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Polynomial spline functions and application to approximation in the space with mixed norm

1. Spline functions with free knots. Let $I = \langle a_1, b_1 \rangle \times \langle a_2, b_2 \rangle$, $-\infty < a_i, b_i < \infty$, be a given closed rectangle in R^2, and $\boldsymbol{m} = (m_1, m_2)$, $\boldsymbol{n} = (n_1, n_2)$ be vectors with non-negative coordinates. Let Δ_n be a partition of I:

$$\begin{aligned} &a_1 = t_0 < t_1 < t_2 < \ldots < t_{n_1} = b_1, \\ \Delta_n\colon\ &a_2 = u_0 < u_1 < u_2 < \ldots < u_{n_2} = b_2. \end{aligned}$$

DEFINITION 1. We denote by $S_{mn}(I)$ the class of spline functions on I satisfying two conditions:

1. Each function from $S_{mn}(I)$ is a polynomial of degree not exceeding m_1 with respect to x and m_2 with respect to y in each subrectangle $\langle t_i, t_{i+1} \rangle \times \times \langle u_k, u_{k+1} \rangle$ from Δ_n

2. Partial derivatives of $s(t, u) \in S_{mn}(I)$ up to $\boldsymbol{m} - \boldsymbol{1} = (m_1 - 1, m_2 - 1)$ are continuous.

Functions from the class $S_{mn}(I)$ are called *spline functions with free knots* if furthermore some knots in I are free from interpolation conditions – these knots are point of joint only.

Now we describe the method for constructing the spline function $s(x, y) \in S_{mn}(I)$ with free knots. It is convenient to construct in the first place the spline function from the class $S_{mm}(I_{jk})$, $I_{jk} \in I$ and then to extend the domain I_{jk} up to I or, what is the same, to increase the number of knots.

Let $I_{jk} = \langle x_j, x_{j+1} \rangle \times \langle y_k, y_{k+1} \rangle$ be given rectangle divided into $m_1 m_2$ parts:

(1) $$x_j = t_{j_0} < t_{j_1} < \ldots < t_{jm_1} = x_{j+1}, \quad y_k = u_{k_0} < u_{k_1} < \ldots < u_{km_2} = y_{k+1}.$$

In this notation the Taylor expansion for $s(x, y) \in S_{mn}(I_{jk})$ takes the following form:

$$(2)\quad s(x,y)=\sum_{i_1=0}^{m_1}\sum_{i_2=0}^{m_2} s^{(i_1,i_2)}(x_j,y_k)(x-x_j)^{i_1}(y-y_k)^{i_2}\frac{1}{i_1!\,i_2!}+$$
$$+\sum_{i_1=0}^{m_1}\frac{1}{i_1!(m_2+1)!}(x-x_j)^{i_1}\sum_{s=1}^{m_2-1}(y-u_{ks})_+^{m_2}[s^{(i_1,m_2)}(x_j,u_{ks})-s^{(i_1,m_2)}(x_j,u_{ks-1})]+$$
$$+\sum_{i_2=0}^{m_2}\frac{1}{i_2!(m_1+1)!}(y-y_k)^{i_2}\sum_{r=1}^{m_1-1}(x-t_{j_r})_+^{m_1}[s^{(m_1,i_2)}(t_{j_r},y_k)-s^{(m_1,i_2)}(t_{j_r-1},y_k)]+$$
$$+\sum_{r=1}^{m_1-1}\sum_{s=1}^{m_2-1}\frac{1}{(m_1+1)!(m_2+1)!}(x-t_{j_r})_+^{m_1}(y-u_{ks})_+^{m_2}[s^{(m_1,m_2)}(t_{j_r},u_{ks})-$$
$$-s^{(m_1,m_2)}(t_{j_r-1},u_{ks})+s^{(m_1,m_2)}(t_{j_r-1},u_{ks-1})-s^{(m_1,m_2)}(t_{j_r},u_{ks-1})].$$

Here we use the notation: $(x-t)_+^m=\max(0,(x-t))^m$. In the construction of expansion (2) we make use of two facts: the partial derivatives of the function $s(x,y)$

$$s^{(m_1,0)}(x,y),\qquad s^{(0,m_2)}(x,y),\qquad s^{(m_1,m_2)}(x,y)$$

must be constant on the subsets $\langle t_{j_r-1},t_{j_r}\rangle\times\langle u_{ks-1},u_{ks}\rangle$ with respect to the first, second or both variables, and the Stieltjes integrals in the rest can be computed without difficulty. For any function $f(x,y)$ with continuous partial derivatives up to $\boldsymbol{m}+\boldsymbol{1}=(m_1+1,m_2+1)$ on I_{jk} we have the following Taylor expansion

$$(3)\quad f(x,y)=\sum_{i_1=0}^{m_1}\sum_{i_2=0}^{m_2} f^{(i_1,i_2)}(x_j,y_k)(x-x_j)^{m_1}(y-y_k)^{i_2}\frac{1}{i_1!\,i_2!}+$$
$$+\sum_{i_1=0}^{m_1}\frac{1}{i_1!\,m_1!}\int_{y_k}^{y_{k+1}} f^{(i_1,m_2+1)}(x_j,u)(y-u)_+^{m_2}(x-x_j)^{i_1}\,du+$$
$$+\sum_{i_2=0}^{m_2}\frac{1}{i_1!\,m_2!}\int_{x_j}^{x_{j+1}} f^{(m_1+1,i_2)}(t,y_k)(x-t)_+^{m_1}(y-y_k)^{i_2}\,dt+$$
$$+\frac{1}{m_1!\,m_2!}\int_{x_j}^{x_{j+1}}\int_{y_k}^{y_{k+1}} f^{(m_1+1,m_2+1)}(t,u)\cdot(x-t)_+^{m_1}(y-y_k)_+^{m_2}.$$

For $s(x,y)$ and $f(x,y)$, given by (2) and (3), we require the following interpolation conditions:

$$(4)\quad \begin{aligned} &s^{(l_1,l_2)}(\xi,\eta)=f^{(l_1,l_2)}(\xi,\eta),\qquad l_1=0,\ldots,m_1-1,\ l_2=0,\ldots,m_2-1,\\ &\xi=x_j,x_{j+1},\qquad \eta=y_k,y_{k+1}.\end{aligned}$$

