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On homotopy in the category of small categories

Abstract. In this paper we prove the equivalence of fraction categories $\mathscr{C}at[\Sigma^{-1}] \rightleftarrows \mathscr{S}et^{\square^{op}}[\mathbf{H}^{-1}]$, where $\mathscr{C}at$ denotes the category of small categories, $\mathscr{S}et^{\square^{op}}$ the category of cubical sets Σ, and $\mathbf{H}$ the appropriate classes of morphisms. Moreover, it is proved that this equivalence is induced by a pair of adjoint functors $\mathscr{C}at \underset{I}{\overset{\mathbf{S}\gamma}{\rightleftarrows}} \mathscr{S}et^{\square^{op}}$.

The proof of a similar result adapted in the simplicial theory, differs essentially and is much simpler than the proof given by D. G. Quillen (see [3], VI, 3).

0. Preliminaries. Let $\square$ be a category; the objects of the category $\square$ are partially ordered sets $[1]^n = [1] \times \underset{n}{\cdots} \times [1]$ for $n \geqslant 0$, where $[1] = \{0 < 1\}$ and $(t_1, \ldots, t_n) \leqslant (t'_1, \ldots, t'_n)$ iff $t_i \leqslant t'_i$ for $i = 1, \ldots, n$. The morphisms of this category are generated by maps (preserving $\leqslant$) $\varepsilon_n^{\delta,i}\colon [1]^{n-1} \to [1]^n$ and $\eta_n^i\colon [1]^{n+1} \to [1]^n$, where $\varepsilon_n^{\delta,i}(t_1, \ldots, t_{n-1}) = (t_1, \ldots, t_{i-1}, \delta, t_i, \ldots, t_{n-1})$ for $i = 1, \ldots, n+1$, $\delta = 0, 1$ and $\eta_n^i(t_1, \ldots, t_{n+1}) = (t_1, \ldots, t_i, \ldots, t_{n+1})$ for $i = 1, \ldots, n+1$.

Not every map preserving $\leqslant$ belongs to the category $\square$.

A functor $\mathbf{X}\colon \square^{op} \to \mathscr{S}et$ is called a *cubical set*; $\mathscr{S}et$ denotes the category of sets.

Put $\mathbf{X}([1]^n) = X_n$, $\mathbf{X}(\varepsilon_n^{\delta,i}) = d_{i,n}^{\delta}$ and $\mathbf{X}(\eta_n^i) = s_{i,n}$. The category $\mathscr{S}et^{\square^{op}}$ is called the *category of cubical sets* (see [5]).

In the Kan extension theory the following theorem is known (see [1], II, 1).

Let $\mathscr{C}$ be a cocomplete and $\mathscr{D}$ a small category. Any functor $\tau\colon \mathscr{D} \to \mathscr{C}$ induces a pair of adjoint functors $\mathbf{E}_{\mathbf{h}}(\tau)\colon \mathscr{S}et^{\mathscr{D}^{op}} \to \mathscr{C}$, $\mathbf{S}_\tau\colon \mathscr{C} \to \mathscr{S}et^{\mathscr{D}^{op}}$, where the functor $\mathbf{E}_{\mathbf{h}}(\tau)$ ($\mathbf{E}_{\mathbf{h}}(\tau)(\mathbf{F}) = \operatorname{colim}_{\mathbf{h}/\mathbf{F}} \tau \cdot \pi$, $\pi\colon \mathbf{h}/\mathbf{F} \to \mathscr{D}$) is the left Kan extension of the functor $\tau\colon \mathscr{D} \to \mathscr{C}$ along Yoneda functor $\mathbf{h}\colon \mathscr{D} \to \mathscr{S}et^{\mathscr{D}^{op}}$ and $\mathbf{S}_\tau$ is the left adjoint functor of $\mathbf{E}_{\mathbf{h}}(\tau)$ ($\mathbf{S}_\tau(C) = \mathscr{C}(\tau -, C)$ for $C \in \operatorname{ob} \mathscr{C}$). Moreover, $\mathbf{E}_{\mathbf{h}}(\tau)\mathbf{h} = \tau$.

