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On the spaces $L_m^{p,\lambda}(\Omega)$ with mixed norms. I

1. Introduction, notations. In this paper we introduce and investigate the space $L_m^{p,\lambda}(\Omega)$ which consists of functions f defined and integrable with mixed powers $p = (p_1, \ldots, p_n)$ in a product $\mathbf{P}_{i=1}^n \Omega_i$, $\Omega_i \subset R^{k_i}$. In the last part of the paper we give, using the properties of $L_m^{p,\lambda}(\Omega)$-spaces, some characterization of the function spaces $C^{m,\alpha}(\Omega)$ of functions satisfying the product's Hölder condition with exponent $\alpha = (\alpha_1, \ldots, \alpha_n)$, with theirs derivatives of certain order. Particular cases of some results of this paper can be found in [2] for $m_i = 0$, in [1] for $i = 1$. Spaces of this type were studied also by G. Stampacchia, V. K. Murthy, F. John, L. Nirenberg and others (for references see [1]).

The index i runs through $1, \ldots, n$ everywhere, unless otherwise stated. Let R be the set of real numbers and $k_i > 0$, an integer, $1 \leqslant p_i < \infty$, $\lambda_i \geqslant 0$. In what follows we shall use vector notations, i.e., $x = (x_1, \ldots, x_n)$, $p = (p_1, \ldots, p_n)$, $\varrho = (\varrho_1, \ldots, \varrho_n)$ etc. Let Ω_i be an open, connected, bounded subset of the real Euclidean space R . Let $d(\Omega_i)$ denote the diameter of Ω_i and let $\Omega = \mathbf{P}_{i=1}^n \Omega_i$, $\bar{\Omega} = \mathbf{P}_{i=1}^n \bar{\Omega}_i$. Let $I(x_i^0, \varrho_i) \subset R^{k_i}$ be the ball with centre at $x_i^0 \in \Omega_i$ and radius ϱ_i, $0 < \varrho_i \leqslant d(\Omega_i)$ and let $S_i = I(x_i^0, \varrho_i) \cap \Omega$, $S = I(x_0, \varrho) \cap \Omega = \mathbf{P}_{i=1}^n S_i$. The measure used is always the Lebesgue measure. We assume that the boundary $\partial\Omega_i$ of Ω_i has Lebesgue measure zero in R^{k_i} and that Ω_i satisfies the condition:

(A) There exist positive constants A_i such that for every $x_i^0 \in \bar{\Omega}_i$ and $\varrho_i \in [0, d(\Omega_i)]$

$$\mu(S_i) \geqslant A_i \varrho_i^{m_k}.$$

To simplify the notations, we shall write, for example:

$$\int_S |f(x)|\, dx = \int_{S_n} \ldots \int_{S_1} |f(x)|\, dx_1 \ldots dx_n,$$

$$\int_{{}_pS} |f(x)|^p\, dx = \|f\|_{L^p(S)}^{p_n} = \int_{S_n} [\ldots(\int_{S_1} |f(x)|^{p_1}\, dx_1)^{p_2/p_1}\, dx_2 \ldots]^{p_n/p_{n-1}}\, dx_n.$$

Supremum, denoted by sup, runs over $x_i^0 \in \bar{\Omega}_i$, $0 < \varrho_i \leqslant d(\Omega_i)$.

Let $l! = l_1! \ldots l_n!$, $l_i! = l_1^i! \ldots l_{k_i}^i!$, $|l| = |l_1| + \ldots + |l_n|$, $|l_i| = l_1^i + \ldots + l_{k_i}^i$, $x^l = x_1^{1l_1^1} x_2^{1l_2^1} \ldots x_{k_n}^{nl_{k_n}^n}$, $k_i = N$. Let $D^l f(x)$ denote the generalized derivative of function $f(x)$ in Sobolev sense and let

$$D^l f(x) = \frac{\partial^{|l|} f(x)}{\partial x_1^{1l_1^1} \ldots \partial x_{k_n}^{nl_{k_n}^n}} \quad \text{if } |l| > 0, \qquad D^l f(x) = f(x) \quad \text{if } l = 0,$$

and $P_m(x)$ a polynomial of variable x with real coefficients. Let P_m denote the set of polynomials $P(x)$ of degree $|m| = \sum_{i=1}^{n} m_i$, i.e., of degree m_i with respect to x_i, m_i a non-negative integer.

We shall denote by $C^m(\bar{\Omega})$ the space of functions defined in $\bar{\Omega}$, which have continuous partial derivatives of order up to and including m_i with respect to x_i, m_i an integer $\geqslant 0$.

We shall denote by $C^{m,\alpha}(\bar{\Omega})$, m_i an integer $\geqslant 0$, $\alpha = (\alpha_1, \ldots, \alpha_n)$, $0 < \alpha_i \leqslant 1$, the subspace of $C^m(\bar{\Omega})$ of functions, the derivative of which of order $|m|$ satisfies the product's Hölder condition in $\bar{\Omega}$ with exponent α.

