ANNALES SOCIETATIS MATHEMATICAE POLONAE
Series I: COMMENTATIONES MATHEMATICAE XXIV (1984)
ROCZNIKI POLSKIEGO TOWARZYSTWA MATEMATYCZNEGO
Seria I: PRACE MATEMATYCZNE XXIV (1984)

ZENON DOPIERAŁA, LUCYNA REMPULSKA (Poznań)

Application of the Abel means of trigonometric Fourier series for differential equations of the Laplace type

Abstract. In this paper we shall prove that the Abel mean of order n of trigonometric Fourier series is the solution of boundary problem for differential equation of the Laplace type in the unit disc.

1. Let (r, φ) be the polar coordinates of point and let X be a domain in the plane. Let m be a fixed non-negative integer. Denote by $C^m(X)$ $(C^0(X) = C(X))$ the class of all real functions $u = u(r, \varphi)$ defined in X and having the partial derivatives $\partial^s u/\partial r^p \partial\varphi^q$, $s = 0, 1, \ldots, m$, continuous in X. Let $K = \{(r, \varphi)\colon |re^{i\varphi}| < 1\}$, $\bar{K} = \{(r, \varphi)\colon |re^{i\varphi}| \leqslant 1\}$ and $K_0 = \{(r, \varphi)\colon 0 < |re^{i\varphi}| < 1\}$.

Considering the functions u of the class $C^{2n+2}(K_0)$ we define the operators ∇^n by the formulae

$$(1)\qquad \nabla^0 u = \Delta u, \qquad \nabla^n u = \Delta(r\nabla^{n-1}u) \qquad (n = 1, 2, \ldots),$$

where Δ is the Laplace operator, i.e.

$$\Delta u = \frac{\partial^2 u}{\partial r^2} + \frac{1}{r}\frac{\partial u}{\partial r} + \frac{1}{r^2}\frac{\partial^2 u}{\partial \varphi^2}.$$

Clearly,

$$(2)\qquad \nabla^n(\alpha u + \beta v) = \alpha\nabla^n u + \beta\nabla^n v$$

$(n = 0, 1, \ldots;\ \alpha, \beta = \text{const})$ for $u, v \in C^{2n+2}(K_0)$.

Using induction, we can prove the following properties of the operator ∇^n

LEMMA 1. *If* $u \in C^\infty(K_0)$ *and* $n = 0, 1, \ldots$, *then*

1° $\nabla^n\left(r\frac{\partial u}{\partial r}\right) = \frac{\partial}{\partial r}(r\nabla^n u) + (n+1)\nabla^n u,$

2° $\nabla^0\left(r^2\frac{\partial u}{\partial r}\right) = \frac{\partial}{\partial r}\left(r^2\Delta u + u + 2r\frac{\partial u}{\partial r}\right),$

3° $\nabla^n\left(r^2\frac{\partial u}{\partial r}\right)=\frac{\partial}{\partial r}\left\{r\nabla^{n-1}\left(r^2\Delta u+u+2r\frac{\partial u}{\partial r}\right)\right\}+$

$$+n\nabla^{n-1}\left(r^2\Delta u+u+2r\frac{\partial u}{\partial r}\right) \quad \text{if } n=1,2,\ldots,$$

4° $\nabla^n(r^2\Delta u)=r\nabla^{n+1}u+2(n+1)\frac{\partial}{\partial r}(r\nabla^n u)+(n+1)^2\nabla^n u.$

Applying Lemma 1 we get

LEMMA 2. *Suppose that* $u\in C^{2n+5}(K_0)$. *If* $\nabla^n u(r,\varphi)=0$ *in* K_0, *then the function*

(3) $$v(r,\varphi)=u(r,\varphi)+\frac{r-r^2}{n+1}\frac{\partial u}{\partial r}$$

satisfies the equation $\nabla^{n+1}v(r,\varphi)=0$ *in* K_0.

Proof. By (2),

$$\nabla^{n+1}v=\nabla^{n+1}u+\frac{1}{n+1}\left\{\nabla^{n+1}\left(r\frac{\partial u}{\partial r}\right)-\nabla^{n+1}\left(r^2\frac{\partial u}{\partial r}\right)\right\}.$$

The condition $\nabla^n u(r,\varphi)=0$ in K_0 and definition (1) imply $\nabla^{n+1}u(r,\varphi)=0$ in K_0. By Lemma 1,

$$\nabla^{n+1}\left(r\frac{\partial u}{\partial r}\right)=0,\quad \nabla^{n+1}\left(r^2\frac{\partial u}{\partial r}\right)=0 \quad \text{in } K_0.$$

Hence, we obtain $\nabla^{n+1}v(r,\varphi)=0$ in K_0. The proof is completed.

Below, we shall apply the following

LEMMA 3. *If* $u\in C^{n+2}(\bar{K})$ *and if*

$$\left(\frac{\partial u}{\partial r}\right)_{r=1}=\ldots=\left(\frac{\partial^n u}{\partial r^n}\right)_{r=1}=0 \quad (\varphi\in\langle 0,2\pi\rangle),$$

then the function v *defined by* (3) *satisfies the conditions*

$$\left(\frac{\partial^p v}{\partial r^p}\right)_{r=1}=0,\quad p=1,2,\ldots,n+1, \textit{ for } \varphi\in\langle 0,2\pi\rangle.$$

2. Let $C_{2\pi}^m$ (m is a fixed non-negative integer) be the class of all 2π-periodic real functions f of variable φ having the derivatives $f^{(p)}$, $p=0,1,\ldots,m$, continuous everywhere ($C_{2\pi}^0=C_{2\pi}$). Let