Assuming in this theorem $\mathscr{D} = \square$, $\mathscr{C} = \mathscr{T}op$, where $\mathscr{T}op$ denotes the category of topological spaces, and the functor $\tau\colon \square \to \mathscr{T}op$ is given by $\tau([1]^n) = \mathbf{I}^n$, where $\mathbf{I}^n$ denotes the n-dimensional cube, $\tau(\varepsilon_n^{\delta,i})\colon \mathbf{I}^{n-1} \to \mathbf{I}^n$ and $\tau(\eta_n^i)\colon \mathbf{I}^{n+1} \to \mathbf{I}^n$ are the canonical maps of cubes, we obtain the pair of adjoint functors $\mathcal{T}op \underset{\mathbf{E}_h(\tau)}{\overset{\mathbf{S}\tau}{\rightleftarrows}} \mathcal{S}et^{\square^{op}}$. The functor $\mathbf{E}_h(\tau)$: $\mathcal{S}et^{\square^{op}} \to \mathcal{T}op$ is called the *geometric realization.* Put $\mathbf{E}_h(\tau) = |\ |$.

One can easily prove the following lemma.

LEMMA. *The pair of adjoint functors* $\mathcal{T}op \underset{|\ |}{\overset{\mathbf{S}\tau}{\rightleftarrows}} \mathcal{S}et^{\square^{op}}$ *preserves the homotopy relation.* ■

1. Cubical sets from categories. Let $\mathcal{D} = \square$, $\mathcal{C} = \mathcal{C}at$, where $\mathcal{C}at$ is the category of small categories, and the functor $\tau: \square \to \mathcal{C}at$ is given by $\tau([1]^n) = [1]^n$, where $[1]^n$ is treated as a category in an obvious way.

From the theorem above we get the pair of adjoint functors $\mathcal{C}at \underset{\mathbf{E}_h(\tau)}{\overset{\mathbf{S}\tau}{\rightleftarrows}} \mathcal{S}et^{\square^{op}}$. The functor $\mathbf{S}_\tau$: $\mathcal{C}at \to \mathcal{S}et^{\square^{op}}$ is called the *cubical nerve.* Put $\mathbf{S}_\tau = \mathbf{N}$ and $\mathbf{E}_h(\tau) = \mathbf{c}$.

Homotopy relation in the category $\mathcal{C}at$ is the smallest congruence relation generated by natural transformations of functors.

The following lemma describes the properties of the pair of functors $\mathcal{C}at \underset{\mathbf{c}}{\overset{\mathbf{N}}{\rightleftarrows}} \mathcal{S}et^{\square^{op}}$.

LEMMA 1.1. (See [2].) 1° $\mathbf{c} \cdot \mathbf{N} = \mathbf{id}_{\mathcal{C}at}$.

2° *The functor* $\mathbf{N}$: $\mathcal{C}at \to \mathcal{S}et^{\square^{op}}$ *is full and faithful.*

3° *Functors* $\mathcal{C}at \underset{\mathbf{c}}{\overset{\mathbf{N}}{\rightleftarrows}} \mathcal{S}et^{\square^{op}}$ *preserve the homotopy relation.*

4° *The cubical set* $\mathbf{N}\mathcal{C}$ *satisfies the Kan extension condition for*:

(a) $n \geqslant 3$, *where* $\mathcal{C}$ *is any category,*

(b) $n \geqslant 2$, *iff every morphism of a category* $\mathcal{C}$ *is mono- and epimorphism,*

(c) $n \geqslant 1$, *iff a category* $\mathcal{C}$ *is a groupoid.* ■

Let θ: $\square \to \mathcal{C}at$ be any functor.

