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On classes of couples of characteristic functions satisfying the condition of Dugué

D. Dugué in [1] was interested in finding couples (φ_1, φ_2) of characteristic functions satisfying the condition

$$\text{(D)} \qquad \frac{\varphi_1(t)+\varphi_2(t)}{2} = \varphi_1(t)\varphi_2(t).$$

He has remarked that the characteristic functions

$$\varphi_1(t) = \frac{1}{1+it}, \qquad \varphi_2(t) = \frac{1}{1-it}$$

satisfy the condition (D).

L. Kubik has given in [2] two classes of couples (φ_1, φ_2) of characteristic functions for which a more general condition

$$\text{(K)} \qquad p\varphi_1(t)+q\varphi_2(t) = \varphi_1(t)\varphi_2(t), \qquad p+q = 1,\ p > 0,\ q > 0$$

holds. The first class is created by characteristic functions

$$\varphi_1(t) = \frac{a}{a+it}, \qquad \varphi_2(t) = \frac{pa}{pa-itq}, \qquad a > 0,$$

while the second one by characteristic functions

$$\varphi_1(t) = q+p\cos bt-ip\sin bt, \qquad \varphi_2(t) = p+q\cos bt+iq\sin bt, \qquad b\in \boldsymbol{R},$$

where $p+q = 1$, $p > 0$, $q > 0$.

In this note we give some new classes of couples (φ_1, φ_2) of characteristic functions satisfying (K) or (D).

EXAMPLE 1. It can easily be stated that (K) is satisfied by the functions

$$\varphi_1(t) = \frac{q}{1-pe^{it}}, \qquad \varphi_2(t) = e^{-it},$$

being the characteristic functions of random variables X_1 and X_2, respectively, which have the discrete distributions

$$P[X_1 = k] = qp^k, \quad k = 0, 1, 2\ldots, \quad P[X_2 = -1] = 1.$$

The random variable $X = X_1 + X_2$ (and $Y = pX_1 + qX_2$) has the characteristic function

$$\varphi(t) = \varphi_1(t)\varphi_2(t) = \frac{q}{(1 - pe^{it})e^{it}}$$

and the discrete distribution

$$P[X = -1] = q, \quad P[X = k] = qp^{k+1}, \quad k = 0, 1, 2, \ldots$$

EXAMPLE 2. Condition (K) is satisfied by functions

$$\varphi_1(t) = \frac{qe^{it}}{(1+q)e^{it} - 1}, \quad \varphi_2(t) = \frac{pe^{it}}{1 - qe^{it}}$$

being the characteristic functions of random variables X_1 and X_2 respectively of geometric distributions

$$P[X_1 = -k] = q\left(\frac{1}{1+q}\right)^k, \quad k = 0, 1, 2, \ldots,$$

$$P[X_2 = k] = pq^{k-1}, \quad k = 1, 2, \ldots$$

The random variable $X = X_1 + X_2$ has the characteristic function

$$\varphi(t) = \varphi_1(t)\varphi_2(t) = \frac{pqe^{2it}}{(1 - qe^{it})[(1+q)e^{it} - 1]}$$

and the discrete distribution

$$P[X = -k] = pq\left(\frac{1}{1+q}\right)^k, \quad k = 0, 1, 2, \ldots,$$

$$P[X = k] = pq^k, \quad k = 1, 2, \ldots$$

EXAMPLE 3. Condition (D) is satisfied by functions

$$\varphi_1(t) = \frac{1}{(1 - it)^2}, \quad \varphi_2(t) = \frac{1}{[(\sqrt{2} - 1) + it][(\sqrt{2} + 1) - it]},$$

being the characteristic functions of absolutely continuous distributions given by the density functions

$$f_1(x) = \begin{cases} 0, & \text{if } x \leqslant 0, \\ xe^{-x}, & \text{if } x > 0, \end{cases} \quad f_2(x) = \begin{cases} \dfrac{1}{2\sqrt{2}} e^{(\sqrt{2} - 1)x}, & \text{if } x \leqslant 0, \\ \dfrac{1}{2\sqrt{2}} e^{-(\sqrt{2} + 1)x}, & \text{if } x > 0, \end{cases}$$

respectively. The random variable $X = X_1 + X_2$ has the characteristic function

$$\varphi(t) = \varphi_1(t)\varphi_2(t) = \frac{1}{(1-it)^2[(\sqrt{2}-1)+it][(\sqrt{2}+1)-it]}$$

and the density function

$$f(x) = \begin{cases} \frac{1}{4\sqrt{2}} e^{(\sqrt{2}-1)}, & \text{if } x \leqslant 0, \\ \frac{1}{2} xe^{-x} + \frac{1}{4\sqrt{2}} e^{-(\sqrt{2}+1)}, & \text{if } x > 0. \end{cases}$$

EXAMPLE 4. Condition (K) is satisfied by functions

$$\varphi_1(t) = q + p\frac{a}{\sqrt{a^2+t^2}} - ip\frac{t}{\sqrt{a^2+t^2}}, \qquad a > 0,$$

$$\varphi_2(t) = p + q\frac{a}{\sqrt{a^2+t^2}} + iq\frac{t}{\sqrt{a^2+t^2}}$$

being, so-called, generalized characteristic functions (see, i.e., [4]) Moreover, one can verify that φ_1 and φ_2 are the generalized characteristic functions of the generalized distributions which can be written respectively as follows:

$$F_1(x) = F_{1d}(x) + F_{1c}(x), \qquad F_2(x) = F_{2d}(x) + F_{2c}(x),$$

where F_{id} and F_{ic}, $i = 1, 2$, are given by the formulas:

$$F_{1d}(x) = \begin{cases} 0, & \text{if } x \leqslant 0, \\ q, & \text{if } x > 0, \end{cases} \qquad F_{2d}(x) = \begin{cases} 0, & \text{if } x \leqslant 0, \\ p, & \text{if } x > 0, \end{cases}$$

$$F_{1c}(x) = \frac{pa}{\pi} \int_{-\infty}^{x} \left(\int_0^\infty \frac{\cos yu}{\sqrt{u^2+a^2}} du \right) dy + \frac{p}{\pi} \int_0^\infty \frac{\cos xu}{\sqrt{u^2+a^2}} du,$$

$$F_{2c}(x) = \frac{qa}{\pi} \int_{-\infty}^{x} \left(\int_0^\infty \frac{\cos yu}{\sqrt{u^2+a^2}} du \right) dy - \frac{q}{\pi} \int_0^\infty \frac{\cos xu}{\sqrt{u^2+a^2}} du.$$

Remark. It is not difficult to note that functions of the form

$$\varphi_1(t) = q + p\psi(t) - ip\chi(t), \qquad \varphi_2(t) = p + q\psi(t) + iq\chi(t),$$

where ψ and χ are real functions, satisfy (K) if and only if $\psi^2(t) + \chi^2(t) = 1$.

That form with $\psi(t) = \cos bt$ and $\chi(t) = \sin bt$ have the characteristic functions from the second class of Kubik and functions considered in Example 4.

References

[1] D. Dugué, *Arithmétique des lois de probabilités*, Mémor. Sci. Math. 137, Paris 1957.
[2] L. Kubik, *Sur un problème de M. D. Dugué*, Pràce Mat. (Comment. Math.) 13 (1969), p. 1–2.
[3] E. Lukacs, *Characteristic functions*, London 1960.
[4] Ju. P. Studnev, *The theory of unlimited divisible laws in the class B*, I, Teor. Verojatnost. i Mat. Statist. 2 (1970), p. 183–192 (in Russian).