The space $C^m(\bar{\Omega})$ is a Banach space with the norm

$$|f|_{m,\bar{\Omega}} = \sum_{\substack{|l_i| \leqslant m_i \\ i=1,\ldots,n}} \sup_{\bar{\Omega}} |D^l f|. \tag{1.1}$$

The space $C^{m,\alpha}(\bar{\Omega})$ is a Banach space (compare [9] p. 26) with the norm

$$|f|_{m,\alpha,\bar{\Omega}} = \sum_{\substack{|l_i| \leqslant m_i \\ i=1,\ldots,n}} \sup_{\bar{\Omega}} |D^l f| + \sup_{|l_i| = m_i} \sup_{\substack{x_i, y_i \in \bar{\Omega}_i \\ x_i \neq y_i}} \frac{|D^l f(x) - D^l f(y)|}{\prod_{i=1}^{n} |x_i - y_i|^{\alpha_i}}. \tag{1.2}$$

For $f \in C^{m,\alpha}(\bar{\Omega})$ we write

$$[f]_{m,\alpha,\bar{\Omega}} = \sup_{|l_i| = m_i} \sup_{\substack{x_i, y_i \in \bar{\Omega}_i \\ x_i \neq y_i}} \frac{|D^l f(x) - D^l f(y)|}{\prod_{i=1}^{n} |x_i - y_i|^{\alpha_i}}. \tag{1.3}$$

The above is a seminorm in $C^{m,\alpha}(\bar{\Omega})$.

Definition. We shall say that the function $f(x) \in L^p(\Omega)$ *belongs to the class* $L_m^{p,\lambda}(\Omega)$ if

$$|||f|||_{m,p,\lambda} = \sup_{\substack{x_i^0 \in \bar{\Omega}_i \\ 0 < \varrho_i \leqslant d(\Omega_i)}} \Big\{ \prod_{i=1}^{n} \varrho_i^{-\lambda_i p_n / p_i} \inf_{P \in P_m} \int_{bS} |f(x) - P(x)|^p dx \Big\}^{1/p_n} < \infty. \tag{1.4}$$

This is a seminorm in $L_m^{p,\lambda}(\Omega)$. The space $L_m^{p,\lambda}(\Omega)$ is a Banach space under the norm

(1.5) $$\|f\|_{m,p,\lambda} = \{\|f\|_{L^p(\Omega)}^{p_n} + |||f|||_{m,p,\lambda}^{p_n}\}^{1/p_n}.$$

The proof proceeds similarly to that of completeness of $L^{p,\lambda}(\Omega)$ or $M^{p,\lambda}(\Omega)$ given in [6], [8].

2. Let $f(x) \in L_m^{p,\lambda}(\Omega)$; we can prove analogously as in n. 3 [1] and I.2. [5] that for every $x_i^0 \in \bar{\Omega}_i$, $\varrho_i \in [0, d(\Omega_i)]$ there exists only one polynomial $P_m(x, x_0, \varrho, p, f)$ such that

(2.1) $$\inf_{P \in P_m} \int_{{}_bS} |f(x) - P(x)|^p dx = \int_{{}_bS} |f(x) - P_m(x, x_0, \varrho, p, f)|^p dx.$$

Let $P(x)$ be the polynomial $P(x) \in P_m$. We can write it in the form

$$P(x) = \sum_{\substack{|l_i| \leq m_i \\ i = 1, \ldots, n}} a_l (|l|!)^{-1} \prod_{i=1}^{n} (x_i - x_i^0)^{l_i}.$$

We shall write also $P_m(x, x_0, \varrho)$ or $P_m(x, x_0, \varrho, p)$ instead of $P_m(x, x_0, \varrho, p, f)$ if there will be no possibility of misunderstanding. We have

(2.2) $$a_l(x_0, \varrho) = [D^l P_m(x, x_0, \varrho)]_{x = x_0}.$$

LEMMA 2.1. *If $f \in L_m^{p,\lambda}(\Omega)$, then there exists a positive constant $c_1(p, \lambda)$ such that*

(2.3) $$\int_{{}_bS''} |P_m(x, x_0, \varrho|2^h) - P_m(x, x_0, \varrho|2^{h+1})|^p dx \leq c_1 |||f|||_{m,p,\lambda}^{p_n} \prod_{i=1}^{n} 2^{-h\lambda_i p_n/p_i} \varrho_i^{\lambda_i p_n/p_i}$$

for every $x_i^0 \in \bar{\Omega}_i$, $0 < \varrho_i \leq d(\Omega_i)$ and positive integer h, where we set

$$S_i' = I(x_i^0, \varrho_i|2^h) \cap \Omega_i, \quad S_i'' = I(x_i^0, \varrho_i|2^{h+1}) \cap \Omega_i, \quad \varrho|2^h = (\varrho_1|2^n, \ldots, \varrho_n|2^n).$$

Proof. Let $x_i^0 \in \bar{\Omega}_i$, $0 < \varrho_i \leq d(\Omega_i)$, h positive integer. For almost all $x \in \mathbf{P}_{i=1}^n S_i$ we have

$$|P_m(x, x_0, \varrho|2^h) - P_m(x, x_0, \varrho|2^{h+1})|^{p_1} \leq 2^{p_1} |P_m(x, x_0, \varrho|2^h) - f(x)|^{p_1} + 2^{p_1} |P_m(x, x_0, \varrho|2^h) - f(x)|^{p_1}.$$

