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Separability and local boundedness of Orlicz spaces of functions with values in separable, linear topological spaces

Abstract. The paper deals with the theory of Orlicz spaces of functions with values in separable, linear topological spaces. In Section 1 we define Orlicz spaces $L^{\Phi}(T, X)$ and a subspace $E^{\Phi}(T, X)$ of the space $L^{\Phi}(T, X)$. Section 2 contains a theorem on separability of the space $E^{\Phi}(T, X)$. In Section 3 we prove a theorem on local boundedness of the space $L^{\Phi}(T, X)$.

1. Preliminaries.

1.1. We assume henceforth that (T, Σ, μ) is a measure space, where Σ is a σ-algebra of subsets of T, μ is a positive, σ-finite, atomless and complete measure on Σ.

1.2. (X, τ) will denote a separable, linear topological space. $\mathscr{B}$ will be always a base of neighbourhoods of θ, where θ is the origin of X. Moreover, Λ will denote the smallest σ algebra of subsets of X containing τ.

1.3. Let $\mathscr{M}(T, X)$ be the set of all functions $f\colon T \to X$ such that for every $U \in \Lambda$, $f^{-1}(U) \in \Sigma$. $\mathscr{M}_0(T, X)$ will be a linear subspace of the set $\mathscr{M}(T, X)$.

1.4. Let $\omega\colon R_+ \to R_+$ be a homeomorphism such that $\omega(1) = 1$ and $\omega(u \cdot v) \geqslant \omega(u) \cdot \omega(v)$ for every $u, v \in R_+$, $(R_+ = [0, +\infty))$.

1.5. $F\colon R_+ \times R_+ \to R_+$ will denote an F-operation, i.e. F have the following properties: F is continuous, $F(\cdot, u)$ and $F(u, \cdot)$ are non-decreasing functions for every fixed $u \in R_+$, $F(u, v) = F(v, u)$, $F(u, 0) = u$ and $F(u, F(v, w)) = F(F(u, v), w)$.

1.6. $r\colon R_+ \to R_+$ is a function satisfying the properties:

$$F(r(u), r(v)) \leqslant r(u+v), \quad r(u) > 0 \quad \text{for every } u > 0 \quad \text{and} \quad \lim_{u \to 0} r(u) = 0.$$

1.7. Definition. A function $\Phi\colon X \times T \to R_+$ is said to be a *Φ-function* on $\mathscr{M}_0(T, X)$, if

(a) Φ is $\Lambda \times \Sigma$ measurable,

(b) $\Phi(\theta, t) = 0$ for almost every $t \in T$,

(c) $\Phi(x, t) = \Phi(-x, t)$ for every $x \in X$ and almost every $t \in T$,

(d) for every $f, g \in \mathcal{M}_0(T, X)$ and $u, v \in R_+$ such that $\omega(u)+\omega(v) \leqslant 1$

$$\int_T \Phi(uf(t)+vg(t), t)\,d\mu \leqslant F\Big(\int_T \Phi(f(t), t)\,d\mu;\ \int_T \Phi(g(t), t)\,d\mu\Big),$$

(e) $\Phi(\cdot, t)\colon X \to R_+$ is continuous for almost every $t \in T$, i.e. there is a set S of measure 0 in Σ such that for every $t \notin S$, $x \in X$ and $\varepsilon > 0$ there is a neighbourhood $U \in \mathscr{B}$ that for every $y \in x+U$

$$|\Phi(x, t)-\Phi(y, t)| < \varepsilon,$$

(f) there is a set $S \in \Sigma$ of measure 0 with the following property: for every $c_1, c_2 > 0$ such that $c_1 \leqslant c_2$, $\Phi(c_1 x, t) \leqslant \Phi(c_2 x, t)$ for every $x \in X$ outside the set S.

1.8. Definition. A Φ-function $\Phi\colon X \times T \to R_+$ is said to be a Φ_1-*function*, if for every set A of finite measure and for every positive number ε there is a neighbourhood $U \in \mathscr{B}$ such that for every $z \in U$,

$$\int_A \Phi(z, t)\,d\mu < \varepsilon.$$

1.9. For every Φ-function $\Phi\colon X \times T \to R_+$ a function $\varrho\colon \mathcal{M}_0(T, X) \to R_+$ defined by $\varrho(f) = \int_T \Phi(f(t), t)\,d\mu$ is an (F, ω)-*pseudomodular on* $\mathcal{M}_0(T, X)$.

1.10 The space

$$L^\Phi(T, X) = \{f \in \mathcal{M}_0(T, X)\colon \lim_{c \to 0} \varrho(cf) = 0\}$$

is an (F, ω)-pseudomodular space. We call the space $L^\Phi(T, X)$ the *Orlicz space* (see [4]).

Moreover, a function $|\cdot|\colon L^\Phi(T, X) \to R_+$ defined by

$$|f| = \inf\left\{u > 0\colon \int_T \Phi\left(\frac{f(t)}{\omega^{-1}(u)}, t\right)d\mu < r(u)\right\}$$

is an F-pseudonorm on $L^\Phi(T, X)$.

1.11 By $\chi_A\colon T \to \{0, 1\}$ we denote an indicator function of the set $A \subset T$.

$P(T, X)$ will denote the space of all functions of the type

$$\sum_{k=1}^{n} x_k \chi_{A_k},$$

where $x_k \in X$, $A_k \in \Sigma$ and $\mu(A_k) < +\infty$ for every $k = 1, 2, \ldots, n$. The space $P(T, X)$ will be called the *space of simple functions.*

We assume henceforth that $P(T, X) \subset \mathcal{M}_0(T, X)$.

1.12. $E^\Phi(T, X)$ will denote the subspace of the space $L^\Phi(T, X)$ which consists of all these functions $f \in L^{\Phi}(T, X)$ for which there is a sequence (g_n) of simple functions that $\lim_{n\to\infty} |f-g_n| = 0$.

1.13. By $L_0^{\Phi}(T, X)$ we denote the set

$$\{f \in \mathcal{M}_0(T, X): \int_T \Phi(f(t), t)d\mu < +\infty\}.$$

Of course, $L_0^{\Phi}(T, X)$ is a subset of the space $L^{\Phi}(T, X)$.

1.14. The F-pseudonorm $|\cdot|$ yields topologies on $L^{\Phi}(T, X)$ and $E^{\Phi}(T, X)$. We shall always consider the spaces $L^{\Phi}(T, X)$ and $E^{\Phi}(T, X)$ with these topologies. Therefore, the space $E^{\Phi}(T, X)$ is the closure of the set $P(T, X)$ in the space $L^{\Phi}(T, X)$.

1.15. By $\tilde{P}(T, X)$ we denote the space of all functions of the type

$$\sum_{k=1}^{\infty} x_k \chi_{A_k},$$

where $(x_k) \subset X$ and the family $\{A_k\}$ is a family of disjoint sets of finite measures.

1.16. A. Kozek in papers [1], [2] investigated spaces L_{Φ} of functions defined on an abstract set T with values in a Banach space X, generated by an N-function Φ: $X \times T \to [0, +\infty]$. Let us suppose that X is a Banach space, $F(u, v) = u+v$, $r(u) = u$ and $\omega(u) = u$. If Φ: $X \times T \to [0, +\infty)$ is a convex Φ-function, then it is an N''-function, too. Moreover, if X is the space of real numbers, then Φ-function Φ: $R \times T \to [0, +\infty)$ is a well-known φ-function with parameter which has been investigated in many papers, for example in [3] and [5].

2. Separability of Orlicz spaces.

2.1. We assume henceforth that D is a countable and dense subset of X and Φ: $X \times T \to R_+$ is a Φ-function on $\mathcal{M}_0(T, X)$. Let

$$P(x) = \{t \in T: \Phi(ax, t) > b\Phi(x, t)\},$$

where $a, b, c > 0$. (It is easy to verify that $P(x) \in \Sigma$.) Functions $\varkappa$: $T \to R_+$ and $\tilde{\varkappa}$: $T \to R_+$ are defined in the following manner

$$\varkappa(t) = \sup_{x\in X} \Phi(cx, t)\chi_{P(x)}(t), \qquad \tilde{\varkappa}(t) = \sup_{x\in D} \Phi(cx, t)\chi_{P(x)}(t).$$

Then we have the following lemma.