Let us observe that the interpolation conditions (4) with $\xi = x_j$, $\eta = y_k$ and notations

$$
\begin{aligned}
&\alpha_{i_1,s} = s^{(i_1,m_2)}(x_j, u_{ks}) - s^{(i_1,m_2)}(x_j, u_{ks-1}), && i_1 = 0, \ldots, m_1 - 1,\\
&\alpha_{i_1,0} = s^{(i_1,m_1)}(x_j, y_k) - f^{(i_1,m_2)}(x_j, y_k), && s = 1, \ldots, m_2 - 1,\\
&\beta_{ri_2} = s^{(m_1,i_2)}(t_{j_r}, y_k) - s^{(m_1,i_2)}(t_{j_r-1}, y_k), && i_2 = 0, \ldots, m_2 - 1,\\
&\beta_{0i_2} = s^{(m_1,i_2)}(x_j, y_k) - f^{(m_1,i_2)}(x_j, y_k), && r = 1, \ldots, m_1 - 1,\\
(5)\quad &\gamma_{rs} = s^{(m_1,m_2)}(t_{j_r}, u_{ks}) - s^{(m_1,m_2)}(t_{j_r-1}, u_{ks}) - s^{(m_1,m_2)}(t_{j_r}, u_{ks-1}) + \\
&\qquad + s^{(m_1,m_2)}(t_{j_r-1}, u_{ks-1}),\\
&\gamma_{00} = s^{(m_1,m_2)}(x_j, y_k) - f^{(m_1,m_2)}(x_j, y_k),\\
&\gamma_{r0} = s^{(m_1,m_2)}(t_{j_r}, y_k) - s^{(m_1,m_2)}(t_{j_r-1}, y_k),\\
&\gamma_{0s} = s^{(m_1,m_2)}(x_j, u_{ks}) - s^{(m_1,m_2)}(x_j, u_{ks-1}),
\end{aligned}
$$

lead us to the following identity:

$$
\begin{aligned}
(6)\quad & f(x, y) - s(x, y)\\
&\equiv \frac{1}{m_2!} \sum_{i_1=0}^{m_1} (x - x_j)^{i_1} \frac{1}{i_1!} \Big[\int_{y_k}^{y_{k+1}} f^{(i_1,m_2+1)}(x_j, u)(y-u)_+^{m_2}\, du - \sum_{s=0}^{m_2-1} \alpha_{i_1,s}(y - u_{ks})_+^{m_2} \Big] +\\
&+ \frac{1}{m_1!} \sum_{i_2=0}^{m_2} (y - y_k)^{i_2} \frac{1}{i_2!} \Big[\int_{x_j}^{x_{j+1}} f^{(m_1+1,i_2)}(t, y_k)(x-t)_+^{m_1}\, dt - \sum_{r=0}^{m_1-1} \beta_{ri_2}(x - t_{j_r})_+^{m_1} \Big] +\\
&+ \frac{1}{m_1!\, m_2!} \Big[\int_{x_j}^{x_{j+1}} \int_{y_k}^{y_{k+1}} f^{(m_1+1,m_2+1)}(t, u)(x-t)_+^{m_1}(y-u)_+^{m_2}\, dt du -\\
&- \sum_{r=0}^{m_1-1} \sum_{s=0}^{m_2-1} \gamma_{rs}(x - t_{j_r})_+^{m_1}(y - u_{ks})_+^{m_2} \Big] + \int_{x_j}^{x_{j+1}} f^{(m_1+1,m_2)}(t, y_k)(x-t)_+^{m_1}(y - y_k)^{m_2}\, dt\\
&+ \int_{y_k}^{y_{k+1}} f^{(m_1,m_2+1)}(x_j, u)(x - x_j)_+^{m_1}(y-u)_+^{m_2}\, du.
\end{aligned}
$$

To calculate the spline function $s(x, y)$ interpolating $f(x, y)$ in the sense of (4), we must compute quantities (5) in the following manner: applying to (6) the interpolation conditions (4) with

$$(\xi, \eta) = (x_j, y_{k+1}),\ (x_{j+1}, y_k),\ (x_{j+1}, y_{k+1}),$$

we get a system of $3m_1m_2$ linear equations. The matrix of this system of equations can be divided into sqare block-matrix with non-zero determinants of Vandermonde-type, so the matrix is non-singular. Similarly as in [3], computing a number of determinants we get the parameters (5). Let $w_r(t)$ and $w_s(u)$ be functions defined as follows:

$$w_{j_r}(t) = \frac{\omega_j(t)}{\omega_j'(t_{j_r})(t-t_{j_r})}, \quad \omega_j(t) = \prod_{r=0}^{m_1} (t-t_{j_r}),$$

$$w_{ks}(u) = \frac{\omega_k(u)}{\omega_k'(u_{ks})(u-u_{ks})}, \quad \omega_k(u) = \prod_{s=0}^{m_2} (u-u_{ks}).$$

Now parameters (5) take the following form:

$$\alpha_{i_1 s} = \int_{y_k}^{y_{k+1}} f^{(i_1,m_2+1)}(x_j, u)\, w_{ks}(u)\, du,$$

$$\beta_{ri_2} = \int_{x_j}^{x_{j+1}} f^{(m_1+1,i_2)}(t, y_k)\, w_{jr}(t)\, dt,$$

$$s, i_2 = 0, \ldots, m_2-1, \quad r, i_1 = 0, \ldots, m_1-1,$$

$$\gamma_{rs} = \int_{x_j}^{x_{j+1}} \int_{y_k}^{y_{k+1}} f^{(m_1+1,m_2+1)}(t, u)\, w_{jr}(t)\, w_{ks}(u)\, dt du,$$

$$\gamma_{0s} = \int_{x_j}^{x_{j+1}} \int_{y_k}^{y_{k+1}} f^{(m_1+1,m_2+1)}(t, u)\, w_{j0}(t)\, w_{ks}(u)\, dt du + \tag{7}$$
$$+ \int_{y_k}^{y_{k+1}} f^{(m_1+1,m_2+1)}(x_j, u)\, w_{ks}(u)\, du,$$

$$\gamma_{r0} = \int_{x_j}^{x_{j+1}} \int_{y_k}^{y_{k+1}} f^{(m_1+1,m_2+1)}(t, u)\, w_{jr}(t)\, w_{k0}(u)\, dt du +$$
$$+ \int_{x_j}^{x_{j+1}} f^{(m_1+1,m_2+1)}(t, y_k)\, w_{jr}(t)\, dt,$$

$$r = 1, \ldots, m_1-1, \quad s = 1, \ldots, m_2-1,$$

$$\gamma_{00} = \int_{x_j}^{x_{j+1}} \int_{y_k}^{y_{k+1}} f^{(m_1+1,m_2+1)}(t, u)\, w_{j0}(t)\, w_{k0}(u)\, dt du +$$
$$+ \int_{x_j}^{x_{j+1}} f^{(m_1+1,m_2)}(t, y_k)\, w_{j0}(t)\, dt + \int_{y_k}^{y_{k+1}} f^{(m_1,m_2+1)}(x_j, u)\, w_{k0}(u)\, du.$$