Integrating both sides of this inequality with respect to all variables x_i on S_i'' successively and taking a suitable power $p_{i+1}|p_i$ we obtain

$$\int_{{}_bS''} |P_m(x, x_0, \varrho|2^h) - P_m(x, x_0, \varrho|2^{h+1})|^p dx \leq 2^{p_n} \int_{{}_bS'} |P_m(x, x_0, \varrho|2^h) - f(x)|^p dx + 2^{p_n} \int_{{}_bS''} |P_m(x, x_0, \varrho|2^{h+1}) - f(x)|^p dx.$$

Then, by (1.4) and (2.1) we have

$$\int_{bS''} |P_m(x, x_0, \varrho|2^h) - P_m(x, x_0, \varrho|2^{h+1})|^p dx$$

$$\leqslant 2^{p_n} |||f|||^{p_n}_{m,p,\lambda} \prod_{i=1}^{n} (2^{-h}\varrho_i)^{\lambda_i p_n/p_i} + 2^{p_n} |||f|||^{p_n}_{m,p,\lambda} \prod_{i=1}^{n} (2^{-h-1}\varrho_i)^{\lambda_i p_n/p_i}$$

$$= 2^{p_n}\left(1 + \prod_{i=1}^{n} 2^{-\lambda_i p_n/p_i}\right) |||f|||^{p_n}_{m,p,\lambda} \prod_{i=1}^{n} (2^{-h}\varrho_i)^{\lambda_i p_n/p_i}.$$

Hence the thesis follows.

LEMMA 2.2. *If condition* (A) *is satisfied,* $f \in L^{p,\lambda}_m(\Omega)$, *then for every pair of points* $x^0_i, y^0_i \in \bar{\Omega}_i$ *and for every system of* N *numbers* $l = (l^1_1, \ldots, l^n_{k_n})$ *with* $|l_i| = m_i$ *there holds the inequality*

$$(2.4)\qquad |a_l(x_0, 2|x_0 - y_0|) - a_l(y_0, 2|x_0 - y_0|)|^{p_n}$$

$$\leqslant c_2(l, p, k)\, 2^{p_n+1} \prod_{i=1}^{n} 2^{\lambda_i p_n/p_i} |||f|||^{p_n}_{m,p,\lambda} \prod_{i=1}^{n} |x^0_i - y^0_i|^{\left(\frac{\lambda_i}{p_i} - \frac{k_i}{p_i} - |m_i|\right)p_n}.$$

Proof. Let $x^0_i, y^0_i \in \bar{\Omega}_i$. We set $\varrho_i = |x^0_i - y^0_i|$ and $S'''_i = S'_i \cap S''_i = [I(x^0_i, 2\varrho_i) \cap \Omega_i] \cap [I(y^0_i, 2\varrho_i) \cap \Omega_i]$ for almost all $x \in \mathbf{P}^n_{i=1} S'''_i$ we have

$$|P_m(x, x_0, 2\varrho) - P_m(y, y_0, 2\varrho)|^{p_1}$$

$$\leqslant 2^{p_1} |P_m(x, x_0, 2\varrho) - f(x)|^{p_1} + 2^{p_1} |P_m(y, y_0, 2\varrho) - f(x)|^{p_1}.$$

Integrating both sides of this inequality with respect to all variables x_i on $S_i \subset S'''_i$, taking a suitable power p_{i+1}/p_i we get, arguing analogously as in the proof of the Lemma 2.1

$$(2.5)\qquad \int_{bS\cdot} |P_m(x, x_0, 2\varrho) - P_m(y, y_0, 2\varrho)|^p dx$$

$$\leqslant 2^{p_n} \int_{S'} |P_m(x, x_0, 2\varrho) - f(x)|^p dx + 2^{p_n} \int_{bS''} |P_m(y, y_0, 2\varrho) - f(x)|^p dx$$

$$\leqslant \prod_{i=1}^{n} 2^{p_n + \lambda_i p_n/p_i + 1} |||f|||^{p_n}_{m,p,\lambda} \prod_{i=1}^{n} \varrho^{i p_n/p_i}.$$

Then, if (2.2) holds, apply Lemma 1 [3] to the polynomial $P(x) = P_m(x, x_0, 2\varrho) - P_m(y, y_0, 2\varrho)$ and we obtain

$$(2.6)\qquad |a_l(x_0, 2\varrho) - a_l(y_0, 2\varrho)|^{p_n}$$

$$\leqslant c_3 \cdot \prod_{i=1}^{n} \varrho_i^{-\left(\frac{k_i}{p_i} + m_i\right)p_n} \int_{bS} |P_m(x, x_0, 2\varrho) - P_m(y, y_0, 2\varrho)|^p dx.$$

By (2.5), (2.6) we get the thesis.