LEMMA 1.2. *If there is a natural transformation of functors* χ: $\theta \to \tau$: $\square \to \mathcal{C}at$, *then the functor* $\mathbf{S}_\theta$: $\mathcal{C}at \to \mathcal{S}et^{\square^{op}}$ *preserves the homotopy relation.*

Proof. Of course, it is enough to prove that a natural transformation of functors $(\varphi$: $\mathbf{F} \to \mathbf{G})$: $\mathcal{C} \to \mathcal{S}$ determines a homotopy of maps $\mathbf{S}_\theta(\mathbf{F}) \simeq \mathbf{S}_\theta(\mathbf{G})$: $\mathbf{S}_\theta(\mathcal{C}) \to \mathbf{S}_\theta(\mathcal{S})$. We define a sequence of maps h_n: $\mathbf{S}_\theta(\mathcal{C})_n \to \mathbf{S}_\theta(\mathcal{S})_{n+1}$ such that:

(a) $d_1^0 h_n = \mathbf{S}_\theta(\mathbf{F})_n$,

(b) $d_1^1 h_n = \mathbf{S}_\theta(\mathbf{G})_n$,

(c) $d_i^\delta h_n = h_{n-1} d_{i-1}^\delta$ for $1 < i \leqslant n+1$,

(d) $s_{i+1} h_n = h_{n+1} s_i$ for $1 \leqslant i \leqslant n+1$ (see [6]).

Let $\mathbf{H}$: $[1] \times \mathcal{C} \to \mathcal{D}$ be the functor determined by the natural transformation φ: $\mathbf{F} \to \mathbf{G}$. It is easy to prove that the maps given by $h_n(\alpha) = \mathbf{H}(\chi([1])\theta(\eta_1^2 \ldots \eta_n^2), \alpha \cdot \theta(\eta_n^1))$ for $\alpha \in \mathbf{S}_\theta(\mathcal{C})_n$ and $n \geqslant 0$, satisfy conditions (a)–(d). ■

Applying this lemma and adapting the proof of the appropriate theorem in simplicial theory (see [5]), we get

THEOREM 1.3. *Let* $\theta\colon \square \to \mathscr{C}at$ *be a functor such that:*

1° *there exists a natural transformation of functors* $(\chi\colon \theta \to \tau)\colon \square \to \mathscr{C}at$,

2° the functor $\chi([1]^k)\colon \theta([1]^k) \to \tau([1]^k)$ *is a homotopy equivalence.*

Then the induced natural transformation of functors $\tilde{\chi}\colon \mathbf{N} \to \mathbf{S}_\theta\colon \mathscr{C}at \to \mathscr{S}et^{\square^{op}}$ *is a weak homotopy equivalence, i.e. for every category* $\mathscr{C} \in ob\mathscr{C}at$ *the map* $|\tilde{\chi}(C)|\colon |\mathbf{N}\mathscr{C}| \to |\mathbf{S}_\theta\mathscr{C}|$ *is a homotopy equivalence.* ∎

EXAMPLE 1.4. For the functor $\gamma\colon \square \to \mathscr{C}at$ given on objects by $\gamma([1]^n) = \square/[1]^n$, where $\square/[1]^n$ is the comma category and in the obvious way on morphisms, there exists the natural transformation $\chi\colon \gamma \to \tau$ determined by the sequence of functors $\chi_n\colon \square/[1]^n \to [1]^n$ for $n \geqslant 0$ given by $\chi_n(\alpha\colon [1]^k \to [1]^n) = \alpha(1, \ldots, 1)$ and $\chi_n(\beta\colon a \to a') = \big(\alpha(1, \ldots, 1) \leqslant \alpha'(1, \ldots, 1)\big)$. Categories $\square/[1]^n$ and $[1]^n$ have terminal objects, so functors $\chi_n\colon \square/[1]^n \to [1]^n$ are homotopy equivalences. From Theorem 1.3 we get that the induced natural transformation of functors $\tilde{\chi}\colon \mathbf{N} \to \mathbf{S}_\gamma\colon \mathscr{C}at \to \mathscr{S}et^{\square^{op}}$ is a weak homotopy equivalence.

Let $\mathbf{E}_h(\gamma)\colon \mathscr{S}et^{\square^{op}} \to \mathscr{C}at$ be the left adjoint functor of $\mathbf{S}_\gamma\colon \mathscr{C}at \to \mathscr{S}et^{\square^{op}}$. Put $\mathbf{E}_h(\gamma) = \boldsymbol{\Gamma}$. From the definition of functor $\boldsymbol{\Gamma}$ it follows that $ob\,\boldsymbol{\Gamma}(\mathbf{X}) = \coprod_{n \geqslant 0} X_n$, and morphisms of the category $\boldsymbol{\Gamma}(\mathbf{X})$ are generated by triples $(\varepsilon^{\delta,i}, d_i^\delta x, x)\colon d_i^\delta x \to x$ and $(\eta^i, s_i x, x)\colon s_i x \to x$ for $\mathbf{X} \in ob\mathscr{S}et^{\square^{op}}$.