2.2 Lemma. *$\varkappa(t)$ is equal to $\tilde{\varkappa}(t)$ for almost every $t \in T$.*

Proof. Since Φ is a Φ-function, there is a set $S \in \Sigma$, $\mu(S) = 0$, such that for every $t \notin S$, $x \in X$ and $\varepsilon > 0$ there is a neighbourhood U of the origin θ

with

$$|\Phi(x, t) - \Phi(y, t)| < \varepsilon$$

for every $y \in x+U$ (condition 1.7 (e)).

(a) We shall show that if $t \notin S$ and $t \in P(x)$ for some $x \in X$, then for every $V \in \mathscr{B}$ there is $y \in D \cap (x+V)$ such that $t \in P(y)$.

Indeed, let $t \in P(x)$ and $V \in \mathscr{B}$. Then there is an $\eta > 0$ such that

$$b\Phi(x, t) < b\Phi(x, t) + \eta < \Phi(ax, t) - \eta < \Phi(ax, t).$$

Now, putting $\varepsilon = \eta/b$ in the first case and $\varepsilon = \eta$ in the second one, condition 1.7 (e) implies that

$$\exists\, U \in \mathscr{B}\ \forall\, z \in x+U \quad |\Phi(x, t) - \Phi(z, t)| < \eta/b,$$

and

$$\exists\, W \in \mathscr{B}\ \forall\, z \in ax+W\ |\Phi(ax, t) - \Phi(z, t)| < \eta.$$

Moreover, since D is a countable and dense subset of X, there is a $y \in D$ such that $y \in x + (a^{-1} W \cap U \cap V)$. Hence

$$b\Phi(y, t) < b\Phi(x, t) + \eta < \Phi(ax, t) - \eta < \Phi(ay, t).$$

Thus $t \in P(y)$.

(b) Now, we shall prove that

$$(*) \qquad \varkappa(t) = \tilde{\varkappa}(t)$$

for every $t \notin S$.

It is obvious that for every $t \in T$, $\tilde{\varkappa}(t) \leqslant \varkappa(t)$.

If $\varkappa(t) = 0$, then $\tilde{\varkappa}(t) = 0$ and equality $(*)$ is true.

If $\varkappa(t) = +\infty$, then there is a sequence (x_n) of elements of the space X such that for every n we have $\Phi(cx_n, t) > n$ and $t \in P(x_n)$. But condition 1.7 (e) implies that for every n there is a set $V_n \in \mathscr{B}$ such that for every $z \in cx_n + V_n$,

$$\Phi(cx_n, t) - 1/n < \Phi(z, t).$$

By the first part of this proof, we obtain that for every n we can find $y_n \in D \cap (x_n + c^{-1} V_n)$ such that $t \in P(y_n)$. Moreover, for every n

$$\Phi(cx_n, t) - 1/n < \Phi(cy_n, t).$$

Thus

$$\tilde{\varkappa}(t) \geqslant \sup_n \Phi(cy_n, t) \geqslant \sup_n \big(\Phi(cx_n, t) - 1/n\big) = +\infty.$$

Hence equality $(*)$ is true.

Now, let us suppose that $0 < \varkappa(t) < +\infty$. Then there is a sequence $(x_n) \subset X$ such that $t \in P(x_n)$ and $\Phi(cx_n, t) > \varkappa(t) - 1/2n$ for every n. By condition 1.7 (e) we obtain that for every n there is a set $W_n \in \mathscr{B}$ such that $|\Phi(cx_n, t) - \Phi(z, t)| < 1/2n$ for every $z \in cx_n + W_n$.

Now, we apply the first part of this proof. For every n we can find $y_n \in D \cap (x_n + c^{-1} W_n)$ such that $t \in P(y_n)$. Hence

$$|\varkappa(t) - \Phi(cy_n, t)| \leqslant |\varkappa(t) - \Phi(cx_n, t)| + |\Phi(cx_n, t) - \Phi(cy_n, t)| \leqslant 1/2n + 1/2n = 1/n.$$

Thus

$$\tilde{\varkappa}(t) \geqslant \sup_n \Phi(cy_n, t) \geqslant \sup_n \big(\varkappa(t) - 1/n\big) = \varkappa(t)$$

and also in this case equality $(*)$ is true.

2.3. Corollary. *The function $\varkappa$ is Σ-measurable.*

Now we shall prove four lemmas. They will be used in proofs of Proposition 2.9 and Theorem 2.12.

2.4. Lemma. *If there is a constant $a > 0$ such that*

$$\int_T \varkappa_{a,2}(t)\, d\mu < +\infty,$$

where

$$\varkappa_{a,2}(t) = \sup_{x \in X} \Phi(2x, t)\, \chi_{P_a(x)}(t)$$

and

$$P_a(x) = \{t \in T\colon\ \Phi(2x, t) > a\Phi(x, t)\},$$

then

$$(*) \qquad \int_T \Phi(2^n f(t), t)\, d\mu \leqslant a^n \int_T \Phi\big(f(t), t\big)\, d\mu + \sum_{k=1}^{n} a^{n-k} \int_T \varkappa_a(t)\, d\mu$$

for every natural number n and $f \in \mathscr{M}_0(T, X)$.

Proof. Let f be an arbitrary function from $\mathscr{M}_0(T, X)$. Putting

$$A = \big\{t \in T\colon\ \Phi\big(2f(t), t\big) \leqslant a\Phi\big(f(t), t\big)\big\}$$

we have $A \in \Sigma$ and $T \setminus A = \big\{t \in T\colon\ t \in P_a\big(f(t)\big)\big\}$. Hence

$$\int_T \Phi\,(2f(t), t)\, d\mu = \int_A \Phi(2f(t), t)\, d\mu + \int_{T\setminus A} \Phi(2f(t), t)\, d\mu.$$

Thus

$$\int_T \Phi(2f(t), t)\, d\mu \leqslant a \int_T \Phi\big(f(t), t\big)\, d\mu + \int_T \varkappa_{a,2}(t)\, d\mu.$$

Now, let us suppose that inequality $(*)$ is true for some n and let

$$B = \big\{t \in T\colon\ \Phi\big(2^{n+1} f(t), t\big) \leqslant a\Phi\big(2^n f(t), t\big)\big\}.$$

Then $B \in \Sigma$ and $T \setminus B = \{t \in T\colon\ t \in P_a(2^n f(t))\}$. Hence

$$\begin{aligned}\int_T \Phi(2^{n+1}f(t),\, t)\,d\mu &= \int_B \Phi(2^{n+1}f(t),\, t)\,d\mu + \int_{T\setminus B} \Phi(2^{n+1}f(t),\, t)\,d\mu \\ &\leqslant a \int_T \Phi(2^n f(t),\, t)\,d\mu + \int_T \varkappa_{a,2}(t)\,d\mu \\ &\leqslant a^{n+1} \int_T \Phi(f(t),\, t)\,d\mu + \sum_{k=1}^{n} a^{n-k+1} \int_T \varkappa_{a,2}(t)\,d\mu + \\ &\quad + \int_T \varkappa_{a,2}(t)\,d\mu \\ &= a^{n+1} \int_T \Phi(f(t),\, t)\,d\mu + \sum_{k=1}^{n+1} a^{n+1-k} \int_T \varkappa_{a,2}(t)\,d\mu.\end{aligned}$$

Thus inequality $(*)$ is true for every n.