LEMMA 2.3. *If condition* (A) *is satisfied*, $f \in L_m^{p,\lambda}(\Omega)$, *then there exists a positive constant* $c_4(m, p, \varrho, k, A)$ *such that*

$$(2.7)\qquad |a_l(x_0, \varrho) - a_l(x_0, \varrho|(2^h)| \leqslant c_4 |||f|||_{m,p,\lambda} \sum_{i=0}^{h-1} \prod_{j=1}^{n} (2^{-i}\varrho_j)^{\frac{\lambda_j}{p_j} - \frac{k_j}{p_j} - |l_j|}$$

for every $x^0 \in \bar{\Omega}_i$, $0 < \varrho_i \leqslant d(\Omega_i)$, *positive integer* h *and* $|l_i| \leqslant m_i$.

Proof. We have

$$(2.8)\qquad |a_l(x_0, \varrho) - a_l(x_0, \varrho|2^h)| \leqslant \sum_{i=0}^{h-1} |a_l(x_0, \varrho|2^i) - a_l(x_0, \varrho|2^{i+1})|$$

$$= \sum_{i=0}^{h-1} |\{D^l[P_m(x, x_0, \varrho|2^i) - P_m(x, x_0, \varrho|2^{i+1})]\}_{x=x_0}|.$$

Applying Lemma 1 [4] to the polynomial $P(x) = P_m(x, x_0, \varrho|2^i) - P_m(x, x_0, \varrho|2^{i+1})$ we deduce

$$(2.9)\qquad |a_l(x_0, \varrho) - a_l(x_0, \varrho|2^i)|$$

$$\leqslant c_3 \prod_{j=1}^{n} \varrho_j^{\frac{-k_j}{p_j} - |l_j|} \cdot 2^{(i+1)(\frac{k_j}{p_j} - |l_j|)} \sum_{i=0}^{h-1} \Big[\int_{bS'} |P_m(x, x_0, \varrho|2^i) - P_m(x, x_0, \varrho|2^{i+1})|^p dx\Big]^{1/p_n}.$$

After applying (2.3) to the above inequality we get the thesis.

LEMMA 2.4. *If condition* (A) *is satisfied*, $f \in L_m^{p,\lambda}(\Omega)$, $k_i + p_i < \lambda_i \leqslant k_i + (s_i + 1)p_i$, s_i — *positive integer*, $s_i \leqslant m_i$, *then there exists a system of functions* $\{v_l(x_0)\}$, $|l_i| \leqslant s_i$, *defined in* $\bar{\Omega}$ *such that for* $x_i^0 \in \bar{\Omega}_i$, $0 < \varrho_i \leqslant d(\Omega_i)$, $|l_i| \leqslant s_i$, *we have*

$$(2.10)\qquad |a_l(x_0, \varrho) - v_l(x_0)| \leqslant c_5(\lambda, p, m, k, A) |||f|||_{m,p,\lambda} \prod_{j=1}^{n} \varrho_j^{\frac{\lambda_j}{p_j} - \frac{k_j}{p_j} - |l_j|}$$

and

$$(2.11)\qquad \lim_{\substack{\varrho_i \to 0 \\ i=1,\ldots,n}} a_l(x_0, \varrho) = v_l(x_0)$$

uniformly with respect to x_0.

Proof. We fix l, ϱ, x_0 with $|l_i| \leqslant s_i$, $x_i^0 \in \bar{\Omega}_i$, $0 < \varrho_i \leqslant d(\Omega_i)$. Let h and h' be two positive integers, for example $h > h'$; then we apply Lemma 2.3, setting $\varrho_j 2^{-h'}$ instead of ϱ_j and we obtain

$$(2.12)\qquad |a_l(x_0, \varrho 2^{-h'}) - a_l(x_0, \varrho 2^{-h})| \leqslant c_4 |||f|||_{m,p,\lambda} \sum_{i=h'}^{h-1} \prod_{j=1}^{n} (2^i \varrho_j)^{\frac{\lambda_j}{p_j} - \frac{k_j}{p_j} - |l_j|}.$$

By assumption, the series $\sum_{i=h'}^{\infty} \prod_{j=1}^{n} (2^i)^{\frac{k_j}{p_j}+|l_j|-\frac{\lambda_j}{p_j}}$ is convergent. We deduce, taking into account (2.12) and convergence of this series, that the sequence $\{a_l(x_0, \varrho 2^{-h})\}$ satisfies the Cauchy condition and thus it is convergent with $h \to \infty$. We observe that this limit is independent of ϱ_i, $0 < \varrho_i \leqslant d(\Omega_i)$. Let ϱ_i^1 and ϱ_i^2 be two real numbers satisfying the relation $0 < \varrho_i^1 \leqslant \varrho_2^i \leqslant d(\Omega_i)$. Applying Lemma 1 [3] and definition of the class $L_m^{p,\lambda}(\Omega)$, setting $S_j' = I(x_j^0, \varrho_j^1|2^i) \cap \Omega_j$, $S_j'' = I(x_j^0, \varrho_j^2|2^i) \cap \Omega_j$, we get