The spline function given by above construction belongs to the class $S_{mm}(I_{jk})$, the number of knots and the degree of polynomial are defined by the same vector $\boldsymbol{m}$. Now we will extend the construction of the function from the disscused class to a class defined on the whole domain I:

$$s(x, y)|_{(x,y)\in I_{j,k}} = S(x, y), \quad s(x, y) \in S_{mm}(I_{jk}),$$

and then we shall prove that such spline functions $S(x, y)$ are elements of the class $S_{mn}(I)$, $\boldsymbol{n} = \boldsymbol{m} \cdot \boldsymbol{s}$, $\boldsymbol{s} = (s_1, s_2)$, s_i – integers.

Let us make a partition of the domain I

(8) $a_1 = x_0 < x_1 < \ldots < x_{s_1} = b_1, \quad a_2 = y_0 < y_1 < \ldots < y_{s_2} = b_2,$

and for all j, k:

$$x_j = t_{j0} < t_{j1} < \ldots < t_{jm_1} = x_{j+1}, \quad y_k = u_{k_0} < u_{k_1} < \ldots < u_{km_2} = y_{k+1}.$$

We only need to prove that the spline function $S(x, y)$ is continuously differentiable on I. From the construction described above immediately follows that the derivatives $S^{(l_1,l_2)}(x, y)$ for $(x, y) \in (x_j, x_{j+1}) \times (y_k, y_{k+1})$, $l_i = 0, \ldots, m_i - 1$, are continuous functions. We shall prove continuity of this partial derivatives on the intervals which are bounds of the subdomains $I_{j,k}$. Let us consider two neighbouring subdomains $I_{j,k}$ and $I_{j+1,k}$. We ought to prove

(9) $$S^{(l_1,l_2)}(x, y)\Big|_{\substack{(x,y)\in I_{j,k} \\ x=x_{j+1}}} = S^{(l_1,l_2)}(x, y)\Big|_{\substack{(x,y)\in I_{j+1,k} \\ x=x_{j+1}}}, \quad l_i = 0, \ldots, m_i - 1,\ i = 1, 2.$$

The partial derivatives from (9) take on $I_{j,k}$ and $I_{j+1,k}$ for $x = x_{j+1}$ the following forms:

(10) $$\begin{aligned} &S^{(l_1,l_2)}(x, y)\Big|_{\substack{(x,y)\in I_{j,k} \\ x=x_{j+1}}} \\ &= \sum_{i_1=l_1}^{m_1} \sum_{i_2=l_2}^{m_2} S^{(i_1,i_2)}(x_j, y_k)(x_{j+1}-x_j)^{i_1-l_1}(y-y_k)^{i_2-l_2}\frac{1}{(i_1-l_1)!(i_2-l_2)!} + \\ &+ \sum_{i_1=l_1}^{m_1} (x_{j+1}-x_j)^{i_1-l_1}\frac{1}{(i_1-l_1)!}\sum_{s=0}^{m_2}(y-u_{ks})_+^{m_2-l_2}\int_{y_k}^{y_{+1}} f^{(i_1,m_2+1)}(x_j, u)\,w_{ks}(u)\,du + \\ &+ \sum_{i_2=l_2}^{m_2} (y-y_k)^{i_2-l_2}\frac{1}{(i_2-l_2)!}\sum_{r=0}^{m_1-1}(x_{j+1}-t_{j_r})_+^{m_1-l_1}\int_{x_j}^{x_{j+1}} f^{(m_1+1,i_2)}(t, y_k)\,w_{j_r}(t)\,dt + \\ &+ \sum_{r=0}^{m_1-1}\sum_{s=0}^{m_2-1}(x_{j+1}-x_{j_r})_+^{m_1-l_1}(y-u_{ks})_+^{m_2-l_2}\frac{1}{(m_1-l_1)!(m_2-l_2)!}\times \\ &\qquad\times \int_{x_j}^{x_{j+1}}\int_{y_k}^{y_{k+1}} f^{(m_1+1,m_2+1)}(t, u)\,w_{ks}(u)\cdot w_{j_r}(t)\,du\,dt, \end{aligned}$$

(11) $$\begin{aligned} &S^{(l_1,l_2)}(x, y)\Big|_{\substack{(x,y)\in I_{j+1,k} \\ x=x_{j+1}}} \\ &= \sum_{i_2=l_2}^{m_2} S^{(l_1,i_2)}(x_{j+1}, y_k)(y-y_k)^{i_2-l_2}\frac{1}{(i_2-l_2)!} + \\ &+ \sum_{s=0}^{m_2-1}(y-u_{ks})_+^{m_2-l_2}\frac{1}{(m_2-l_2)!}\int_{y_k}^{y_{k+1}} f^{(l_1,m_2+1)}(x_{j+1}, u)\,w_{ks}(u)\,du. \end{aligned}$$

Now let us consider the functions $f^{(0,l_2)}(x, y_k)$, $l_i = 0, \ldots, m_i - 1$, and $\int_{y_k}^{y_{k+1}} f^{(0,m_2+1)}(x, u)\,w_{ks}(u)\,du$, and one dimensional spline functions which interpolate them. We denote by $P_{l_2}(x)$ the spline function of degree m_1 with free knots and with the interpolation conditions

$$f^{(l_1,l_2)}(x_j, y_k) = P_{l_2}^{(l_1)}(x_j), \quad f^{(l_1,l_2)}(x_{j+1}, y_k) = P_{l_2}^{(l_1)}(x_{j+1}), \quad l_1 = 0, \ldots, m_1-1.$$