In [5] P. Turpin considered a function $s_{\infty,\lambda,\alpha}\colon T \to [0, +\infty]$ defined by $s_{\infty,\lambda,\alpha}(t) = \sup\{x \geqslant 0\colon \Phi(\lambda x, t) > \alpha\Phi(x, t)\}$, where $\lambda, \alpha > 0$ and Φ is a φ-function with parameter. Putting $\lambda = 2$, $\alpha = a$, $X = R$, $F(u, v) = u+v$, $r(u) = u$, and $\omega(u) = u$ we have $\varkappa_{a,2}(t) = \Phi(2s_{\infty,2,a}(t), t)$, where Φ is a Φ-function.

2.5. Lemma. *Let c be a positive real number and let us put*

$$T = \bigcup_{n=1}^{\infty} T_n \quad \text{where } T_1 \subset T_2 \subset \ldots,\ \mu(T_n) < +\infty,$$

$$D = \{y_n\colon n \in \mathcal{N}\},$$

$$\varkappa_{m,c}(t) = \sup_n \Phi(cy_n, t)\,\chi_{P_m(y_n)}(t),$$

$$\varkappa_{m,n,c}(t) = \sup_{k \leqslant n} \Phi(cy_k, t)\,\chi_{P_m(y_k)}(t)\cdot\chi_{T_n}(t),$$

where m, n are natural numbers and $P_m(y_k) = \{t \in T\colon \Phi(2y_k, t) > 2^m\Phi(y_k, t)\}$.

If

$$\int_T \varkappa_{m,c}(t)\,d\mu = +\infty$$

for every $m \in \mathcal{N}$, then for an arbitrary sequence (a_k) of real positive numbers there are sequences (m_k), (n_k) of natural numbers, $\lim_{k\to\infty} m_k = +\infty$, and a countable family of disjoint sets $G_k \in \Sigma$ of finite measures such that

$$\int_{G_k} \varkappa_{m_k,n_k,c}(t)\,d\mu = a_k$$

for every $k \in \mathcal{N}$.

Proof. Since $\int_T \varkappa_{1,c}(t)\,d\mu = +\infty$ there is an $n_1 \in \mathcal{N}$ such that $\int_T \varkappa_{1,n_1,c}(t)\,d\mu \geqslant 2a_1$. The measure μ is atomless and $\mu(T_{n_1}) < +\infty$; moreover, $0 \leqslant \varkappa_{1,n_1,c}(t) < +\infty$ for almost every t. Thus, there is a set $A_1 \subset T_{n_1}$ such that $\int_{A_1} \varkappa_{1,n_1,c}(t)\,d\mu = a_1$.

There is also a set $B_1 \in \Sigma$, $B_1 \subset T_{n_1} \setminus A_1$, such that $\int_{B_1} \varkappa_{1,n_1,c}(t)\,d\mu = a_1$. (Indeed, if $\int_{T\setminus A_1} \varkappa_{1,n_1,c}(t)\,d\mu < a_1$, then $\int_T \varkappa_{1,n_1,c}(t)\,d\mu < 2a_1$ and this contradicts the assumption.) Moreover,

$$\int_{T\setminus A_1} \varkappa_{m,c}(t)\,d\mu + \int_{A_1} \varkappa_{m,c}(t)\,d\mu = +\infty$$

for every $m > 1$.

Let us write $m_1 = 1$,

$$I_2' = \{m > 1:\ \int_{A_1} \varkappa_{m,c}(t)\,d\mu = +\infty\}, \qquad I_2'' = \{m > 1:\ \int_{T\setminus A_1} \varkappa_{m,c}(t)\,d\mu = +\infty\}.$$

Of course, $I_2' \cup I_2'' = \{m \in \mathcal{N}:\ m > 1\}$, thus one of these sets is infinite. Let us denote this set briefly by I_2.

Moreover, let us write

$$G_1 = \begin{cases} A_1 & \text{if } I_2 = I_2'', \\ B_1 & \text{if } I_2 = I_2', \end{cases} \qquad F_1 = \begin{cases} T\setminus A_1 & \text{if } I_2 = I_2'', \\ A_1 & \text{if } I_2 = I_2'. \end{cases}$$

Then $\mu(G_1) < +\infty$ and $\int_{F_1} \varkappa_{m,c}(t)\,d\mu = +\infty$ for every $m \in I_2$.

Let $m_2 = \min I_2$. Then $\int_{F_1} \varkappa_{m_2,c}(t)\,d\mu = +\infty$, and hence there is an $n_2 \in \mathcal{N}$ such that $\int_{F_1} \varkappa_{m_2,n_2,c}(t)\,d\mu \geqslant 2a_2$. Thus there are sets $A_2 \subset F_1 \cap T_{n_2}$, $B_2 \subset F_1 \cap T_{n_2}$, $A_2 \cap B_2 = \emptyset$ such that

$$\int_{A_2} \varkappa_{m_2,n_2,c}(t)\,d\mu = \int_{B_2} \varkappa_{m_2,n_2,c}(t)\,d\mu = a_2.$$

Moreover,

$$\int_{A_2} \varkappa_{m,c}(t)\,d\mu + \int_{F_1\setminus A_2} \varkappa_{m,c}(t)\,d\mu = +\infty$$

for every $m \in I_2$.

Analogously, we write

$$I_3' = \{m \in I_2 \setminus \{m_2\}:\ \int_{A_2} \varkappa_{m,c}(t)\,d\mu = +\infty\},$$

$$I_3'' = \{m \in I_2 \setminus \{m_2\}:\ \int_{F_1\setminus A_2} \varkappa_{m,c}(t)\,d\mu = +\infty\},$$

where $m_2 > 1$; we choose this set (from I_3' and I_3'') which is infinite and we denote it by I_3.

Now, let us put

$$G_2 = \begin{cases} A_2 & \text{if } I_3 = I_3'', \\ B_2 & \text{if } I_3 = I_3', \end{cases} \qquad F_2 = \begin{cases} T\setminus A_2 & \text{if } I_3 = I_3'', \\ A_2 & \text{if } I_3 = I_3'. \end{cases}$$

Then $\int\limits_{F_2} \varkappa_{m,c}(t)\,d\mu = +\infty$ for every $m \in I_3$. Moreover, $\mu(G_2) < +\infty$, $G_2 \cap G_1 = \emptyset$, $m_2 > m_1$ and $\int\limits_{G_2} \varkappa_{m_2,n_2,c}(t)\,d\mu = a_2$.

Let us put $m_3 = \min I_3$.

Continuing this process, we obtain the sequences (m_k), (n_k), (G_k) possessing all required properties.

2.6. Lemma. *For every $n \in \mathcal{N}$, $m \in \mathcal{N}$ and $c > 0$ there is a simple function $f\colon T \to X$ such that for every $t \in T$*

$$\Phi(cf(t), t) = \varkappa_{m,n,c}(t),$$

where the function $\varkappa_{m,n,c}$ is defined in the same way as in Lemma 2.5.

Proof. Let us put

$$A_1 = \{t \in T_n\colon \varkappa_{m,n,c}(t) = \Phi(cy_1, t)\chi_{P_m(y_1)}(t)\},$$

$$A_k = \{t \in T_n \setminus \bigcup_{p=1}^{k-1} A_p\colon \varkappa_{m,n,c}(t) = \Phi(cy_k, t)\chi_{P_m(y_k)}(t)\}$$

for $k = 2, 3, \ldots, n$. Since functions $\varkappa$ and $\Phi(cy_k, \cdot)$ are measurable $A_k \in \Sigma$ and $\mu(A_k) < +\infty$ for each $k = 1, 2, \ldots, n$.

Moreover, we have $A_i \cap A_k = \emptyset$ for $i \neq k$. Let

$$f(t) = \sum_{k=1}^{n} y_k \chi_{A_k}(t)\chi_{P_m(y_k)}(t).$$

Then

$$\Phi(cf(t), t) = \sum_{k=1}^{n} \Phi(cy_k, t)\chi_{A_k}(t)\chi_{P_m(y_k)}(t) = \varkappa_{m,n,c}(t).$$

2.7. Lemma. *If*

$$\int\limits_T \varkappa_{m,2}(t)\,d\mu = +\infty$$

for every natural number m, then there is a sequence (f_k) of simple functions such that the following conditions hold:

(a) $\int\limits_T \Phi(f_k(t), t)\,d\mu \leqslant r(1)/2^k$ *for every* k,

(b) $\int\limits_T \Phi(2f_k(t), t)\,d\mu \geqslant r(1)$ *for every* k,

(c) $\{t \in T\colon f_k(t) \neq \theta\} \cap \{t \in T\colon f_i(t) \neq \theta\} = \emptyset$ *for* $i \neq k$,

where $\varkappa_{m,2}$ is defined in Lemma 2.5 (for $c = 2$) and r is the function defined in 1.6.