(4) $$\tfrac{1}{2}a_0+\sum_{k=1}^{\infty}(a_k\cos k\varphi+b_k\sin k\varphi)=\sum_{k=0}^{\infty}T_k(\varphi;f)$$

be the Fourier series of function $f\in C_{2\pi}$. Consider the Abel means of series (4) defined in [1]. As in [1], let D^n be the differential operator, defined for the functions r^k $(k = 0, 1, \ldots)$ by the formula

(5) $$D^0(r^k) = r^k, \quad D^n(r^k) = D^{n-1}(r^k) + \frac{r-r^2}{n}\frac{d}{dr} D^{n-1}(r^k)$$

$(n = 1, 2, \ldots)$. Let

(6) $$P(r, \varphi; n, f) = \sum_{k=0}^{\infty} D^n(r^k)\, T_k(\varphi; f)$$

$(r \in \langle 0, 1), \varphi \in (-\infty, +\infty), n = 0, 1, \ldots)$ be the Abel mean of order n of series (4). In [1] it is proved that if $f \in C_{2\pi}$ and $n = 0, 1, \ldots,$ then

$$P(r, \varphi; n, f) = (1-r)^{n+1} \sum_{k=0}^{\infty} \binom{n+k}{n} r^k \sum_{p=0}^{k} T_p(\varphi; f)$$

and $\max_{\varphi} |P(r, \varphi; n, f) - f(\varphi)| \to 0$ if $r \to 1-$ (see [3], p. 241).

By (5),

(7) $$P(r, \varphi; n, f) = P(r, \varphi; n-1, f) + \frac{r-r^2}{n}\frac{\partial}{\partial r} P(r, \varphi; n-1, f)$$

in the unit disc K. Using the induction and (7), we obtain

LEMMA 4. *If $f \in C_{2\pi}$ and $n = 1, 2, \ldots,$ then*

(8) $$P(r, \varphi; n, f) = P(r, \varphi; 0, f) + \sum_{k=1}^{n} W_k(r; n) \frac{\partial^k}{\partial r^k} P(r, \varphi; 0, f)$$

in the unit disc K, where $W_k(r; n)$ are some algebraic polynomials of order $\leqslant 2n$.

Clearly, if $f \in C_{2\pi}^1$ and $n = 0, 1, \ldots,$ then the Abel mean of order n can be defined in the unit disc K. Moreover, by (6),

(9) $$P(1, \varphi; n, f) = \sum_{k=0}^{\infty} T_k(\varphi; f) = f(\varphi)$$

$(\varphi \in \langle 0, 2\pi \rangle)$. If $f \in C_{2\pi}^{n+2}$, then (7) and (8) hold in $\bar{K}$.

3. Now, we shall give the theorem on the solution of equation $V^n u = 0$.

THEOREM. *The Abel mean $P(n, f) = P(r, \varphi; n, f)$ of order n, $n = 1, 2, \ldots,$ of trigonometric Fourier series of function $f \in C_{2\pi}^{2n+2}$ has the following properties:*

1° $P(n, f) \in C^n(\bar{K})$,

2° $P(1, \varphi; n, f) = f(\varphi)$ $(\varphi \in \langle 0, 2\pi \rangle)$,

3° $V^n P(r, \varphi; n, f) = 0$ *in the domain* K_0,

4° $\left(\frac{\partial^q}{\partial r^q} P(r, \varphi; n, f)\right)_{r=1} = 0$ *for* $q = 1, 2, \ldots, n$; $\varphi \in \langle 0, 2\pi \rangle$.

Proof. First, we shall prove condition 1°. It is known that $a_k, b_k = O(k^{-2n-2})$ if $f \in C_{2\pi}^{2n+2}$. Hence, the Abel mean $P(0, f) = P(r, \varphi; 0, f)$ of trigonometric Fourier series of function $f \in C_{2\pi}^{2n+2}$ is the function of the class $C^\infty(K)$ and $C^{2n}(\bar{K})$. By (8),

$$(10) \qquad P(N,f) \in C^\infty(K), \qquad P(N,f) \in C^{2n-N}(\bar{K}) \qquad \text{for } N = 1, 2, \ldots, n.$$

Condition 2° in the case $n = 1, 2, \ldots$ holds by (9).

As it is known ([2], p. 279), conditions 1°–3° are satisfied for the Abel mean $P(0,f) = P(r, \varphi; 0, f)$ of trigonometric Fourier series of function $f \in C_{2\pi}^1$. Hence, if $f \in C_{2\pi}^{2n+2}$, then $\nabla^0 P(r, \varphi; 0, f) = 0$ in K_0.

Applying (7), (10) and Lemma 2, we obtain

$$(11) \qquad \nabla^n P(r, \varphi; n, f) = 0 \quad \text{in } K_0.$$

By (7), we get $\left(\frac{\partial}{\partial r} P(r, \varphi; 1, f)\right)_{r=1} = 0$ for $\varphi \in \langle 0, 2\pi\rangle$. Applying (7), (10) and Lemma 3, we obtain 4°. This result and (9)–(11) prove our thesis.

References

[1] Z. Dopierała, L. Rempulska, *On the summability of series by harmonic methods*, Comment. Math. 23 (1983), 11–25.

[2] A. Tichonow, A. Samarski, *Równania fizyki matematycznej*, Warszawa 1963 (*Equations of mathematical physics*, Russian ed., Moscow 1953).

[3] A. Zygmund, *Trigonometric series* (Russian ed.), Moscow 1965.