It was shown in [3] that $P_{l_2}(x)$ takes the following form:

$$P_{l_2}(x) = \sum_{i_1=0}^{m_1} f^{(i_1,l_2)}(x_j, y_k)(x-x_j)^{i_1}\frac{1}{i_1!} + \\ +\frac{1}{m_1!}\sum_{r=0}^{m_1-1}(x-t_{j_r})_+^{m_1}\int_{x_j}^{x_{j+1}} f^{(m_1+1,l_2)}(t, y_k)w_{j_r}(t)\,dt.$$

After l_1-times, $l_1 = 0, \ldots, m_1-1$, differentiation of $P_{l_2}(x)$ and application of the interpolation conditions in $x = x_{j+1}$ we get the equalities

$$f^{(l_1,l_2)}(x_{j+1}, y_k) = \sum_{i_1=l_1}^{m_1} f^{(i_1,l_2)}(x_j, y_k)(x_{j+1}-x_j)^{i_1-l_1}\frac{1}{(i_1-l_1)!} + \\ +\sum_{r=0}^{m_1-1}(x_{j+1}-t_{j_r})_+^{m_1-l_1}\frac{1}{(m_1-l_1)!}\int_{x_j}^{x_{j+1}} f^{(m_1+1,l_2)}(t, y_k)w_{j_r}(t)\,dt. \tag{12}$$

In the same way as explained above for the function

$$\int_{y_k}^{y_{k+1}} f^{(0,m_2+1)}(x, t)\,w_{ks}(u)\,du,$$

one can construct the spline function $Q(x)$ of degree m_1 with free knots and with the interpolation conditions

$$\int_{y_k}^{y_{k+1}} f^{(l_1,m_2+1)}(x_j, u)\,w_{ks}(u)\,du = Q^{(l_1)}(x_j),$$

$$\int_{y_k}^{y_{k+1}} f^{(l_1,m_2+1)}(x_{j+1}, u)\,w_{ks}(u)\,du = Q^{(l_1)}(x_{j+1}), \quad l_1 = 0, \ldots, m_1-1.$$

After differentiation in $x = x_{j+1}$ we get:

$$\int_{y_k}^{y_{k+1}} f^{(l_1,m_2+1)}(x_{j+1}, u)\,w_{ks}(u)\,du \\ = \sum_{i_1=l_1}^{m_1}\int_{y_k}^{y_{k+1}} f^{(i_1,m_2+1)}(x_j, u)\,w_{ks}(u)\,du\,(x-x_j)^{i_1-l_1}\frac{1}{(i_1-l_1)!} + \\ +\sum_{r=0}^{m_1-1}(x-t_{j_r})_+^{m_1-l_1}\frac{1}{(m_1-l_1)!}\int_{x_j}^{x_{j+1}}\int_{y_k}^{y_{k+1}} f^{(m_1+1,m_2+1)}(t, u)\,w_{j_r}(t)\,w_{ks}(u)\,du\,dt. \tag{13}$$

Equations (12) and (13) applied in (11) lead us immediately to equations (9).

The same reasoning on subdomains $I_{j,k}$ and $I_{j,k+1}$ leads us to the following conclusion:

$$S^{(l_1,l_2)}(x, y)\Big|_{\substack{(x,y)\in I_{jk}\\ y=y_{k+1}}} = S^{(l_1,l_2)}(x, y)\Big|_{\substack{(x,y)\in I_{j,k+1}\\ y=y_{k+1}}}, \quad l_i = 0, \ldots, m_i-1, \; i = 1, 2, \tag{14}$$

which complete the proof of the continuity of the partial derivatives on I because of free choice of j and k $(0 < j < n_1 - 1,\ 0 < k < n_2 - 1)$.

2. Approximation properties. From (7) and (6) a useful identity can be obtained:

$$
\begin{aligned}
&f(x, y) - S(x, y)\\
&\equiv \frac{1}{m_2!}\sum_{i_1=0}^{m_1}(x-x_j)^{i_1}\frac{1}{i_1!}\int_{y_k}^{y_{k+1}} f^{(i_1, m_2+1)}(x_j, u)\Big[(y-u)_+^{m_2} - \\
&\qquad - \sum_{s=0}^{m_2-1} w_{ks}(u)(y-u_{ks})_+^{m_2}\Big]\,du + \\
&+ \frac{1}{m_1!}\sum_{i_2=0}^{m_2}(y-y_k)^{i_2}\frac{1}{i_2!}\int_{x_j}^{x_{j+1}} f^{(m_1+1, i_2)}(t, y_k)\Big[(x-t)_+^{m_1} - \\
&\qquad - \sum_{r=0}^{m_1-1} w_{j_r}(t)(x-t_{j_r})_+^{m_1}\Big]\,dt + \\
&+ \frac{1}{m_1!\,m_2!}\int_{x_j}^{x_{j+1}}\int_{y_k}^{y_{k+1}} f^{(m_1+1, m_2+1)}(t, u)\Big[(x-t)_+^{m_1}(y-u)_+^{m_2} - \\
&- \sum_{r=0}^{m_1-1}\sum_{s=0}^{m_2-1} w_{j_r}(t)(x-t_{j_r})_+^{m_1} w_{ks}(u)(y-u_{ks})_+^{m_2}\Big]\,dt\,du, \qquad (x, y)\in I_{j,k}.
\end{aligned}
\tag{15}
$$

Now we start with estimation of the difference (15) for $(x, y)\in I_{j,k}$. With the help of the method described in [3], Lemma 2, the following inequalities can be obtained:

$$
\begin{aligned}
&\Big|(x-t)_+^{m_1} - \sum_{r=0}^{m_1-1} w_{j_r}(t)(x-t_{j_r})_+^{m_1}\Big| \leqslant C_{m_1}(x_{j+1}-x_j)^{m_1},\\
&\Big|(y-u)_+^{m_2} - \sum_{s=0}^{m_2-1} w_{ks}(u)(y-u_{ks})_+^{m_2}\Big| \leqslant C_{m_2}(y_{k+1}-y_k)^{m_2},
\end{aligned}
\tag{16}
$$

fulfiled for $x, t\in\langle x_j, x_{j+1}\rangle$, $y, u\in\langle y_k, y_{k+1}\rangle$. The constants C depend on m_1 or m_2 only. Note that the following identity holds: $RT - ab = (R-a)(T-b) + a(T-b) + b(R-a)$. Replacing in it

$$
R = \sum_{r=0}^{m_1-1} w_{j_r}(t)(x-t_{j_r})_+^{m_1}, \qquad T = \sum_{s=0}^{m_2-1} w_{ks}(u)(y-u_{ks})_+^{m_2},
$$