Proof. In the proof we shall consider two cases.

1° Let us suppose that there is a number m_0 such that

$$\int_T \varkappa_{m_0,1}(t)\,d\mu < +\infty,$$

where the function $\varkappa_{m,1}$ is defined in Lemma 2.5 (for $c = 1$).

Putting $c = 2$ and $(a_p) = (r(1))$, Lemma 2.5 implies that there are sequences (m_p), (n_p) of natural numbers and a countable family of disjoint sets $G_p \in \Sigma$ of finite measures such that

$$\int_{G_p} \varkappa_{m_p,n_p,2}(t)\,d\mu = r(1) \quad \text{for every } p.$$

Since $\lim_{m\to\infty} \int_T \varkappa_{m,1}(t)\,d\mu = 0$ (this follows from the Lebesgue theorem), for every k there is a number m_k such that

$$\int_T \varkappa_{m,1}(t)\,d\mu < r(1)/2^k \quad \text{for every } m > m_k.$$

Since $m_p \to +\infty$ for $p \to +\infty$, for every k there is a number p_k such that $m_{p_k} > \max\{m_k, k\}$. Then we have

$$\int_{G_{p_k}} \varkappa_{m_{p_k},n_{p_k},2}(t)\,d\mu = r(1)$$

and

$$\int_{G_{p_k}} \varkappa_{m_{p_k},n_{p_k},1}(t)\,d\mu \leqslant \int_T \varkappa_{m_{p_k},1}(t)\,d\mu \leqslant r(1)/2^k.$$

Putting $c = 1$, Lemma 2.6 implies that there are simple functions $g_k\colon T \to X$ such that $\Phi(g_k(t), t) = \varkappa_{m_{p_k},n_{p_k},1}(t)$ for every $t \in T$. Hence, if $f_k = g_k \chi_{G_{p_k}}$, we have:

$$\int_T \Phi(f_k(t), t)\,d\mu = \int_{G_{p_k}} \varkappa_{m_{p_k},n_{p_k},1}(t)\,d\mu \leqslant r(1)/2^k$$

and

$$\int_T \Phi(2f_k(t), t)\,d\mu = r(1).$$

It is easy to verify that condition (c) holds also.

2° If

$$\int_T \varkappa_{m,1}(t)\,d\mu = +\infty$$

for every natural number m, then taking $c = 1$ and $(a_k) = (r(1)/2^k)$ we obtain, by Lemma 2.5: there are sequences (m_k), (n_k) and a countable family of disjoint sets $G_k \in \Sigma$ of finite measures such that for every k

$$\int_{G_k} \varkappa_{m_k,n_k,1}(t)\,d\mu = r(1)/2^k.$$

Lemma 2.6 (with $c = 1$) implies that there are simple functions $f_k\colon T \to X$ such that every function f_k is of the type

$$f_k(t) = \sum_{i=1}^{n_k} y_i \chi_{A_{k,i}}(t)\chi_{P_{m_k}(y_i)}(t),$$

where

$$A_{k,1} = \{t \in G_k\colon\ \Phi(y_1, t)\chi_{P_{m_k}(y_1)}(t) = \varkappa_{m_k,n_k,1}(t)\}$$

and

$$A_{k,i} = \{t \in G_k \setminus \bigcup_{p=1}^{i-1} A_{k,p}\colon\ \Phi(y_i, t)\chi_{P_{m_k}(y_i)}(t) = \varkappa_{m_k,n_k,1}(t)\}$$

for $i = 2, 3, \ldots, n_k$.

Let us suppose that $f_k(t) \neq \theta$. Then there is exactly one i such that $t \in A_{k,i} \cap P_{m_k}(y_i)$. Thus $f_k(t) = y_i$ and $t \in P_{m_k}(f_k(t))$. Hence

$$\Phi(2f_k(t), t) \geqslant 2^{m_k}\Phi(f_k(t), t).$$

If $f_k(t) = \theta$ then the above inequality is obvious. Thus we have

$$\int_T \Phi(f_k(t), t)\,d\mu = r(1)/2^k,$$

$$\int_T \Phi(2f_k(t), t)\,d\mu \geqslant 2^{m_k}\int_T \Phi(f_k(t), t)\,d\mu \geqslant 2^{m_k}\frac{r(1)}{2^k} \geqslant r(1)$$

for every natural number k.

Condition (c) is obvious.

A function $\Phi\colon X \times T \to R_+$ satisfies the Δ_2-condition if there are a set T_0 of measure zero, a contstant $K > 0$ and an integrable function $h\colon T \to R_+$ such that

$$\Phi(2x, t) \leqslant K\Phi(x, t) + h(t)$$

for every $x \in X$ and $t \in T \setminus T_0$.

2.8. COROLLARY. *A function Φ satisfies the Δ_2-condition if and only if*

$$(*) \qquad \int_T \varkappa_{a,2}(t)\,d\mu < +\infty$$

for some $a > 0$.

Proof. It is easy to verify that condition $(*)$ implies the Δ_2-condition.

Suppose that there is a Φ-function Φ: $X \times T \to R_+$ which satisfies the Δ_2-condition, but $\int_T \varkappa_{m,2}(t)\,d\mu = +\infty$ for all m.

Lemma 2.7 implies that there is a sequence (f_k) such that

$$\int_T \Phi(f_k(t), t)\,d\mu \leqslant r(1)/2^k, \qquad \int_T \Phi(2f_k(t), t)\,d\mu \geqslant r(1)$$

for every k and $F_k \cap F_i = \emptyset$ for $k \neq i$, where $F_k = \{t \in T: f_k(t) \neq \theta\}$. Then

$$\sum_{k=1}^{\infty} \int_T \Phi(2f_k(t), t)\,d\mu = +\infty$$

and

$$\sum_{k=1}^{\infty} \int_T \Phi(f_k(t), t)\,d\mu \leqslant r(1).$$

On the other hand, since Φ satisfies the Δ_2-condition,

$$\begin{aligned}\sum_{k=1}^{\infty} \int_T \Phi(2f_k(t), t)\,d\mu &= \sum_{k=1}^{\infty} \int_{F_k} \Phi(2f_k(t), t)\,d\mu \\ &\leqslant K \sum_{k=1}^{\infty} \int_{F_k} \Phi(f_k(t), t)\,d\mu + \sum_{k=1}^{\infty} \int_{F_k} h(t)\,d\mu \\ &\leqslant K \cdot r(1) + \int_T h(t)\,d\mu < +\infty\end{aligned}$$

and we obtain a contradiction.

2.9. PROPOSITION. *Let us assume that* $\tilde{P}(T, X) \subset \mathcal{M}_0(T, X)$ *and let* Φ: $X \times T \to R_+$ *be a* Φ*-function on* $\mathcal{M}_0(T, X)$. *The equality*

$$L_0^{\Phi}(T, X) = L^{\Phi}(T, X)$$

holds if and only if there is a positive number $a > 0$ *such that*

$$\int_T \varkappa_{a,2}(t)\,d\mu < \infty.$$

Proof. *Sufficiency*. We shall show that $L_0^{\Phi}(T, X)$ is a linear space.

Let $f \in L_0^{\Phi}(T, X)$ and $c \in R$. Then there is a natural number n such that $|c| < 2^n$. Lemma 2.4 implies that

$$\begin{aligned}\int_T \Phi(cf(t), t)\,d\mu &\leqslant \int_T \Phi(2^n f(t), t)\,d\mu \\ &\leqslant a^n \int_T \Phi(f(t), t)\,d\mu + \sum_{k=1}^{n} a^{n-k} \int_T \varkappa_{a,2}(t)\,d\mu < +\infty.\end{aligned}$$

Thus $cf \in L_0^{\Phi}(T, X)$.