$$(2.13) \quad |a_l(x_0, \varrho^1|2^i) - a_l(x_0, \varrho^2|2^i)|$$
$$\leqslant c_3 \prod_{j=1}^{n} [2^i(\varrho_j^1)^{-1}]^{\frac{k_j}{p_j}+|l_j|} \cdot \{ \int_{{}_hS'} |P_m(x, x_0, \varrho^1|2^i) - P_m(x, x_0, \varrho^2|2^i)|^p dx\}^{1/p_n}$$
$$\leqslant c_3 \prod_{j=1}^{n} [2^i(\varrho_j^1)^{-1}]^{\frac{k_j}{p_j}+|l_j|} \cdot \{ \int_{{}_hS'} |P_m(x, x_0, \varrho^1|2^i) - f(x)^p| dx +$$
$$+ \int_{{}_hS''} |P_m(x, x_0, \varrho^2|2^i) - f(x)|^p dx\}^{1/p_n}$$
$$\leqslant c_3\, 2 \cdot |||f|||_{m,p,\lambda} \prod_{j=1}^{n} (\varrho_j^{1\frac{\lambda_j}{p_j}} + \varrho_j^{2\frac{\lambda_j}{p_j}})^{1/p_n} (\varrho_j^1)^{-\frac{k_j}{p_j}-|l_j|} \cdot 2^{i(\frac{k_j}{p_j}+|l_j|-\frac{\lambda_j}{p_j})}.$$

By assumptions of the lemma we get that the latter expression tends to zero for $i \to \infty$. We set then for $x_i^0 \in \bar{\Omega}_i$, $0 < \varrho_i \leqslant d(\Omega_i)$, $|l_i| \leqslant s_i$

$$(2.14) \qquad v_l(x_0) = \lim_{i\to\infty} a_l(x_0, \varrho|2^i),$$

where $v_l(x_0)$ is defined in $\bar{\Omega}$.

By the convergence of the series $\sum_{i=0}^{\infty} \prod_{j=1}^{n} 2^{i(\frac{k_j}{p_j}+|l_j|-\frac{\lambda_j}{p_j})}$ and by (2.7) we deduce for every $|l_j| < s_j$, $x_j^0 \in \bar{\Omega}_j$, $0 < \varrho_j \leqslant d(\Omega_j$, $j = 1, \ldots, n$, $i \geqslant 0$:

$$(2.15) \qquad |a_l(x_0, \varrho) - a_l(x_0, \varrho|2^i)| \leqslant c_6(\lambda, m, k, A)\, |||f|||_{m,p,\lambda} \prod_{j=1}^{n} \varrho_j^{\frac{\lambda_j}{p_j}-\frac{k_j}{p_j}-|l_j|}.$$

From (2.14) and (2.15) we get (2.10) and (2.11).

3. THEOREM 3.1. *If condition* (A) *is satisfied,* $f \in L_m^{p,\lambda}(\Omega)$, $\lambda_i \geqslant k_i + |l_i|\, p_i$, *then the functions* $v_l(x)$ *satisfy the Hölder condition in* $\bar{\Omega}$ *for* $|l_i| = m_i$ *and for every pair of points* $x, y \in \bar{\Omega}$ *there holds the inequality*

$$(3.1) \qquad |v_l(x) - v_l(y)| \leqslant c_7 |||f|||_{m,p,\lambda} \prod_{j=1}^{n} |x_j - y_j|^{\frac{\lambda_j}{p_j}-\frac{k_j}{p_j}-|l_j|}.$$

Proof. Let the integers $l = (l_1^1, l_2^1, \ldots, l_{k_1}^1, \ldots, l_{k_n}^n)$ be fixed, $\sum_{i=1}^{n} k_i = N$,

$|l_i| = m_i$ and let $x_i, y_i \in \bar{\Omega}_i$ be a pair of such points that $|\varrho_i| = |x_i - y_i| \leqslant d(\Omega_i)/2$. We have the inequality

$$(3.2) \quad |v_l(x) - v_l(y)| \leqslant |v_l(x) - a_l(x, 2\varrho)| + |v_l(y) - a_l(y, 2\varrho)| + |a_l(x, 2\varrho) - a_l(y, 2\varrho)|.$$

By (2.10) we also have

$$(3.3) \qquad |v_l(x) - a_l(x, 2\varrho)| \leqslant c_5 |||f|||_{m,p,\lambda} \prod_{i=1}^{n} (2\varrho_i)^{\frac{\lambda_i}{p_i} - \frac{k_i}{p_i} - |l_i|},$$

$$(3.4) \qquad |v_l(y) - a_l(y, 2\varrho)| \leqslant c_5 |||f|||_{m,p,\lambda} \prod_{i=1}^{n} (2\varrho_i)^{\frac{\lambda_i}{p_i} - \frac{k_i}{p_i} - |l_i|}$$

and by (2.4) we get

$$(3.5) \quad |a_l(x, 2\varrho - a_l(y, 2\varrho)| \leqslant c_2^{1/p_n} 2^{1+1/p_n} \cdot \prod_{i=1}^{n} 2^{\lambda_i/p_i} |||f|||_{m,p,\lambda} \prod_{i=1}^{n} |x_i - y_i|^{\frac{\lambda_i}{p_i} - \frac{k_i}{p_i} - |l_i|}.$$

From (3.2), (3.3), (3.4), (3.5) we get (3.1) for every pair of points x_i, y_i, satisfying the relation $|x_i - y_i| \leqslant d(\Omega_i)/2$. If $|x_i - y_i| \geqslant d(\Omega_i)/2$ we argue similarly as in Theorem 2 [2].

