then $L^{\psi}(S, \Sigma, \mu)$ is an ideal in $L^{\vee}$ ([2], Theoren 2.2). Therefore $x^{\vee} \in L^{\psi}(S, \Sigma, \mu)$. This fact gives the equality $L^{\vee} = L^{\psi}(S, \Sigma, \mu)$. It is clear that the topology $\tau^{\vee}$ equals the topology determined by the F-norm $|\cdot|_{\psi}$.

If $L_1^{\vee}$ is the maximal topological extension of $L^{\psi}(S, \Sigma, \mu)$, then $L_1^{\vee} \subset L^{\vee}$ because $L_f^{\psi}(S, \Sigma, \mu)$ is super order dense in $L^{\psi}(S, \Sigma, \mu)$ and thus $L_1^{\vee}$ is a topological extension of $L_f^{\psi}(S, \Sigma, \mu)$. This remark ends the proof.

Added in proof. The topology $\tau^{\vee}$ has the Levi property. Indeed, if $z_\alpha \in L^{\vee}$ and $z_\alpha \uparrow$ is a $\tau^{\vee}$-bounded net then the set $\{z_\alpha\}$ is dominable ([2], Theorem 24.2). Therefore $z_\alpha \uparrow z$, where $z \in L^u$ ([2], Theorem 23.22). Since $\tau^{\vee}$ has a base of neighbourhoods of zero consisting of order closed sets, then it is not difficult to check that $z \in L^{\vee}$.

Now the $\tau^{\vee}$-completeness of $L^{\vee}$ is evident (see [2], Theorem 13.9).

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