$a = (x-t)_+^{m_1}$, $b = (y-u)_+^{m_2}$, the equation (15) can be transformed into such a form, which together with (16) immediately give us:

$$
\begin{aligned}
&|f(x, y) - s(x, y)|\\
&\leqslant \sum_{i_2=0}^{m_2-1}\frac{1}{i_2!}(y_{k+1}-y_k)^{i_2}(x_{j+1}-x_j)^{m_1} C_{m_2}\Big|\int_{x_j}^{x_{j+1}} f^{(m_1+1, i_2)}(t, y_k)\,dt\Big| + \\
&+ \sum_{i_1=0}^{m_1-1}\frac{1}{i_1!}(x_{j+1}-x_j)^{i_1}(y_{k+1}-y_k)^{m_2} C_{m_2}\Big|\int_{y_k}^{y_{k+1}} f^{(i_1, m_2+1)}(x_j, u)\,du\Big| + \\
&+ C_{m_1 m_2}(x_{j+1}-x_j)^{m_1}(y_{k+1}-y_k)^{m_2}\Big|\int_{x_j}^{x_{j+1}}\int_{y_k}^{y_{k+1}} f^{(m_1+1, m_2+1)}(t, u)\,dt\,du\Big|,\\
&\qquad (x, y)\in I_{j,k}.
\end{aligned}
\tag{17}
$$

For the next estimation of (15) furthermore we need

LEMMA 1. *For any function $f(x, y) \in C^{(\alpha,\beta)}(\langle a, b\rangle \times J)$, $\langle a, b\rangle \subset \langle c, d\rangle$, α, β — positive integers, the following inequality holds:*

$$(18)\quad |f^{(p,\beta)}(a, y)| \leqslant C_p!(b-a)^{-p}\left[\int_c^d |f^{(0,\beta)}(x, y)|\,dx + \frac{1}{(\alpha-1)!}(b-a)^{\alpha-1}\int_a^b |f^{(\alpha,\beta)}(x, y)|\,dx\right],$$

where $0 \leqslant p \leqslant \alpha-1$.

Proof. Let us denote by $\Delta_1, \Delta_2, \ldots, \Delta_{2\alpha}$ a uniform partition of the interval $\langle a, b\rangle$ on 2α parts and choose a set of α points $\{x_k\}_{k=1}^{\alpha}$, $x_k \in \Delta_{2k}$. From the Taylor expansions for $f^{(0,\beta)}(x, y)$:

$$f^{(0,\beta)}(x, y) = \sum_{p=0}^{\alpha-1} f^{(p,\beta)}(a, y)(x-a)^p\frac{1}{p!} + \frac{1}{(\alpha-1)!}\int_a^x f^{(\alpha,\beta)}(t, y)(x-t)^{\alpha-1}\,dt,$$

we arrive at a system of linear equations

$$(19)\quad \sum_{p=0}^{\alpha-1} f^{(p,\beta)}(a, y)(x_k-a)^p\frac{1}{p!} = f^{(0,\beta)}(x_k, y) - R(x_k, y), \qquad k = 1, \ldots, \alpha,$$

where

$$R(x, y) = \frac{1}{(\alpha-1)!}\int_a^x f^{(\alpha,\beta)}(t, y)(x-t)^{\alpha-1}\,dt.$$

Let us consider now the determinant of this system:

$$w = \begin{vmatrix} 1 & (x_1-a) & \ldots & (x_1-a)^{\alpha-1}\dfrac{1}{(\alpha-1)!} \\ 1 & (x_2-a) & \ldots & (x_2-a)^{\alpha-1}\dfrac{1}{(\alpha-1)!} \\ \vdots & \vdots & & \vdots \\ 1 & (x_\alpha-a) & \ldots & (x_\alpha-a)^{\alpha-1}\dfrac{1}{(\alpha-1)!} \end{vmatrix}$$

$$= \frac{1}{1!\,2!\ldots(\alpha-1)!}\prod_{\substack{k=1\\k<l}}^{\alpha}(x_k-x_l)$$

$$\geqslant \frac{1}{1!\,2!\ldots(\alpha-1)!}\prod_{\substack{k=1\\k<l}}^{\alpha}\Delta_{2k} = \frac{1}{1!\,2!\ldots(\alpha-1)!}\left[\frac{b-a}{2\alpha}\right]^{\frac{\alpha(\alpha-1)}{2}}$$

and the determinant:

$$w_p = \prod_{k\neq p}^{\alpha}\frac{1}{k!}\begin{vmatrix} 1 & (x_1-a_1) & \dots & (x_1-a)^{p-1}, & f^{(0,\beta)}(x_1,y)-R(x_1,y), & \dots, & (x_1-a)^{\alpha-1} \\ 1 & (x_2-a) & \dots & (x_2-a)^{p-1}, & f^{(0,\beta)}(x_2,y)-R(x_2,y), & \dots, & (x_2-a)^{\alpha-1} \\ \vdots & \vdots & & \vdots & \vdots & \vdots & \\ 1 & (x_\alpha-a) & \dots & (x_\alpha-a)^{p-1}, & f^{(0,\beta)}(x_\alpha,y)-R(x_\alpha,y), & \dots, & (x_\alpha-a)^{\alpha-1} \end{vmatrix}$$

$$\leqslant \prod_{k\neq p}^{\alpha}\frac{1}{k!}\sum_{i=1}^{\alpha}[|f^{(0,\beta)}(x_i,y)|+|R(x_i,y)|]\,C_\alpha\left[\frac{b-a}{2\alpha}\right]^{\frac{\alpha(\alpha-1)}{2}-p}.$$

Because of $f^{(p,\beta)}(a,y) = w_p/w$ we get for $f^{(p,\beta)}(a,y)$ the following estimation:

$$|f^{(p,\beta)}(a,y)| \leqslant C_\alpha p!(b-a)^{-p}\sum_{i=1}^{\alpha}[|f^{(0,\beta)}(x_i,y)|+|R(x_i,y)|], \qquad p = 0,\dots,\alpha-1.$$

The left-hand side of the above inequalities does not depend on x_k, $k = 1,\dots,\alpha$, so that

$$|f^{(p,\beta)}(a,y)| \leqslant Cp!(b-a)^{-p}\left[\frac{\alpha}{d-c}\int_c^d |f^{(0,\beta)}(x,y)|\,dx + \frac{1}{(\alpha-1)!}(b-a)^{\alpha-1}\int_a^b |f^{(\alpha,\beta)}(x,y)|\,dx\right].$$

The method used in the proof of Lemma 1 was given in paper [2], p. 162.