Now, we assume that $f, g \in L_0^\Phi(T, X)$. There is a natural number n such that $1/\omega^{-1}(\frac{1}{2}) \leqslant 2^n$. Applying Lemma 2.4, we obtain

$$I = \int_T \Phi(f(t)+g(t), t)\,d\mu = \int_T \Phi\left(\frac{1}{\omega^{-1}(\frac{1}{2})}\omega^{-1}(\tfrac{1}{2})(f(t)+g(t)), t\right)d\mu$$

$$\leqslant \int_T \Phi\Big(2^n\big(\omega^{-1}(\tfrac{1}{2})(f(t)+g(t))\big), t\Big)\,d\mu.$$

Therefore

$$I \leqslant a^n \int_T \Phi(\omega^{-1}(\tfrac{1}{2})f(t)+\omega^{-1}(\tfrac{1}{2})g(t), t)\,d\mu + \sum_{k=1}^{n} a^{n-k} \int_T \varkappa_{a,2}(t)\,d\mu$$

$$\leqslant a^n F\Big(\int_T \Phi(f(t), t)\,d\mu;\ \int_T \Phi(g(t), t)\,d\mu\Big) + \sum_{k=1}^{n} a^{n-k} \int_T \varkappa_{a,2}(t)\,d\mu < +\infty.$$

Thus $L_0^\Phi(T, X)$ is a linear space.

Necessity. If

$$\int_T \varkappa_{a,2}(t)\,d\mu = +\infty$$

for every $a > 0$, then Lemma 2.7 implies that there is a countable family $f_k\colon T \to X$ of simple functions such that

$$\int_T \Phi(f_k(t), t)\,d\mu \leqslant r(1)/2^k, \qquad \int_T \Phi(2f_k(t), t)\,d\mu \geqslant r(1)$$

for every k and

$$\{t \in T\colon f_i(t) \neq \theta\} \cap \{t \in T\colon f_k(t) \neq \theta\} = \emptyset$$

for $i \neq k$.

Let us put

$$g(t) = \sum_{k=1}^{\infty} f_k(t).$$

Then $g \in \mathcal{M}_0(T, X)$ and

$$\int_T \Phi(g(t), t)\,d\mu = \sum_{k=1}^{\infty} \int_T \Phi(f_k(t), t)\,d\mu \leqslant r(1);$$

hence $g \in L_0^\Phi(T, X)$.

On the other hand,

$$\int_T \Phi(2g(t), t)\,d\mu = \sum_{k=1}^{\infty} \int_T \Phi(2f_k(t), t)\,d\mu = +\infty;$$

thus $2g \notin L_0^{\Phi}(T, X)$ and this contradicts the assumption.

2.10. Lemma. *Let us assume that the following conditions hold:*

(a) *Φ is a Φ_1-function on $\mathcal{M}_0(T, X)$,*

(b) *there is a positive number a such that*

$$\int_T \varkappa_{a,2}(t)\,d\mu < +\infty,$$

(c) *the measure μ is separable.*

Then there is a countable family of sets Σ' such that for an arbitrary set $A \in \Sigma$ of finite measure, for every $x \in X$ and $\varepsilon > 0$ there is a set $B \in \Sigma'$ such that

$$\int_{A \div B} \Phi(x, t)\,d\mu < \varepsilon,$$

where by $A \div B$ we denote the set $(A \setminus B) \cup (B \setminus A)$.

Proof. Since the measure μ is separable, there is a countable family of sets Σ' such that for any set $A \in \Sigma$ of finite measure and for every $\varepsilon > 0$ we can find a set $B \in \Sigma'$ that $\mu(A \div B) < \varepsilon$.

Let $x \in X$, $A \in \Sigma$, $\mu(A) < +\infty$ and let ε be a positive number. For every natural number n there is a set $B_n \in \Sigma'$ such that

$$\mu(A \div B_n) < 1/2^n.$$

Let us put

$$E = A \cup \bigcup_{n=1}^{\infty} (B_n \setminus A).$$

Then

$$\mu(E) \leqslant \mu(A) + \sum_{n=1}^{\infty} \mu(B_n \setminus A) \leqslant 1 + \mu(A) < +\infty.$$

Since Φ is a Φ_1-function, $\lim_{c \to 0} \int_T \Phi(cx\chi_E(t), t)\,d\mu = 0$. Thus, $cx\chi_E \in L^{\Phi}(T, X)$ for sufficiently small c. Therefore, by condition (b) and Proposition 2.9, $x\chi_E \in L_0^{\Phi}(T, X)$. Thus for every $\varepsilon > 0$ there is a positive number η such that

$$\int_{G \cap E} \Phi(x, t)\,d\mu < \varepsilon$$

for every measurable set G such that $\mu(G) < \eta$. Taking n so large that $1/2^n < \eta$, we obtain $A \div B_n \subset E$, $\mu(A \div B_n) < 1/2^n$. Thus

$$\int_{A \div B_n} \Phi(x, t)\,d\mu < \varepsilon.$$

2.11. COROLLARY. *If conditions* (a), (b) *and* (c) *of Lemma* 2.10 *are fulfilled, then the space* $P(T, X)$ *of simple functions is separable.*

Proof. Let $D = \{y_k\colon k = 1, 2, \ldots\}$, $x \in X$, $\varepsilon > 0$ and let A be a set of finite measure. Since Φ is a Φ_1-function, there is a neighbourhood U of the origin θ such that

$$\forall\, z \in U \ \int_A \Phi(z, t)\, d\mu < \tfrac{1}{3} r(\varepsilon).$$

Since the space X is separable, there is an $y \in D$ such that

$$y \in x + \omega^{-1}(\varepsilon)\, U.$$

Lemma 2.10 implies that we can find a set $B \in \Sigma'$ such that

$$\int_{A \div B} \Phi\Big(\frac{x}{\omega^{-1}(\varepsilon)}, t\Big) d\mu < \tfrac{1}{3} r(\varepsilon) \quad \text{and} \quad \int_{A \div B} \Phi\Big(\frac{y}{\omega^{-1}(\varepsilon)}, t\Big) d\mu < \tfrac{1}{3} r(\varepsilon).$$

Then we have

$$\int_T \Phi\Big(\frac{1}{\omega^{-1}(\varepsilon)}\big(x\chi_A(t) - y\chi_B(t)\big), t\Big) d\mu$$

$$= \int_{A \cap B} \Phi\Big(\frac{1}{\omega^{-1}(\varepsilon)}(x-y), t\Big) d\mu + \int_{A \setminus B} \Phi\Big(\frac{x}{\omega^{-1}(\varepsilon)}, t\Big) d\mu + \int_{B \setminus A} \Phi\Big(\frac{y}{\omega^{-1}(\varepsilon)}, t\Big) d\mu$$

$$\leqslant \tfrac{1}{3} r(\varepsilon) + \tfrac{1}{3} r(\varepsilon) + \tfrac{1}{3} r(\varepsilon) = r(\varepsilon).$$

Thus $|x\chi_A - y\chi_B| < \varepsilon$. So we have proved that the set of functions of the type

$$\sum_{k=1}^{n} y_k \chi_{B_k},$$

where $y_k \in D$ and $B_k \in \Sigma'$, is dense in the space $P(T, X)$. Since this set is also countable, the space $P(T, X)$ is separable.