LEMMA 1.5. *The functor* $\boldsymbol{\Gamma}\colon \mathscr{S}et^{\square^{op}} \to \mathscr{C}at$ *preserves the homotopy relation.*

Proof. Let $h_n\colon X_n \to Y_{n+1}$ be the sequence of maps determined by the homotopy of maps $f^0 \simeq f^1\colon \mathbf{X} \to \mathbf{Y}$. Let us consider the functor $\mathbf{H}\colon \boldsymbol{\Gamma}(\mathbf{X}) \to \boldsymbol{\Gamma}(\mathbf{Y})$ given by $\mathbf{H}(x) = h_n(x)$ for $x \in X_n \subset ob\boldsymbol{\Gamma}(\mathbf{X})$ and $n \geqslant 0$, $\mathbf{H}(\varepsilon_n^{\delta,i}, d_i x, x) = \big(\varepsilon_{n+1}^{\delta,i+1}, d_{i+1}^\delta \mathbf{H}(x), \mathbf{H}(x)\big)$ and $\mathbf{H}(\eta_n^i, s_i x, x) = \big(\eta_n^{i+1}, s_{i+1}\mathbf{H}(x), \mathbf{H}(x)\big)$.

Then the natural transformations of functors $\boldsymbol{\Gamma}(f^0) \xrightarrow{\varphi^0} \mathbf{H} \xleftarrow{\varphi^1} \boldsymbol{\Gamma}(f^1)$ given by $\varphi^\delta(x) = \big(\varepsilon^{\delta,1}, d_1^\delta \mathbf{H}(x), \mathbf{H}(x)\big)\colon d_1^\delta \mathbf{H}(x) = f^\delta(x) \to \mathbf{H}(x)$ for $x \in ob\boldsymbol{\Gamma}(\mathbf{X})$ and $\delta = 0, 1$ induce the homotopy of functors $\boldsymbol{\Gamma}(f^0)$ and $\boldsymbol{\Gamma}(f^1)$. ∎

Let us observe that if $* \in ob\mathscr{S}et^{\square^{op}}$ is the terminal object, then the category $\boldsymbol{\Gamma}(*)$ has the terminal object. So, if a cubical set $\mathbf{X}$ is homotopy trivial, then the category $\boldsymbol{\Gamma}(\mathbf{X})$ is also homotopy trivial.

2. A homotopy inverse for the functor of cubical nerve. Applying the definition of the left Kan extension it is easy to observe that for the diagram of functors

$$\begin{array}{ccccc} \mathscr{A} & \xrightarrow{\mathbf{T}} & \mathscr{B} & \xrightarrow{\mathbf{F}} & \mathscr{C} \\ {\scriptstyle \mathbf{K}}\big\downarrow & & & & \\ \mathscr{D} & & & & \end{array}$$

the associated transformation $\mathbf{T} \to \mathbf{E}_{\mathbf{K}}(\mathbf{T}) \cdot \mathbf{K}$ determines the transformation $\mathbf{F} \cdot \mathbf{T} \to \mathbf{F} \cdot \mathbf{E}_{\mathbf{K}}(\mathbf{T}) \cdot \mathbf{K}$ that induces the transformation of functors $\varrho\colon \mathbf{E}_{\mathbf{K}}(\mathbf{F} \cdot \mathbf{T}) \to \mathbf{F} \cdot \mathbf{E}_{\mathbf{K}}(\mathbf{T})$.

In particular, for the diagram of functors

$$\begin{array}{ccccc} \square & \xrightarrow{\gamma} & \mathscr{C}at & \xrightarrow{\mathbf{Ar}(\mathscr{C},\, -)} & \mathscr{C}at \\ {\scriptstyle \mathbf{h}}\big\downarrow & & & & \\ \mathscr{S}et^{\square^{op}} & & & & \end{array}$$

where $\mathscr{C} \in ob\mathscr{C}at$, we get the natural transformation $\varrho\colon \mathbf{E}_{\mathbf{h}}(\mathbf{Ar}(\mathscr{C}, -) \cdot \gamma) \to \mathbf{Ar}(\mathscr{C}, -) \cdot \mathbf{E}_{\mathbf{h}}(\gamma) = \mathbf{Ar}(\mathscr{C}, -) \cdot \boldsymbol{\Gamma}$.