We set $(0) = (0, \ldots, 0) \in R^N$, $e_j = (0, \ldots, 0, 1, 0, \ldots, 0) \in R^N$ – the j-th coordinate is equal to 1, the others are equal zero.

THEOREM 3.2. *If condition* (A) *is satisfied,* $f \in L_m^{p,\lambda}(\Omega)$, $m_i \geqslant 1$, $\lambda_i \geqslant k_i + |l_i| p_i$, *then for every system of N positive integers* $(l_1^1, \ldots, l_{k_n}^n)$, $|l_i| \leqslant m_i - 1$ *the function* $v_l(x)$ *has partial derivatives of order one in* Ω *and for every* $x \in \Omega$ *there holds the equality*

$$(3.6) \qquad \frac{\partial v_l}{\partial x_j} = v_{(l+e_j)}(x), \quad j = 1, \ldots, N,$$

where x_j *denotes the coordinate of* $x = (x_1^1, \ldots, x_{k_n}^n)$.

Proof. We know from Theorem 3.1 that the functions $v_l(x)$, $|l_i| = m_i$ satisfy the Hölder condition and hence, in particular, are continuous in $\bar{\Omega}$ (see e.g. [10], p. 123–125). We shall prove Theorem 3.2 by induction, if we prove (3.6), by assumption that $v_{(l+e_j\delta)}(x)$ are continuous in $\bar{\Omega}$ for $\delta = 1, 2, \ldots, |m| - |l|$. Let l be a system of N positive integers such that $|l_i| \leqslant m_i - 1$, let j be a fixed positive integer, $1 \leqslant j \leqslant N$ and suppose that $v_{(l+e_j\delta)}(x)$ are continuous in $\bar{\Omega}$ for $\delta = 1, 2, \ldots, |m| - |l|$. Let $x_i^0 \in \Omega_i$ and let ϱ_i be real numbers sufficiently small such that $I(x_i^0, |\varrho_i|) \subset \Omega_i$. Let d be a real number such that $d = \operatorname{diam}\left[\mathbf{P}_{i=1}^{n} I(x_i, |\varrho_i|)\right] \subset \Omega$. Taking into account (2.2) we can write

$$(3.7)\qquad [a_l(x_0+e_jd,\ 2|\varrho|)-a_l(x_0,\ 2|\varrho|)]\,d^{-1} = d^{-1}\{D^l[P_m(x,\ x_0+e_jd,\ 2|\varrho|)-P_m(x,\ x_0,\ 2|\varrho|)]\}_{x=x_0}- -\sum_{1}^{|m|-|l|}(-1)^{\delta}(\delta!)^{-1}d^{\delta-1}a_{(l+\delta e_j)}(x_0+e_jd,\ 2|\varrho|).$$

By Lemma 1 [4] and (2.5) where $S_i = I(x_i^0, |\varrho_i|)\cap\Omega_i$, we obtain

$$(3.8)\qquad |d^{-1}\{D^l[P_m(x,\ x_0+e_jd,\ 2|\varrho|)-P_m(x,\ x_0,\ 2|\varrho|)]\}_{x=x_0}| \leqslant c_3\prod_{i=1}^{n}|\varrho_i|^{\frac{-k_i}{p_i}-|l_i|}\Big\{\int_{{}_bS}|P_m(x,\ x_0+e_jd,\ 2|\varrho|)-P_m(x,\ x_0,\ 2|\varrho|)|^p\,dx\Big\}^{1/p_n} \leqslant c_3\prod_{1}^{n}2^{1+1/p_n+\lambda_i/p_i}|||f|||_{m,p,\lambda}\prod_{i=1}^{n}|\varrho_i|^{\frac{\lambda_i}{p_i}-\frac{k_i}{p_i}-|l_i|}.$$

We have for every $1\leqslant\delta\leqslant|m|-|l|$

$$(3.9)\qquad |a_{(l+\delta e_j)}(x_0+e_jd,\ 2|\varrho|)-v_{(l+\delta e_j)}(x_0)| \leqslant |a_{(l+\delta e_j)}(x_0+e_jd,\ 2|\varrho|)-v_{(l+\delta e_j)}(x_0+e_jd)|+|v_{(l+\delta e_j)}(x_0+e_jd)-v_{(l+\delta e_j)}(x_0)|.$$

Applying (2.10) we get

$$(3.10)\qquad |a_{(l+\delta e_j)}(x_0+e_jd,\ 2|\varrho|)-v_{(l+\delta e_j)}(x_0+e_jd)| \leqslant c_5\prod_{i=1}^{n}(2|\varrho_i|)^{[\lambda_i-k_i-(|l_i|+\delta)p_i]/p_i}\cdot|||f|||_{m,p,\lambda}.$$