2.12. THEOREM. (1) *If* X *is a separable space and*

(i) $\Phi\colon X \times T \to R_+$ *is a* Φ_1*-function,*

(ii) *the measure* μ *is separable,*

(iii) *there is a positive number* a *such that*

$$\int_T \varkappa_{a,2}(t)\, d\mu < +\infty,$$

then $E^{\Phi}(T, X)$ *is a separable space.*

(2a) *If the space* $L^{\Phi}(T, X)$ *is separable and* $\tilde{P}(T, X) \subset \mathcal{M}_0(T, X)$ *then condition* (iii) *holds.*

(2b) *If the space* $E^{\Phi}(T, X)$ *is separable and there is* $x \in X$ *and a set* $S \in \Sigma$, $\mu(S) = 0$, *such that*

(iv) $$\inf_{t \notin S} \Phi(x, t) > 0,$$

then the measure μ *is also separable.*

Proof. (1) Let $f \in E^{\Phi}(T, X)$, then there is a sequence (g_n) of simple functions such that $|f - g_n| < \varepsilon/2$ for $n \geqslant n_0$. Corollary 2.11 implies that for some function $h\colon T \to X$ of the type

(∗) $$h = \sum_{i=1}^{m} y_i \chi_{B_i},$$

where $y_i \in D$, $B_i \in \Sigma'$ for $i = 1, 2, \ldots, m$, we have $|g_{n_0} - h| < \varepsilon/2$. Thus

$$|f - h| \leqslant |f - g_{n_0}| + |g_{n_0} - h| \leqslant \varepsilon/2 + \varepsilon/2 = \varepsilon.$$

So the countable set of all functions of type (∗) is dense in the space $E^{\Phi}(T, X)$. Thus the space $E^{\Phi}(T, X)$ is separable.

(2a) Let us suppose that

$$\int_T \varkappa_{a,2}(t)\, d\mu = +\infty$$

for every $a > 0$. Lemma 2.7 implies that there is a sequence (f_k) of simple functions such that

$$\int_T \Phi(f_k(t), t)\, d\mu \leqslant r(1)/2^k, \qquad \int_T \Phi(2f_k(t), t)\, d\mu \geqslant r(1)$$

for every k and

$$\{t \in T\colon f_i(t) \neq \theta\} \cap \{t \in T\colon f_k(t) \neq \theta\} = \emptyset \quad \text{for } i \neq k.$$

Let us denote by M the set of all functions of the type

$$\sum_{k=1}^{\infty} \xi_k f_k,$$

where $(\xi_k)_{k=1}^{\infty} = \xi \in s$, s being the space of all bounded sequences. Taking $c = 1/\|\xi\| = 1/\sup_k |\xi_k|$ we have

$$\int_T \Phi\Big(c \sum_{k=1}^{\infty} \xi_k f_k(t), t\Big)\, d\mu = \sum_{k=1}^{\infty} \int_T \Phi(c\xi_k f_k(t), t)\, d\mu$$
$$\leqslant \sum_{k=1}^{\infty} \int_T \Phi(f_k(t), t)\, d\mu < +\infty,$$

for every $\xi \in s$, $\xi \neq 0$. Thus the space $M \subset L^{\Phi}(T, X)$.

Moreover, the space M with topology determined by the F-pseudonorm $|\cdot|$ is complete since the mapping from M into s defined by

$$\sum_{k=1}^{\infty} a_k g_k \mapsto (a_k)_{k=1}^{\infty}$$

is a linear isomorphism. Thus the space M is a closed linear subspace of $L^{\Phi}(T, X)$.

The space $L^{\Phi}(T, X)$ is separable, hence the space M is also separable. Let us suppose that the set

$$(*) \qquad \Big\{\sum_{k=1}^{\infty} \xi_k^{(n)} f_k \colon (\xi_k^{(n)})_{k=1}^{\infty} = \xi^{(n)} \in s,\ n \in \mathcal{N}\Big\}$$

is dense in the space M. We denote it by Q.

Since the space s is not separable, there is a sequence $\eta = (\eta_k)_{k=1}^{\infty} \in s$ and a positive number $\varepsilon > 0$ such that

$$\|\eta - \xi^{(n)}\| \geqslant \varepsilon$$

for every n. Thus for every n there is an index p_n such that

$$|\eta_{p_n} - \xi_{p_n}^{(n)}| \geqslant \varepsilon.$$

Let us assume that $\varepsilon < 2$ and let d be a positive number such that $d = \omega(\varepsilon/2)$. Then $d \leqslant \omega(1) = 1$ and

$$\int_T \Phi\left(\frac{\sum_{k=1}^{\infty} (\eta_k - \xi_k^{(n)}) f_k(t)}{\omega^{-1}(d)}, t\right) d\mu \geqslant \int_T \Phi\left(\frac{\eta_{p_n} - \xi_{p_n}^{(n)}}{\varepsilon} 2f_{p_n}(t), t\right) d\mu$$

$$\geqslant \int_T \Phi(2f_{p_n}(t), t)\, d\mu \geqslant r(1) \geqslant r(d).$$

Thus, there is a sequence $\eta \in s$ and a number $d > 0$ such that

$$\Big|\sum_{k=1}^{\infty} (\eta_k - \xi_k^{(n)}) f_k\Big| = \Big|\sum_{k=1}^{\infty} \eta_k f_k - \sum_{k=1}^{\infty} \xi_k^{(n)} f_k\Big| \geqslant d$$

for every n. This contradicts the assumption that the set Q is dense in the space M.

Hence condition (iii) holds.

(2b) Let us suppose that the measure μ is not separable. Then there are a non-countable family of sets $(A_a)_{a\in I}$ of finite measures and a positive constant c such that $\mu(A_a \div A_b) > c$ for every $a, b \in I$, $a \neq b$.

Let x be an element of the space X for which condition (iv) is fulfilled. Then

$$\int_T \Phi(x\chi_{A_a}(t) - x\chi_{A_b}(t), t)\, d\mu = \int_{A_a \div A_b} \Phi(x, t)\, d\mu$$

$$\geqslant \mu(A_a \div A_b) \inf_{t\notin S} \Phi(x, t) > c \inf_{t\notin S} \Phi(x, t).$$

Let d be a positive number less than 1 such that

$$r(d) < c \inf_{t\notin S} \Phi(x, t).$$

Then $\omega^{-1}(d) < \omega^{-1}(1) = 1$ and

$$\int_T \Phi\left(\frac{x\chi_{A_a}(t) - x\chi_{A_b}(t)}{\omega^{-1}(d)}, t \right) d\mu \geqslant \int_T \Phi(x\chi_{A_a}(t) - x\chi_{A_b}(t), t)\, d\mu$$

$$\geqslant c \cdot \inf_{t\notin S} \Phi(x, t) > r(d).$$

Thus

$$|x\chi_{A_a} - x\chi_{A_b}| > d$$

for every $a, b \in I$, $a \neq b$. So the set $(x\chi_{A_a})_{a\in I}$ is uncountable and is not dense in $E^\Phi(T, X)$. This contradicts the assumption that the space $E^\Phi(T, X)$ is separable.

Hence the measure μ is separable.

Let $F(u, v) = u+v$, $\omega(u) = u$ and $r(u) = u$.

2.13. Corollary. *If the measure μ is separable, X is a separable Banach space, Φ is an N''-function with finite values, continuous at* 0 *and for every set A of finite measure and $\varepsilon > 0$ there is a real number $\eta > 0$ such that*

$$\int_A \Phi(x, t)\, d\mu < \varepsilon \quad \textit{for } \|x\| < \eta,$$

then the space E_Φ (see [2]) *is separable if and only if Φ satisfies the Δ_2-condition.*

2.14. Corollary. *If μ is a separable measure, $\Phi\colon R \times T \to R_+$ is a φ-function with parameter and $\chi_A \in L^\Phi(T, R)$ for every set A of finite measure, then the space $E^\Phi(T, R)$ is separable if and only if Φ satisfies the Δ_2-condition.*

3. Local boundedness of Orlicz spaces.

3.1. THEOREM. *Let us assume that the following condition hold:*

1° *the space X is locally bounded and separable,*

2° *there is a set S, $\mu(S) = 0$, such that for every $t \notin S$, if a sequence (x_n) is such that $\sup_n \Phi(x_n, t) < +\infty$, then it is bounded.*

Then the Orlicz space $L^\Phi(T, X)$ is locally bounded if and only if

$$\exists_{q>0} \forall_{a>0} \exists_{b>0} \exists_{c>0} \int_T \delta_{a,b,c,q}(t)\, d\mu < +\infty, \tag{v}$$

where

$$\delta_{a,b,c,q}(t) = \sup_{x\in X} \Phi(cx, t)\, \chi_{P_{a,b,q}(x)}(t)$$

and

$$P_{a,b,q}(x) = \left\{ t \in T:\ \Phi\left(\frac{b\omega^{-1}(q)}{\omega^{-1}(aq)} x,\ t \right) > \frac{r(aq)}{r(q)} \Phi(x, t) \right\}.$$

Proof. Let us assume that the Orlicz space $L^\Phi(T, X)$ is locally bounded. Then it is easy to verify that there is a constant $q_0 > 0$ such that for every $a > 0$ we can find a number $b > 0$ such that

$$b\, \overline{K(q_0)} \subset K(aq_0), \tag{1}$$

where $K(q)$ denotes the set $\{f \in L^\Phi(T, X):\ |f| < q\}$ and the set $\overline{K(q)}$ is the closure of $K(q)$.