With the help of Lemma 1 ($\alpha = m_1+1$, $\beta = m_2+1$, $J = \langle a_2, b_2\rangle$, $\langle a, b\rangle = \langle x_j, x_{j+1}\rangle$, $\langle c, d\rangle = \langle a_1, b_1\rangle$) and because of (17) we have for the difference $|f(x,y)-S(x,y)|$ the following inequality:

$$\begin{aligned}|f(x,y)-S(x,y)| \leqslant C_{m_1}\Delta x_j^{m_1}\sum_{s=0}^{m_2-1}\int_{x_2}^{x_{j+1}}\Big[\int_{a_2}^{b_2}|f^{(m_1+1,0)}(t,u)|\,du &+\\ +\frac{1}{m_1!}\Delta y_k^{m_2}\int_{y_k}^{y_{k+1}}|f^{(m_1+1,m_2+1)}(t,u)|\,du\Big]\,dt &+\\ +C_{m_2}\Delta y_k^{m_2}\sum_{r=0}^{m_1-1}\int_{y_k}^{y_{k+1}}\Big[\int_{a_1}^{b_1}|f^{(0,m_2+1)}(t,u)|\,dt &+\\ +\frac{1}{m_2!}\Delta x_j^{m_1}\int_{x_j}^{x_{j+1}}|f^{(m_1+1,m_2+1)}(t,u)|\,dt\Big]\,du &+\\ +C_{m_1m_2}\Delta x_j^{m_1}\Delta y_k^{m_2}\int_{x_j}^{x_{j+1}}\int_{y_k}^{y_{k+1}}|f^{(m_1+1,m_2+1)}(t,u)|\,dt\,du&,\end{aligned}$$

$$\Delta x_j = x_{j+1}-x_j,\ \Delta y_k = y_{k+1}-y_k,\ (x,y)\in I_{j,k},$$

and finally:

$$(20)\quad |f(x, y)-S(x, y)|$$

$$\leqslant C_{m_1} \Delta x_j^{m_1} \int_{x_j}^{x_{j+1}} \int_{a_2}^{b_2} |f^{(m_1+1,0)}(t, u)|\, dt du + C_{m_2} \Delta y_k^{m_2} \int_{y_k}^{y_{k+1}} \int_{a_1}^{a_2} |f^{(0,m_2+1)}(t, u)|\, dt du +$$

$$+ C_{m_1 m_2} \Delta x_j^{m_1} \Delta y_k^{m_2} \int_{x_j}^{x_{j+1}} \int_{y_k}^{y_{k+1}} |f^{(m_1+1,m_2+1)}(t, u)|\, dt du,$$

$$(x, y) \in I_{j,k}.$$

The constants depend on m_1 and m_2 only.

3. Existence of optimal mesh. Let us denote:

$$F(x_i, x_{i+1}, y_k, y_{k+1}) = \Delta x_i^{\alpha} \Delta y_k^{\beta} \int_{x_i}^{x_{i+1}} \int_{y_k}^{y_{k+1}} |f_1(x, y)|\, dx dy +$$

$$+ \Delta x_i^{\alpha} \int_{x_i}^{x_{i+1}} \int_{c}^{d} |f_2(x, y)|\, dx dy + \Delta y_k^{\beta} \int_{y_k}^{y_{k+1}} \int_{a}^{b} |f_3(x, y)|\, dx dy,$$

$$i = 0, \ldots, n_1-1, \; k = 0, \ldots, n_2-1.$$

LEMMA 2. *Let f_i, $i = 1, 2, 3$, be given integrable functions on $M = \langle a, b\rangle \times \langle c, d\rangle$. For arbitrary α, β and fixed $n_1 \geqslant n_2 > 0$ there exists such a partition $\{x_i^*, y_k^*\}_{i=0}^{n_1}{}_{k=0}^{n_2}$ of the set M that the following equation is satisfied*

$$\bigwedge_{0 \leqslant i < n_1} \bigvee_{0 \leqslant k_i < n_2} \inf_{\{x_i, y_k\}_{i=0}^{n_1}{}_{k=0}^{n_2}} \max_{i,k} F(x_i, x_{i+1}, y_k, y_{k+1})$$

$$= F(x_i^*, x_{i+1}^*, y_k^*, y_{k+1}^*) \equiv A,$$

and the following inequality holds:

$$(21)\quad F(x_i^*, x_{i+1}^*, y_k^*, y_{k+1}^*) \leqslant \min(n_1, n_2)\big[n_1^{-(\alpha+1)} n_2^{-(\beta+1)} \iint_M |f_1(x, y)|\, dx dy +$$

$$+ n_1^{-(\alpha+1)} \iint_M |f_2(x, y)|\, dx dy + n_2^{-(\beta+1)} \iint_M |f_3(x, y)|\, dx dy\big].$$

Proof. The proof consists of two parts. In the first part one should prove the existence of an optimal mesh $\{x_i^*, y_k^*\}_{i=0}^{n_1}{}_{i=0}^{n_2}$ and the existence of at least n_1 subsets $\langle x_i^*, x_{i+1}^*\rangle \times \langle y_k^*, y_{k+1}^*\rangle$, $i = 0, \ldots, n_1-1$, $k = 0, \ldots, n_2-1$ with the property $F(x_i^*, x_{i+1}^*, y_{k_i}^*, y_{k_i+1}^*) \equiv A$. The function $F(x_i, x_{i+1}, y_k, y_{k+1})$ is continuous with respect to $x_i, x_{i+1}, y_k, y_{k+1}$, $x_i \in \langle a, b\rangle$, $i = 0, \ldots, n$, $y_k \in \langle c, d\rangle$, $k = 0, \ldots, n_2$, so that the infimum is attained on the compact domain in $R^{n_1+n_2-2}$

$$(22)\quad a = x_0 \leqslant x_1 \leqslant x_2 \leqslant \ldots \leqslant x_{n_1} = b, \qquad c = y_0 \leqslant y_1 \leqslant y_2 \leqslant \ldots \leqslant y_{n_2} = d.$$

Let us denote

$$F(x_i^*, x_{i+1}^*, y_k^*, y_{k+1}^*) = A_{ik}^*,$$

where $\{x_i^*, y_k^*\}_{i=0}^{n_1} {}_{k=0}^{n_2}$ is an optimal mesh on M. Suppose that there exists a number i_0, such that for all k

$$A_{i_0,k} < A.$$

Note, that if $x_i = x_{i+1}$ for some i the following inequalities hold

$$\bigvee_{0 \leqslant j < n_1} \bigwedge_{0 \leqslant k < n_2} F(x_i^*, x_{i+1}^*, y_k^*, y_{k+1}^*) < F(x_j^*, x_{j+1}^*, y_k^*, y_{k+1}^*).$$

Such index j exists because of $a = x_0 \neq x_{n_1} = b$.

