ANNALES SOCIETATIS MATHEMATICAE POLONAE
Series I: COMMENTATIONES MATHEMATICAE XXIV (1984)
ROCZNIKI POLSKIEGO TOWARZYSTWA MATEMATYCZNEGO
Seria I: PRACE MATEMATYCZNE XXIV (1984)

H. Hudzik, J. Musielak and R. Urbański (Poznań)

A totally non-atomic set-valued measure

Abstract. A necessary and sufficient condition for a vector-valued measure with values in a locally convex topological vector space given by I. Tweddle [3] is generalized to the case of multimeasures.

Let E be a Hausdorff locally convex topological vector space over the reals, with topology τ, and let E' be the strong dual of E. For any sequence of sets $E_n \subset E$, $\sum_{n\geq 1} E_n$ will mean the set of all $x\in E$ of the form $x = \sum_{n\geq 1} x_n$ with $x_n \in E_n$ for every n, the series being convergent in the sense of the topology τ. The series $\sum_{n\geq 1} E_n$ will be called (τ)-*convergent*, if for any sequence $(y_n)_1^\infty$ with $y_n \in E_n$ for every n the series $\sum_{n\geq 1} y_n$ is convergent in the topology τ; then we shall write $(\tau)\sum_{n\geq 1} E_n$.

Let S be a non-void set and let Σ be a σ-algebra of subsets of S. In this paper there will be considered a set-valued measure $M\colon \Sigma \to E$ assigning to every element A in Σ a non-empty subsets $M(A)$ of E in such a way that $M(A) = (\tau)\sum_{n\geq 1} M(A_n)$, whenever A is the disjoint union of the sets A_n in Σ.

We denote by $\delta^*(\cdot|\ A)$ the support function of a subset E_1 of E:

$$\delta^*(x'|\ E_1) = \sup\{\langle x, x'\rangle\colon x\in E_1\}, \qquad x'\in E'.$$

It is easy to prove that for every $x'\in E'$ the mapping $A \to \delta^*(x'|\ M(A))$ is a measure with values in $]-\infty, \infty]$.

Let us denote $M^\vee(A) = \bigcup\{M(B)\colon \Sigma \ni B \subset A\}$. A set $A\in\Sigma$ is called an *atom* of M, if $M^\vee(A) \neq \{0\}$ and for every $B\subset A$ either $M(B) = \{0\}$ or $M^\vee(A\setminus B) = \{0\}$. M is said to be *non-atomic* if it has no atoms. M is called *totally non-atomic*, if for each $x'\in E'$ the set-valued measure $x'\circ M(A) = x'[M(A)]$ is non-atomic.

Theorem. *Let $\mathscr{U}$ be a base of absolutely convex neighbourhoods of the orgin in E. A set-valued measure $M\colon \Sigma \to E$ is totally non-atomic, if and only if, for each $U\in\mathscr{U}$ there exists a non-negative, non-atomic, finite measure $\mu\colon \Sigma \to \mathbf{R}^+$ with respect to which each function $\delta^*(x'|\ M(\cdot))$ with $x'\in U^\circ$ (U° being the polar of U) is absolutely continuous.*

Proof. The sufficiency follows from the fact that if a scalar-valued measure is absolutely continuous with respect to μ, then the total variation of this scalar-valued measure is also absolutely continuous with respect to μ and, moreover, it is non-atomic. Hence the total variation $\delta^*(x'|\ M(\cdot))$ is non-atomic for arbitrary $x' \in E'$, but this implies M to be totally non-atomic.

In order to prove the necessity, let us first remark that $\delta^*(x'|\ M(\cdot))$ is a measure on Σ. We shall prove that the family $\{\delta^*(x'|\ M(\cdot)):\ x' \in U^\circ\}$, where $U \in \mathcal{U}$, is a family of uniformly countably additive measures. It is sufficient to prove that this family is uniformly exhausting. Let us suppose the converse. Then there exist a sequence $(A_n)_1^\infty$ of pairwise disjoint sets in Σ, an $\varepsilon > 0$ and a sequence of functionals $x'_n \in U^\circ$ such that $|\delta^*(x'_n|\ M(A_n))| \geqslant \varepsilon$, i.e. there exist $x_n \in M(A_n)$ for every n such that $|\langle x_n, x'_n \rangle| \geqslant \varepsilon$ for $n = 1, 2, \ldots$ Thus $\left|\left\langle \frac{1}{\varepsilon} x_n, x'_n \right\rangle\right| \geqslant 1$, where $x'_n \in U^\circ$. But this implies $\frac{1}{\varepsilon} x_n \notin U$ for $n = 1, 2, \ldots$ Consequently, writing $U_1 = \varepsilon U$, we have $x_n \notin U_1$ for all n. However, $x_n \in M(A_n)$ and so $\sum_{n \geqslant 1} x_n$ is convergent, which gives a contradiction.

Hence, by Theorem 10.7 of [1], there exists a non-negative, finite measure μ such that $\delta^*(x'|\ M(\cdot))$ with $x' \in U^\circ$ is absolutely continuous with respect to Σ.

Now, let us suppose that $A \in \Sigma$ is an atom of μ and let $x' \in U^\circ$, $B \subset A$, $B \in \Sigma$. Then $\mu(B) = 0$ implies $|\delta^*(x'|\ M(B))| = 0$ and $\mu(A \setminus B) = 0$ implies $|\delta^*(x'|\ M(A \setminus B))| = 0$. Hence, if M is totally non-atomic, then $|\delta^*(x'|\ M(A))| = 0$, since otherwise A would be an atom of $|\delta^*(x'|\ M(\cdot))|$.

Let us suppose that (see [2]) has a countable set $\{A_n \in \Sigma:\ n \in N\}$ of pairwise disjoint atoms. We define a finite measure μ_1 in the following manner:

$$\mu_1(A) = \mu(A \setminus \bigcup_{n \in N} A_n) \quad \text{for } A \in \Sigma.$$

Then μ_1 is a non-negative, non-atomic measure on Σ and for every $\delta^*(x'|\ M(\cdot))$ with respect to μ_1. We have for $\mu_1(A) = 0$:

$$\delta^*(x'|\ M(A)) = \delta^*\big(x'|\ M(E \setminus \bigcup_{n \in N} A_n)\big) + \sum_{n \in N} \delta^*(x'|\ M(A \cap A_n)) = 0,$$

because otherwise we should have $\mu(A \cap A_n) = 0$ or $E \cap A_n$ would be an atom of μ.

Corollary. *Let E be metrizable. M is totally non-atomic, if and only if, there exists a non-negative, non-atomic, finite measure on Σ such that $\delta^*(x'|\ M(\cdot))$ is absolutely continuous with respect to this measure for arbitrary $x' \in E'$.*

The proof is analogous to that in [3].

References

[1] L. Drewnowski, *Topological rings of sets, continuous set functions, integration* III, Bull. Acad. Polon. Sci. Ser. Sci. Math. Astr. 20 No 6 (1972), 439–445.
[2] R. Sikorski, *Funckje rzeczywiste*, t. I, Warszawa 1958.
[3] I. Tweddle, *The range of a vector-valued measure*, Glasgow Math. J. 13 (1972), 64–68.

INSTITUTE OF MATHEMATICS, A. MICKIEWICZ UNIVERSITY, POZNAŃ
INSTITUTE OF MATHEMATICS, POLISH ACADEMY OF SCIENCES, POZNAŃ