Let us suppose that for every $q > 0$ there is $a > 0$ such that

$$\int_T \delta_{a,b,c,q}(t)\, d\mu = +\infty \tag{2}$$

for every $b > 0$ and $c > 0$.

Let us put in condition (2) $q = q_0$. For this q there is a positive number a such that condition (2) holds for every $b > 0$ and $c > 0$. For this number a there is a positive number b such that condition (1) is true.

For these numbers q, a, b and for $c = 1$ we denote by ψ_n the function

$$\sup_{1\leqslant k\leqslant n} \Phi(y_k, t)\, \chi_{P_{a,b,q}(y_k)}(t)\, \chi_{T_n}(t),$$

where the set $\{y_k:\ k \in \mathcal{N}\}$ is dense in the space X and $T = \bigcup_{n=1}^{\infty} T_n$, $\mu(T_n) < \infty$. Then $\psi_n(t)$ is a non-decreasing sequence for every $t \in T$ and tends to $\delta_{a,b,1,q}(t)$.

By Beppo–Levi theorem, we obtain

$$\lim_{n\to\infty} \int_T \psi_n(t)\,d\mu = \int_T \delta_{a,b,1,q}(t)\,d\mu = +\infty .$$

Thus,

$$\int_T \psi_n(t)\,d\mu > r(q)$$

for sufficiently large n.

Since the measure μ is atomless and $\mu(T_n) < +\infty$, there is a set $A \in \Sigma$ such that $A \subset \{t \in T\colon \psi_n(t) \neq 0\}$ and

$$\int_A \psi_n(t)\,d\mu = r(q).$$

In an analogous fashion as in Lemma 2.6, we can find a measurable function $f\colon T \to X$ such that $\Phi(f(t), t) = \psi_n(t)$ for every $t \in T$. Then we have

$$\int_T \Phi\left(\frac{\omega^{-1}(q) f(t)\chi_A(t)}{\omega^{-1}(q)}, t\right) d\mu = \int_A \Phi(f(t), t)\,d\mu = \int_A \psi_n(t)\,d\mu = r(q).$$

Hence

$$|\omega^{-1}(q) f\chi_A| \leqslant q,$$

so $\omega^{-1}(q) f\chi_A \in \overline{K(q)}$.

On the other hand,

$$\int_T \Phi\left(\frac{b\omega^{-1}(q) f(t)\chi_A(t)}{\omega^{-1}(aq)}, t\right) d\mu = \int_A \Phi\left(\frac{b\omega^{-1}(q)}{\omega^{-1}(aq)} f(t), t\right) d\mu$$

$$> \frac{r(aq)}{r(q)} \int_A \Phi(f(t), t)\,d\mu = \frac{r(aq)}{r(q)} r(q) = r(aq).$$

Thus $b\omega^{-1}(q) f\chi_A \notin K(aq)$. This contradicts condition (1).

Hence condition (v) holds.

Conversely, let condition (v) hold. We shall show that the set $K(q)$ is bounded, i.e. for every $d > 0$ there is a number $e > 0$ such that $eK(q) \subset K(dq)$.

Let d be an arbitrary positive number. Since $r(\cdot)$ is continuous at 0, we can find a number $a \in (0, d)$ such that $2r(aq) < r(dq)$. By the assumption, there are numbers $b > 0$, $c > 0$ such that

$$\int_T \delta_{a,b,c,q}(t)\,d\mu < +\infty .$$

Let us put

$$\varphi_u(t) = \delta_{a,b,u,q}(t),$$

where $u > 0$. Then $\lim\limits_{u\to 0} \varphi_u(t) = 0$ for almost every $t \in T$.

Indeed, let ε be an arbitrary positive number. The assumptions imply that there hold the conditions:

(3) there is a measurable set S_1, $\mu(S_1) = 0$, such that for every $t \notin S_1$ if a sequence $\Phi(x_n, t)$ is bounded, then the sequence (x_n) is bounded, i.e. there is a neighbourhood V_t of the origin such that $x_n \in V_t$ for every n,

(4) there is a measurable set S_2, $\mu(S_2) = 0$, such that for every $t \notin S_2$ and $\varepsilon > 0$ there is a set $W_t \in \mathscr{B}$ such that $\Phi(z, t) < \varepsilon$ for every $z \in W_t$ (condition 1.7 (e)).

By Z we denote the set $\{t \in T\colon \varphi_c(t) = +\infty\}$. Since

$$\int_T \varphi_c(t)\, d\mu < +\infty,$$

we have $\mu(Z) = 0$.

Now, let t be an arbitrary element of T which does not belong to $S_1 \cup S_2 \cup Z$.

If $\varphi_c(t) = 0$, then $0 \leqslant \varphi_u(t) \leqslant \varphi_c(t) = 0$ for every $u \leqslant c$, thus $\varphi_u(t)$ tends to 0.

If $\varphi_c(t) \neq 0$, then putting $I_t = \{n \in \mathcal{N}\colon t \in P_{a,b,q}(y_n)\}$ we have $t \notin S_1$ and $\sup\limits_{n\in I_t} \Phi(cy_n, t) < +\infty$. Hence by (3), there is a set $U_t \in \mathscr{B}$ such that $y_n \in U_t$ for every $n \in I_t$. Moreover, by condition (4), ($t \notin S_2$) there is a set $W_t \in \mathscr{B}$ such that $\Phi(z, t) < \varepsilon$ for every $z \in W_t$. Since the space X is locally bounded, there is a number $u_t > 0$ such that $u_t U_t \subset W_t$. Hence $\Phi(uy_n, t) < \varepsilon$ for every $0 < u \leqslant u_t$ and every $n \in I_t$, i.e.

$$\varphi_u(t) = \sup_{n\in I_t} \Phi(uy_n, t)\chi_{P_{a,b,q}(y_n)}(t) < \varepsilon.$$

Thus $\varphi_u(t)$ tends to 0.

Therefore, by the Lebesgue theorem, we obtain

$$\lim_{u\to 0} \int_T \varphi_u(t)\, d\mu = 0.$$

Hence there is a number $u > 0$ such that

$$\int_T \varphi_u(t)\, d\mu < r(aq).$$

Now, let

$$e = \min\left\{ b, \frac{u\omega^{-1}(aq)}{\omega^{-1}(q)} \right\}.$$

We shall show that $eK(q) \subset K(dq)$.