From (3.9), (3.10) and from continuity of the function $v_{(l+e_j\delta)}(x)$ for $\delta = 1, 2, \ldots, |m|-|l|$ it follows

$$(3.11)\qquad \lim_{d\to 0}a_{(l+e_j\delta)}(x_0+e_jd,\ 2|\varrho|) = v_{(l+e_j\delta)}(x_0),\qquad \delta = 1, 2, \ldots, |m|-|l|$$

($d\to 0$ when $\varrho_i\to 0$). By (3.7), (3.8), (3.11) we deduce the existence of the finite limes

$$\lim_{d\to 0}d^{-1}[a_l(x_0+e_jd,\ 2|\varrho|)-a_l(x_0,\ 2|\varrho|)]$$

and there holds

$$(3.12)\qquad \lim_{d\to 0}d^{-1}[a_l(x_0+e_jd,\ 2|\varrho|)-a_l(x_0,\ 2|\varrho|)] = v_{(l+e_j)}(x_0)$$

uniformly with respect to x_0.

We have, by (2.10) and by relation $|d|^n \geqslant \prod_{i=1}^{n} |\varrho_i|$

$$(3.13)\quad |d^{-1}[v_l(x_0+e_jd)-a_l(x_0+e_jd,\ 2|\varrho|)]| \leqslant c_5 \prod_{i=1}^{n} 2^{\frac{\lambda_i}{p_i}-\frac{k_i}{p_i}-|l_i|} |||f|||_{m,p,\lambda} \prod_{i=1}^{n} |\varrho_i|^{\frac{\lambda_i}{p_i}-\frac{k_i}{p_i}-|l_i|-\frac{1}{n}},$$

$$(3.14)\quad |d^{-1}[v_l(x_0)-a_l(x_0,\ 2|\varrho|)]| \leqslant c_5 \prod_{i=1}^{n} 2^{\frac{\lambda_i}{p_i}-\frac{k_i}{p_i}-|l_i|} |||f|||_{m,p,\lambda} \prod_{i=1}^{n} |\varrho_i|^{\frac{\lambda_i}{p_i}-\frac{k_i}{p_i}-|l_i|-\frac{1}{n}}.$$

We observe that, for $\varrho_i \to 0$, the left-hand side of the above inequalities tends to zero. We have the equality

$$(3.15)\quad d^{-1}[v_l(x_0+e_jd)-v_l(x_0)] = d^{-1}[v_l(x_0+e_jd)-a_l(x_0+e_jd,\ 2|\varrho|)] + d^{-1}[a_l(x_0,\ 2|\varrho|)-v_l(x_0)] + d^{-1}[a_l(x_0+e_jd,\ 2|\varrho|)-a_l(x_0,\ 2|\varrho|)].$$

From (3.13), (3.14), (3.15) it follows

$$(3.16)\quad \lim_{d\to 0} d^{-1}[v_l(x_0+e_jd)-v_l(x_0)] = \lim_{d\to 0} d^{-1}[a_l(x_0+e_jd,\ 2|\varrho|)-a_l(x_0,\ 2|\varrho|)]$$

and hence (3.6).

The conclusion of Theorems 3.1 and 3.2 is

THEOREM 3.3. *If condition* (A) *is satisfied,* $f \in L_m^{p,\lambda}(\Omega)$, $\lambda_i > k_i + m_i p_i$, *then the function* $v_{(0)}(x) \in C^{m,\alpha}(\bar{\Omega})$, $\alpha_i = (\lambda_i - k_i - m_i p_i) p_i^{-1}$ *and*

$$D^l v_{(0)}(x) = v_l(x) \quad \textit{for every } x \in \Omega,\ |l_i| \leqslant m_i.$$

Remark 3.1. Let condition (A) be satisfied, $f \in L_m^{p,\lambda}(\Omega)$, $m_i \geqslant 1$. Assume that there exists an integer, s_i, $0 \leqslant s_i \leqslant m_i - 1$ such that the degree of the polynomial $P_m(x, x_0, \varrho)$ is $\leqslant |s|$ for every $x_i^0 \in \bar{\Omega}_i$ and $\varrho_i \in [0, d(\Omega_i)]$. Then we deduce that $v_{(0)}(x)$ is a polynomial of degree $\leqslant |s|$. It is true, since: if $a_l(x_0, \varrho) = 0$ for $|l_i| > s_i$, for every $x_i^0 \in \bar{\Omega}_i$, $\varrho_i \in [0, d(\Omega_i)]$, then $v_l(x)$ for $|l_i| > s_i$ are identically equal zero in $\bar{\Omega}$ and by Theorem 3.3, we have the conclusion.

Remark 3.2. If $f \in L_m^{p,\lambda}(\Omega)$, $\lambda_i > k_i + (m_i+1) p_i$, then we deduce by (3.1) that $v_l(x)$, $|l_i| = m_i$, are constant and thus, by Theorem 3.3, $v_{(0)}(x)$ is polynomial of degree $\leqslant |m|$.