The function $F(x_i, x_{i+1}, y_k, y_{k+1})$ is non-decreasing with respect to variables x_i, y_k and non-increasing with respect to x_{i+1}, y_{k+1}. Because of this property we can choose $x_{i_0} < x_{i_0}^*$ and $x_{i_{0+1}} > x_{i_{0+1}}^*$, such that

$$F(x_{i_0}, x_{i_0+1}, y_k^*, y_{k+1}^*) = A_{i_0k} > A_{i_0k}^{*'}, \quad A_{i_0k} < A.$$

Also, for such well-chosen $x_{i_0-1} < x_{i_0-1}^*$, $x_{i_0+2} > x_{i_0+2}^*$ one can obtain

$$F(x_{i_0-1}, x_{i_0}, y_k^*, y_{k+1}^*) = A_{i_0k-1,k} < A_{i_0-1,k}^*,$$
$$F(x_{i_0+2}, x_{i_0+1}, y_k^*, y_{k+1}^*) = A_{i0+1,k} < A_{i_0+1,k}^*,$$

or equivalently: for all k: $A_{jk} < A$, $j = i_0-1, i_0, i_0+1$. The same process as described above for all indices i leads us to a new mesh $\{x_i, y_k^*\}_{i=0}^{n_1} {}_{k=0}^{n_2}$ which has a property inconsistent with the assumption that the mesh $\{x_i^*, y_k^*\}$ is optimal. Suppose now, there exists an index k_0, such that, for all i, $A_{ik_0} < A$. An analogous argument leads us to a contradiction with the assumption that the mesh $\{x_i^*, y_k^*\}_{i=0}^{n_1} {}_{k=0}^{n_2}$ is optimal.

The final corollary is: the value

$$A = \inf_{\{x_i, y_k\}_{i=0}^{n_1} {}_{k=0}^{n_2}} \max_{k,i} F(x_i, x_{i+1}, y_k, y_{k+1})$$

is attained on at least n_1 subsets from the optimal mesh $\{x_i^*, y_k^*\}_{i=0}^{n_1} {}_{k=0}^{n_2}$ and for every i ($i = 0, \ldots, n_1$) there exists an index k_i that the following equations hold:

1° $F(x_i^*, x_{i+1}^*, y_{k_i}^*, y_{k_i+1}^*) = A$,

2° $\{k_1, \ldots, k_{n_1}\} = \{1, 2, \ldots, n_2\}$.

Note, that the solution of the problem

$$\inf_{\{x_i, y_k\}_{i=0}^{n_1} {}_{k=0}^{n_2}} \max_{i,k} F(x_i, x_{i+1}, y_k, y_{k+1})$$

cannot be attained on the bound of the compact domain (22), it means that in optimal mesh $\{x_i^*, y_k^*\}$ all the knots differ from each other. The assumption that $x_i^* = x_{i+1}^*$ for some indices i is equivalent to the following

$$\bigwedge_{0 \leq k < n_2} F(x_i^*, x_{i+1}^*, y_k^*, y_{k+1}^*) = \Delta y_k^\beta \int_{y_k}^{y_{k+1}} \int_a^b |f_3|\, dxdy,$$

and

$$\bigvee_{0 \leq j < n_1} \bigwedge_{0 \leq k < n_2} F(x_i^*, x_{i+1}^*, y_k^*, y_{k+1}^*) < F(x_j^*, x_{j+1}^*, y_k^*, y_{k+1}^*),$$

but it means that the considering mesh is not optimal.

In the second part of our proof we show that inequality (21) is satisfied.

Let us note, if $\sum_{i=1}^{n_1} \Delta x_i = c < \infty$, then $\sum_{i=1}^{n_1} \Delta x_i^{-\alpha} \geq n_1^{\alpha+1} \cdot c^{-\alpha}$. This inequality and the definition of the value A lead us to the inequality:

$$\begin{aligned} A \cdot \sum_i \Delta x_i^{-\alpha} \Delta y_{k_i}^{-\beta} & \\ \leq n_1^{-(\alpha+1)} n_2^{-(\beta+1)} & \sum_i \Delta x_i^{-\alpha} \sum_k \Delta y_k^{-\beta} \sum_i \int_{x_i}^{x_{i+1}} \int_{y_{k_i}}^{y_{k_i+1}} |f_1(x, y)|\, dxdy + \\ & + n_2^{-(\beta+1)} \sum_k \Delta y_k^{-\beta} \sum_i \Delta x_i^{-\alpha} \int_{y_k}^{y_{k+1}} \int_a^b |f_3(x, y)|\, dxdy + \\ & + n_1^{-(\alpha+1)} \sum_i \Delta x_i^{-\alpha} \sum_i \Delta y_{k_i}^{-\beta} \int_{x_i}^{x_{i+1}} \int_c^d |f_2(x, y)| \end{aligned}$$

and

$$A \cdot \frac{\sum_i \Delta x_i^{-\alpha} \Delta y_{k_i}^{-\beta}}{\sum_i \Delta x_i^{-\alpha} \sum_k \Delta y_k^{-\beta}} \leq C_{1,2} n_1^{-(\alpha+1)} n_2^{-(\beta+1)} \iint_M |f_1(x, y)|\, dxdy + \tag{23}$$

$$+ C_1 n_1^{-(\alpha+1)} \iint_M |f_2(x, y)|\, dxdy + C_2 n_2^{-(\beta+1)} \iint_M |f_3(x, y)|\, dxdy.$$

Left-hand side of (23) attains the minimum for $\Delta y_k = 1/n_2$, where $k = 0, \ldots, n_2 - 1$, and for some choice of Δx_i, $i = 0, \ldots, n_1 - 1$, satisfying the following equation:

$$\sum_{j \in D_k} \Delta x_j^{-\alpha} = \frac{1}{n_2} \sum_i \Delta x_i^{-\alpha}, \quad k = 0, \ldots, n_2 - 1.$$

The sets D_k contain the values of index j from the set $\{1, 2, \ldots, n_1\}$ for which $F(x_j^*, x_{j+1}^*, y_k^*, y_{k+1}^*) = A$. The inequality (23) immediately gives (21).