Let $f \in K(q)$. By G_f we denote the set

$$\left\{ t \in T\colon \Phi\left(\frac{uf(t)}{\omega^{-1}(q)}, t \right) \leqslant \varphi_u(t) \right\}.$$

Then we have

$$\int\limits_{S_1 \cup S_2 \cup Z} \Phi\left(\frac{ef(t)}{\omega^{-1}(aq)}, t \right) d\mu = 0, \tag{5}$$

$$\int\limits_{G_f \setminus (S_1 \cup S_2 \cup Z)} \Phi\left(\frac{ef(t)}{\omega^{-1}(aq)}, t \right) d\mu \leqslant \int\limits_{G_f} \Phi\left(\frac{uf(t)}{\omega^{-1}(q)}, t \right) d\mu \leqslant \int\limits_{T} \varphi_u(t)\, d\mu < r(aq), \tag{6}$$

$$\int\limits_{T \setminus (G_f \cup S_1 \cup S_2 \cup Z)} \Phi\left(\frac{ef(t)}{\omega^{-1}(aq)}, t \right) d\mu \leqslant \int\limits_{T \setminus G_f} \Phi\left(\frac{bf(t)}{\omega^{-1}(aq)}, t \right) d\mu.$$

If $t \notin G_f$, then

$$\varphi_u(t) = \sup_{x \in X} \Phi(ux, t)\, \chi_{P_{a,b,q}(x)}(t) < \Phi\left(\frac{uf(t)}{\omega^{-1}(q)}, t \right).$$

Hence $t \notin P_{a,b,q}\big(f(t)/\omega^{-1}(q)\big)$. Therefore

$$\Phi\left(\frac{bf(t)}{\omega^{-1}(aq)}, t \right) = \Phi\left(\frac{b\omega^{-1}(q)}{\omega^{-1}(aq)} \frac{f(t)}{\omega^{-1}(q)}, t \right) \leqslant \frac{r(aq)}{r(q)} \Phi\left(\frac{f(t)}{\omega^{-1}(q)}, t \right).$$

Thus

$$\int\limits_{T \setminus G_f} \Phi\left(\frac{ef(t)}{\omega^{-1}(aq)}, t \right) d\mu \leqslant \frac{r(aq)}{r(q)} \int\limits_{T} \Phi\left(\frac{f(t)}{\omega^{-1}(q)}, t \right) d\mu < \frac{r(aq)}{r(q)} r(q) = r(aq). \tag{7}$$

Adding inequalities (5), (6) and (7), we obtain

$$\int_T \Phi\left(\frac{ef(t)}{\omega^{-1}(aq)}, t\right) d\mu \leqslant 2r(aq).$$

Since $0 < a < d$, we have

$$\int_T \Phi\left(\frac{ef(t)}{\omega^{-1}(dq)}, t\right) d\mu \leqslant \int_T \Phi\left(\frac{ef(t)}{\omega^{-1}(aq)}, t\right) d\mu \leqslant 2r(aq) < r(dq).$$

Thus $ef \in K(dq)$.

Hence we have proved that the Orlicz space $L^{\Phi}(T, X)$ is locally bounded.

3.2. Corollary. *If X is a separable and locally bounded space, Φ: $X \times T \to R_+$ is a convex Φ-function, and assumption 2° of Theorem 3.1 holds, then the space $L^{\Phi}(T, X)$ is locally bounded.*

Proof. We shall show that

$$\forall_{q>0} \ \forall_{a>0} \ \exists_{b>0} \ \forall_{x \in X} \ \mu(P_{a,b,q}(x)) = 0.$$

Let $q > 0$ and $x \in X$. If a is a positive real number such that $r(aq)/r(q) \leqslant 1$, then putting $b = \omega^{-1}(aq) r(aq)/\omega^{-1}(q) r(q)$ we obtain

$$\Phi\left(b\frac{\omega^{-1}(q)}{\omega^{-1}(aq)}x, t\right) = \Phi\left(\frac{r(aq)}{r(q)}x, t\right) \leqslant \frac{r(aq)}{r(q)}\Phi(x, t)$$

for every $x \in X$ and almost every $t \in T$. Thus $\mu(P_{a,b,q}(x)) = 0$.

Suppose that $r(aq)/r(q) > 1$. If $b = \omega^{-1}(aq)/\omega^{-1}(q)$, then

$$\Phi\left(b\frac{\omega^{-1}(q)}{\omega^{-1}(aq)}x, t\right) = \Phi(x, t) < \frac{r(aq)}{r(q)}\Phi(x, t)$$

for every $x \in X$ and almost every $t \in T$. Hence $\mu(P_{a,b,q}(x)) = 0$ in this case also. Thus

$$\forall_{q>0} \ \forall_{a>0} \ \exists_{b>0} \ \forall_{c>0} \ \delta_{a,b,c,q}(t) = 0$$

for almost every $t \in T$ and obviously condition (v) holds.

3.3. Let us put $X = R$, $F(u, v) = u+v$, $\omega(u) = u$ and $r(u) = u$. Then we obtain (see 2.4)

$$\delta_{a,b,c,q}(t) = \Phi \quad (cs_{\chi,\lambda,\alpha}(t), t),$$

where

$$\lambda = \frac{b\omega^{-1}(q)}{\omega^{-1}(aq)} \quad \text{and} \quad \alpha = \frac{r(aq)}{r(q)}.$$

Thus condition (v) is equivalent to the following one

(vi) $$\underset{q>0}{\exists}\ \underset{\alpha>0}{\forall}\ \underset{\lambda>0}{\exists}\ s_{\infty,\lambda,\alpha}\in L^{\Phi}(T, R).$$

3.4. COROLLARY. *If Φ is a φ-function with parameter and*

$$\lim_{x\to\infty} \Phi(x, t) = +\infty$$

for almost every $t\in T$, then the Orlicz space $L^{\Phi}(T, R)$ is locally bounded if and only if condition (vi) *holds* (see [5]).

3.5. EXAMPLE. *Let X be the space C of all continuous functions from R into R with topology τ determined by the base $\mathscr{B}$ which consists of sets of the form*

$$V(x_1, \ldots, x_n, \varepsilon) = \{f\in C\colon \underset{1\leqslant k\leqslant n}{\forall} |f(x_k)| < \varepsilon\},$$

where $x_1, \ldots, x_n\in X$ and $\varepsilon > 0$. Of course the space C is separable and non-metrizable.

Let (R, Σ, μ) be a measure space, where by μ we denote the Lebesgue measure on R and Σ denotes the σ-algebra of all Lebesgue measurable sets. Moreover, let $F(u, v) = u+v$, $r(u) = u$ and $\omega(u) = u$.

We define the function $\Phi\colon C\times R\to R_+$ in the following manner

$$\Phi(x, t) = |x(t)|.$$

Then Φ is a Φ_1-function on $\mathscr{M}_0(R, C)$ with respect to the F-operation F, where $\mathscr{M}_0(R, C)$ is a linear subspace of the set of functions $f\colon R\to C$ fulfilling the property: the set $f^{-1}(U)$ is measurable for any $U\in\tau$.

3.6. EXAMPLE. Let X be the space $L^p(0, 1)$ of all p-integrable functions ($p\in(0, 1)$) with the topology given by the F-norm

$$|x| = \int_0^1 |x(s)|^p\, ds.$$

The space $L^p(0, 1)$ is separable and locally bounded.

Let (T, Σ, μ) be a measure space and let $F(u, v) = u+v$, $r(u) = u$ and $\omega(u) = u$. Let $h\colon T\to R_+$ be an integrable function and let us assume that $\inf_{t\in T} h(t) > 0$.

The function $\Phi\colon L^p(0, 1)\times T\to R_+$, $\Phi(x, t) = |x|\cdot h(t)$ is a p-convex Φ_1-function on $\mathscr{M}_0(T, L^p(0, 1))$ with respect to the F-operation F.

Since Φ is a Φ_1-function and satisfies the Δ_2-condition, the space $E^{\Phi}(T, L^p(0, 1))$ is separable if and only if the measure μ is separable (see Theorem 2.12).

Moreover, $\sup_n \Phi(x_n, t) < +\infty$ if and only if $\sup_n |x_n| < +\infty$, thus assumption 2° of Theorem 3.1 holds. Further, since Φ is a p-convex Φ-function, $\delta_{a,b,c,q}(t) = 0$ for almost every $t \in T$ (compare Corollary 3.2).

Hence the Orlicz space $L^{\Phi}(T, L^p(0, 1))$ is locally bounded.

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