It is not difficult to prove the following lemma.

LEMMA 2.1. *If the category $\mathscr{C} \in ob\mathscr{C}at$ has a terminal object, then the natural transformation of functors $\varrho\colon \mathbf{E}_{\mathbf{h}}(\mathbf{Ar}(\mathscr{C}, -) \cdot \gamma) \to \mathbf{Ar}(\mathscr{C}, -) \cdot \boldsymbol{\Gamma}$ is an isomorphism.* ∎

Let us observe that the category $\tau([1]^k) = [1]^k$ has a terminal object, for any $k \geqslant 0$. So we have the isomorphism of functors $\varrho\colon \mathbf{E}_{\mathbf{h}}(\mathbf{Ar}([1]^k, -) \cdot \gamma) \to \mathbf{Ar}([1]^k, -) \cdot \boldsymbol{\Gamma}$.

Hence we get the following corollary.

COROLLARY 2.2. *The isomorphism of functors $\varrho\colon \mathbf{E}_{\mathbf{h}}(\mathbf{Ar}([1]^k, -) \cdot \gamma) \to \mathbf{Ar}([1]^k, -) \cdot \boldsymbol{\Gamma}$ induces the isomorphism $\tilde{\varrho}\colon \mathbf{E}_{\mathbf{h}}(\mathbf{N} \cdot \gamma) \to \mathbf{N} \cdot \boldsymbol{\Gamma}$, where $\mathbf{N}\colon \mathscr{C}at \to \mathscr{S}et^{\square^{op}}$ is the functor of cubical nerve.* ∎

In the further considerations the following remark will be necessary.

Let $\mathbf{I}$ be the unit interval. The functor $\mathbf{I} \times -\colon \mathscr{T}op \to \mathscr{T}op$ has a right adjoint functor, so $\mathbf{E}_{\mathbf{h}}(\mathbf{I} \times \theta) = \mathbf{I} \times \mathbf{E}_{\mathbf{h}}(\theta)$ for any functor $\theta\colon \square \to \mathscr{T}op$, where $\mathbf{h}\colon \square \to \mathscr{S}et^{\square^{op}}$ is Yoneda functor.

Remark 2.3. Let $\theta, \theta'\colon \square \to \mathscr{T}op$ be functors. Then a natural transformation $\mathbf{H}\colon \mathbf{I} \times \theta \to \theta'$ induces the transformation $\mathbf{E}_{\mathbf{h}}(\mathbf{H})\colon \mathbf{I} \times \mathbf{E}_{\mathbf{h}}(\theta) \to \mathbf{E}_{\mathbf{h}}(\theta')$.

THEOREM 2.4. *The pair of functors $\mathscr{C}at \underset{\boldsymbol{\Gamma}}{\overset{\mathbf{N}}{\rightleftarrows}} \mathscr{S}et^{\square^{op}}$ induces the equivalence of fraction categories $\mathscr{C}at[\Sigma^{-1}] \gtreqless \mathscr{S}et^{\square^{op}}[\mathbf{H}^{-1}]$, where $\mathbf{H}$ is the class of all maps f in $\mathscr{S}et^{\square^{op}}$ such that $|f|$ is a homotopy equivalence and Σ is the class of all functors $\mathbf{F}$ such that $\mathbf{NF} \in \mathbf{H}$.*

Proof. Let $\tilde{\chi}\colon \mathbf{N} \to \mathbf{S}_\gamma$ be the natural transformation of functors given in Example 1.4. From the adjunction of functors $\mathscr{C}at \underset{\Gamma}{\overset{\mathbf{S}\gamma}{\rightleftarrows}} \mathscr{S}et^{\square^{op}}$ we get the natural transformation $\varphi\colon \boldsymbol{\Gamma} \cdot \mathbf{S}_\gamma \to \mathbf{id}_{\mathscr{C}at}$. Hence we have the transformation $\boldsymbol{\Phi}\colon \boldsymbol{\Gamma} \cdot \mathbf{N} \xrightarrow{\Gamma\tilde{\chi}} \boldsymbol{\Gamma} \cdot \mathbf{S}_\gamma \to \mathbf{id}_{\mathscr{C}at}$. The transformation of functors $\boldsymbol{\Phi}\colon \boldsymbol{\Gamma} \cdot \mathbf{N} \to \mathbf{id}_{\mathscr{C}at}$ will be described explicitly.