4. Order of approximation.

DEFINITION 2. Let $k = (k_1, k_2)$, $p = (p_1, p_2)$ be vectors with integer components, and $\Omega \subset R^2$ be a given domain. We denote by $S^{kp}W(\Omega)$ the set of all functions f for which the norm

$$\|f\|_{S^{kp}W(\Omega)} \equiv \|f\|_{L_p(\Omega)} + \sum_{\substack{\alpha_i = \varepsilon_i k_i \\ \varepsilon_1 + \varepsilon_2 > 0}} \|f^{(\alpha_1, \alpha_2)}\|_{L_p(\Omega)}$$

is finite. Here $\varepsilon_i = 0, 1$ $(i = 1, 2)$.

Such spaces are called *spaces with dominating mixed derivatives*, see e.g. [1].

Denote by $p(f)$ the seminorm in $S^{kp}W(\Omega)$

$$p(f) = \sum_{\substack{\alpha_i = \varepsilon_i k_i \\ \varepsilon_1 + \varepsilon_2 > 0}} \|f^{(\alpha_1, \alpha_2)}\|_{L_p(\Omega)}.$$

Now we denote by $S^{kp}W(\Omega, L)$, $L > 0$ – constant, the class of all functions $f \in S^{kp}W(\Omega)$ with absolutely continuous (with respect to variables x and y) derivatives $f^{(\alpha_1, \alpha_2)}(x, y)$, $\alpha_i = k_i - \varepsilon_i$, $\varepsilon_i = 0, 1$, $\varepsilon_1 + \varepsilon_2 = 1$ and $p(f) \leqslant L$.

THEOREM 1. *Let* $f \in S^{m+1,p}W(I, 1)$, $I = \langle a_1, b_1 \rangle \times \langle a_2, b_2 \rangle$, $-\infty < a_i, b_i < \infty$, $1 < p_i, q_i < \infty$, $m_1, m_2 > 0$. *For* $n_i = m_i s_i$, s_i – *integer* $(i = 1, 2)$ *the following inequality holds*:

$$\begin{aligned} E_{mn}(f)_p &\equiv \inf_{s \in S^{mn}(I)} \|f - s\|_{L_p(I)} \\ &\leqslant \min(n_1, n_2) \cdot [C_{m_1 m_2} n_1^{-(m_2+1)} n_2^{-(m_2+1)} \|f^{(m_1+1, m_2+1)}\|_{L_q(I)} + \\ &\quad + C_{m_1} n_1^{-(m_1+1)} \|f^{(m_1+1, 0)}\|_{L_q(I)} + C_{m_2} n_2^{-(m_2+1)} \|f^{(0, m_2+1)}\|_{L_q(I)}]. \end{aligned}$$

Proof. Let us consider a partition $I = \bigcup_{i,k} I_{i,k}$:

$$a_1 = x_0 < x_1 < \ldots < x_{s_1} = b_1, \quad a_2 = y_0 < y_1 < \ldots < y_{s_2} = b_2,$$
$$I_{i,k} = \langle x_i, x_{i+1} \rangle \times \langle y_k, y_{k+1} \rangle,$$

and define the spline function $S(x, y)$ on I as follows: $s(x, y) \in S_{mn}(I_{i,k})$ on each subset $I_{i,k}$ (see construction in 2). The number of knots used in the construction of $S(x, y)$ on I is given by $n = (n_1, n_2)$, and $S(x, y)$ is piecewise a polynomial of degree $m = (m_1, m_2)$. On each subset $I_{i,k}$ inequality (20) for the difference $|S(x, y) - f(x, y)|$ holds. Because of Lemma 2 with $\alpha = m_1$, $\beta = m_2$, (for all i, k as in the lemma)

$$f_1(x, y) = f^{(m_1+1, m_2+1)}(x, y), \quad f_2(x, y) = f^{(m_1+1, 0)}(x, y),$$
$$f_3(x, y) = f^{(0, m_2+1)}(x, y),$$

we can choose such a partition of the set I that the following inequality holds:

(24) $$|f(x, y)-s(x, y)| \leqslant \min(s_1, s_2)\Big[C_{m_1 m_2} s_1^{-(m_1+1)} s_2^{-(m_2+1)} \iint_I |f^{(m_1+1, m_2+1)}(t, u)|\, dt du + C_{m_1} s_1^{-(m_1+1)} \iint_I |f^{(m_1+1, 0)}(t, u)|\, dt du + C_{m_2} s_2^{-(m_2+1)} \iint_I |f^{(0, m_2+1)}(t, u)|\, dt du\Big].$$

Since $n_i = m_i \cdot s_i$ $(i = 1, 2)$, the inequality (24) takes the form:

(25) $$|f(x, y)-S(x, y)| \leqslant \min(n_1, n_2)\Big[C'_{m_1 m_2} n_1^{-(m_1+1)} n_2^{-(m_2+1)} \iint_I f^{(m_1+1, m_2+1)}(t, u)\, dt du + C'_{m_1} n_1^{-(m_1+1)} \iint_I |f^{(m_1+1, 0)}(t, u)|\, dt du + C'_{m_2} n_2^{-(m_2+1)} \iint_I |f^{(0, m_2+1)}(t, u)|\, dt du\Big].$$

The right-hand side of (25) depends neither on x nor y and, moreover, $\|f\|_{L_1} \leqslant M \cdot \|f\|_{L_\varphi}$, so we obtain:

$$E_{mn}(f)_\varphi \leqslant \min(n_1, n_2)\big[C''_{m_1 m_2} n_1^{-(m_1+1)} n_2^{-(m_2+1)} \|f^{(m_1+1, m_2+1)}\|_{L_\varphi(I)} + C''_{m_2} n_1^{-(m_1+1)} \|f^{(m_1+1, 0)}\|_{L_\varphi(I)} + C''_{m_2} n_2^{-(m_2+1)} \|f^{(0, m_2+1)}\|_{L_\varphi(I)}\big],$$

and the proof is finished.

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