Let us observe that, for $\mathscr{C} \in ob\mathscr{C}at$, $ob\boldsymbol{\Gamma}\mathbf{N}(\mathscr{C}) = \coprod_{n \geqslant 0} \mathbf{Ar}([1]^n, \mathscr{C})$. The functor $\boldsymbol{\Phi}(\mathscr{C})$: $\Gamma\mathbf{N}(\mathscr{C}) \to \mathscr{C}$ is given by $\boldsymbol{\Phi}(\mathscr{C}) = g(1, \ldots, 1)$ for $g \in ob\,\boldsymbol{\Gamma}\mathbf{N}(\mathscr{C})$, and the morphism $(\alpha; f, g)$: $f \to g$ of the category $\Gamma\mathbf{N}(\mathscr{C})$, $\boldsymbol{\Phi}(\mathscr{C})(\alpha; f, g) = g\big(\alpha(1, \ldots, 1) \leqslant (1, \ldots, 1)\big)$: $\boldsymbol{\Phi}(\mathscr{C})(f) = f(1, \ldots, 1) \to \boldsymbol{\Phi}(\mathscr{C})(g) = g(1, \ldots, 1)$. Moreover, the natural transformation χ: $\gamma \to \tau$ induces the transformation $\psi = \mathbf{E}_{\mathbf{h}}(\chi)$: $\mathbf{E}_{\mathbf{h}}(\chi)$: $\mathbf{E}_{\mathbf{h}}(\gamma) = \boldsymbol{\Gamma} \to \mathbf{E}_{\mathbf{h}}(\tau) = \mathbf{c}$.

Using the adjunction of functors $\mathscr{C}at \underset{\Gamma}{\overset{\mathrm{S}\gamma}{\rightleftarrows}} \mathscr{S}et^{\square^{op}}$ we get the natural transformation ϑ: $\mathbf{id}_{\mathscr{S}et^{\square^{op}}} \to \mathbf{S}_\gamma \cdot \boldsymbol{\Gamma}$. Hence we have the transformation θ: $\mathbf{id}_{\mathscr{S}et^{\square^{op}}} \xrightarrow{\vartheta} \mathbf{S}_\gamma \cdot \boldsymbol{\Gamma} \xrightarrow{\mathrm{S}\gamma \cdot \psi} \mathbf{S}_\gamma \cdot \mathbf{c}$. It is easy to observe that $\Phi = \psi\mathbf{N}$: $\Gamma\mathbf{N} \to \mathbf{c} \cdot \mathbf{N} = \mathbf{id}_{\mathscr{C}at}$ and $\tilde{\chi} = \theta \cdot \mathbf{N}$: $\mathbf{N}_\iota \to \mathbf{S}_\gamma \cdot \mathbf{c} \cdot \mathbf{N} = \mathbf{S}_\gamma$.

The functors $\mathbf{n}$: $\square \xrightarrow{\tau} \mathscr{C}at \xrightarrow{\mathrm{N}} \mathscr{S}et^{\square^{op}}$ and $\mathbf{h}$: $\square \to \mathscr{S}et^{\square^{op}}$ are not equal (not every map preserving $\leqslant$ belongs to the category $\square$), however, there is a natural transformation i: $h \hookrightarrow \mathbf{n}$. Using the natural transformation $\boldsymbol{\Phi}$: $\boldsymbol{\Gamma} \cdot \mathbf{N} \to \mathbf{id}_{\mathscr{C}at}$, we get the transformation $\mathbf{N}\boldsymbol{\Phi}\tau$: $\mathbf{N}\boldsymbol{\Gamma}\mathbf{N}\tau = \mathbf{N}\boldsymbol{\Gamma}\mathbf{n} \to \mathbf{N}\tau = \mathbf{n}$.

Hence we get the natural transformation δ: $\mathbf{N}\boldsymbol{\Gamma}\mathbf{h} = \mathbf{N}\gamma \xrightarrow{\mathbf{N}\Gamma i} \mathbf{N}\boldsymbol{\Gamma}\mathbf{n} \xrightarrow{\mathbf{N}\Phi\tau} \mathbf{n}$. Then the natural transformations $\mathbf{N} \cdot \gamma \xrightarrow{\delta} \mathbf{n} \xleftarrow{i} \mathbf{h}$ induce the transformations of suitable Kan extensions $\mathbf{E}_{\mathbf{h}}(\mathbf{N}\gamma) \xrightarrow{\mathbf{E}_{\mathbf{h}}(\delta)} \mathbf{E}_{\mathbf{h}}(\mathbf{n}) \xleftarrow{\mathbf{E}_{\mathbf{h}}(i)} \mathbf{E}_{\mathbf{h}}(\mathbf{h})$.

Let us observe that $\mathbf{E}_{\mathbf{h}}(\mathbf{h}) = \mathbf{id}_{\mathscr{S}et^{\square^{op}}}$ and, from Corollary 2.2, $\mathbf{E}_{\mathbf{h}}(\mathbf{N}\gamma) \tilde{\to} \mathbf{N}\boldsymbol{\Gamma}$. Puting $\mathbf{E}_{\mathbf{h}}(\delta) = \Delta$, $\mathbf{E}_{\mathbf{h}}(\mathbf{n}) = \mathbf{E}$ and $\mathbf{E}_{\mathbf{h}}(i) = \mathbf{I}$, we get the transformations $\mathbf{N}\boldsymbol{\Gamma} \xrightarrow{\Delta} \mathbf{E} \xleftarrow{\mathbf{I}} \mathbf{id}_{\mathscr{S}et^{\square^{op}}}$.

The category $\tau([1]^k) = [1]^k$ has a terminal object, so the cubical set $\mathbf{N}\tau([1]^k) = \mathbf{n}([1]^k)$ is homotopy trivial. Hence, from Lemma 1.4, the category $\boldsymbol{\Gamma}\mathbf{N}\tau([1]^k) = \boldsymbol{\Gamma}\mathbf{n}([1]^k)$ is homotopy trivial and we get the homotopy triviality of the cubical set $\mathbf{N}\boldsymbol{\Gamma}\mathbf{N}\tau([1]^k) = \mathbf{N}\boldsymbol{\Gamma}\mathbf{n}([1]^k)$. The category $\gamma([1]^k) = \square/[1]^k$ has also a terminal object, so the cubical set $\mathbf{N}\gamma([1]^k)$ is homotopy trivial.

Hence the topological spaces $|\mathbf{N}\boldsymbol{\Gamma}\mathbf{n}([1]^k)|$, $|\mathbf{N}\gamma([1]^k)|$ are homotopy trivial; moreover, $|\mathbf{h}([1]^k)| = \mathbf{I}^k$ is also homotopy trivial, where $|\ |$: $\mathscr{S}et^{\square^{op}} \to \mathscr{T}op$ is the geometric realization functor. Lastly, for $[1]^k \in ob\square$ the continuous maps $|\mathbf{N}\gamma([1]^k)| \xrightarrow{|\delta([1]^k)|} |\mathbf{n}([1]^k)| \xleftarrow{|i([1]^k)|} |\mathbf{h}([1]^k)|$, as the maps of homotopy trivial topological spaces, are homotopy equivalences. Using Remark 2.3 we get that for any $\mathbf{X} \in ob\mathscr{S}et^{\square^{op}}$ the continuous maps $|\mathbf{N}\boldsymbol{\Gamma}(\mathbf{X})| \xrightarrow{|\Delta(\mathbf{X})|} |\mathbf{E}(\mathbf{X})| \xleftarrow{|\mathbf{I}(\mathbf{X})|} |\mathbf{X}|$ are homotopy equivalences.

Let $\mathbf{H}$ be the class of all maps f in $\mathcal{S}et^{\square^{op}}$ such that $|f|$ is a homotopy equivalence and Σ the class of all functors $\mathbf{F}$ such that $\mathbf{NF} \in \mathbf{H}$. From the commutative diagram

$$\begin{array}{ccccc} \mathbf{N}\Gamma(\mathbf{X}) & \xrightarrow{\Delta(\mathbf{X})} & \mathbf{E}(\mathbf{X}) & \xleftarrow{\mathbf{I}(\mathbf{X})} & \mathbf{X} \\ \downarrow{\scriptstyle \mathbf{N}\Gamma(f)} & & \downarrow{\scriptstyle \mathbf{E}(f)} & & \downarrow{\scriptstyle f} \\ \mathbf{N}\Gamma(\mathbf{Y}) & \xrightarrow{\Delta(\mathbf{Y})} & \mathbf{E}(\mathbf{Y}) & \xleftarrow{\mathbf{I}(\mathbf{Y})} & \mathbf{Y} \end{array}$$

for any map of cubical sets $f\colon \mathbf{X} \to \mathbf{Y}$ and from the above considerations it follows that $f \in \mathbf{H}$ iff $\Gamma(f) \in \Sigma$. Hence the functors $\mathcal{C}at \underset{\Gamma}{\overset{\mathbf{N}}{\rightleftarrows}} \mathcal{S}et^{\square^{op}}$ induce the pair of functors on the appropriate fraction categories $\mathcal{C}at[\Sigma^{-1}] \underset{\tilde{\Gamma}}{\overset{\tilde{N}}{\rightleftarrows}} \mathcal{S}et^{\square^{op}}[\mathbf{H}^{-1}]$.

It is known that for any $\mathbf{X} \in ob\mathcal{S}et^{\square^{op}}$ the maps of cubical sets $\mathbf{N}\Gamma(\mathbf{X}) \xrightarrow{\Delta(\mathbf{X})} \mathbf{E}(\mathbf{X}) \xleftarrow{\mathbf{I}(\mathbf{X})} \mathbf{X}$ belongs to the class $\mathbf{H}$. Note that for any category $\mathcal{C} \in ob\mathcal{C}at$ the sequence of maps $f_n\colon (\mathbf{E}(\mathbf{N}\mathcal{C}))_n = \coprod_{k \geqslant 0} \mathbf{Ar}([1]^n, [1]^k) \times \mathbf{Ar}([1]^k, \mathcal{C})/\sim \; \to \mathbf{Ar}([1]^n, \mathcal{C})$ given by $f_n([\alpha, \beta])/\sim \; = \beta \cdot \alpha$ for $n \geqslant 0$, determines the map of cubical sets $f\colon \mathbf{E}(\mathbf{N}\mathcal{C}) \to \mathbf{N}\mathcal{C}$ such that the following diagrams

$$\begin{array}{ccc} \mathbf{E}(\mathbf{N}\mathcal{C}) & \xrightarrow{f} & \mathbf{N}\mathcal{C} \\ \downarrow{\scriptstyle \mathbf{I}(\mathbf{N}\mathcal{C})} & \nearrow_{=} & \\ \mathbf{N}\mathcal{C} & & \end{array} \qquad \text{and} \qquad \begin{array}{ccc} \mathbf{N}\Gamma(\mathbf{N}\mathcal{C}) & \xrightarrow{\Delta(\mathbf{N}\mathcal{C})} & \mathbf{E}(\mathbf{N}\mathcal{C}) \\ \downarrow{\scriptstyle \mathbf{N}(\Phi\mathcal{C})} & \swarrow_{f} & \\ \mathbf{N}\mathcal{C} & & \end{array}$$

commute.

Hence the functors $\Phi(\mathcal{C})\colon \Gamma\mathbf{N}(\mathcal{C}) \to \mathcal{C}$ induced by the natural transformation belong to the class Σ.

From this it follows that the functors $\mathcal{C}at[\Sigma^{-1}] \underset{\tilde{\Gamma}}{\overset{\tilde{N}}{\rightleftarrows}} \mathcal{S}et^{\square^{op}}[\mathbf{H}^{-1}]$ determine the equivalence of the appropriate fraction categories. ■

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