4. THEOREM 4.1. *If condition* (A) *is satisfied,* $f \in L_m^{p,\lambda}(\Omega)$, $k_i + m_i p_i < \lambda_i \leqslant k_i + (m_i+1) p_i$, *then* $f \in C^{m,\alpha}(\Omega)$, *and there holds the inequality*

$$(4.1)\quad [f]_{m,\alpha} \leqslant c_8 |||f|||_{m,p,\lambda}.$$

If $\lambda_i > k_i + (m_i+1)p_i$, then f coincides in Ω with a polynomial of degree $\leqslant |m|$.

Proof. Taking into consideration Theorem 3.3 and Remark 3.1, it is sufficient to prove, with assumptions of the theorem, that $f(x)$ coincides, for almost all $x \in \Omega$, with $v_{(0)}(x)$, i.e., with $\lim\limits_{\varrho_i \to 0} a_{(0)}(x, \varrho)$. We have proved in Lemma 4 [2] that for almost all $x_0 \in \Omega$,

$$\lim_{\varrho_1 \to 0} \dots \lim_{\varrho_n \to 0} \prod_{i=1}^{n} [\mu(S_i)]^{-p_n/p_i} \int_{{}_bS} |f(x)-f(x_0)|^p\, dx = 0. \tag{4.2}$$

Let $x_0 \in \Omega$ be such that (4.2) is satisfied; for almost all $x \in \Omega$ we have

$$|a_{(0)}(x_0, \varrho)-f(x_0)|^{p_1} \leqslant c_8(p) \{|P_m(x, x_0, \varrho)-a_{(0)}(x_0, \varrho)|^{p_1} + \\ + |P_m(x, x_0, \varrho)-f(x)|^{p_1} + |f(x)-(x_0)|^{p_1}\}.$$

Integrating both sides of this inequality with respect to all variables x_i on S_i, and taking a suitable power p_{i+1}/p_i we get

$$\begin{aligned} |a_{(0)}(x_0, \varrho)-f(x_0)|^{p_n} &\leqslant c_8(p) \prod_{i=1}^{n} (A_i \varrho_i^{m_i})^{-p_n/p_i} \int_{{}_bS} |P_m(x, x_0, \varrho)-a_{(0)}(x_0, \varrho)|^p\, dx + \\ &+ c_8(p) \prod_{i=1}^{n} (A_i \varrho_i^{m_i})^{-p_n/p_i} \int_{{}_bS} |P_m(x, x_0, \varrho)-f(x)|^p\, dx + \\ &+ c_8(p) \prod_{i=1}^{n} [\mu(S_i)]^{-p_n/p_i} \int_{{}_bS} |f(x)-f(x_0)|^p\, dx. \end{aligned} \tag{4.3}$$

Consider now the third part of the right-hand side of (4.3). First (see the proof of the Lemma 2 [3]), we have

$$\begin{aligned} c_8(p) \prod_{i=1}^{n} (A_i \varrho_i^{m_i})^{-p_n/p_i} \int_{{}_bS} |P_m(x, x_0, \varrho)-a_{(0)}(x_0, \varrho)|^p\, dx \\ \leqslant c_9(A, k, p, m) \prod_{i=1}^{n} (A_i \varrho_i^{m_i})^{-p_n/p_i} \sum_{1 \leqslant |l| \leqslant |m|} |a_l(x_0, \varrho)|^{p_n} \prod_{i=1}^{n} \varrho_i^{|l_i| p_n}. \end{aligned} \tag{4.4}$$

Hence and by (2.11) we deduce that (4.4) is infinitesimal with ϱ_i. Then, by definition of class $L_m^{p,\lambda}(\Omega)$, we obtain

$$\begin{aligned} c_8(p) \prod_{i=1}^{n} (A_i \varrho_i^{m_i})^{-p_n/p_i} \int_{{}_bS} |P_m(x, x_0, \varrho)-f(x)|^p\, dx \\ \leqslant c_8(p) \prod_{i=1}^{n} (A_i \varrho_i^{\lambda_i - m_i})^{-p_n/p_i} |||f|||_{m,p,\lambda} \end{aligned} \tag{4.5}$$

and hence (4.5) is infinitesimal with ϱ_i.

The third part of the right-hand side of (4.3) is infinitesimal with ϱ_i, by

(4.2). Then (4.3) follows, for almost all $x_0 \in \Omega$ and

$$\lim_{\varrho_i \to 0} a_{(0)}(x_0, \varrho) = f(x_0).$$

By (3.1) and by the equality $f(x) = v_{(0)}(x)$, which holds almost everywhere in Ω, we deduce (4.1).

Applying Theorem 4.1 and Theorem 1 [7], we can deduce part (a) of the following

THEOREM 4.2 (a) *If condition* (A) *is satisfied,* $f \in L_m^{p,\lambda}(\Omega)$, m_i *— positive integers,* $k_i + s_i p_i < \lambda_i < k_i + s_i + p_i$, $0 \leqslant s_i \leqslant m_i$, *then* $f \in C^{s,\alpha}(\bar{\Omega})$, $\alpha_i = (\lambda_i - k_i - s_i p_i)/p_i$ *and there holds the inequality*

$$[f]_{s,\alpha} \leqslant c_{10} |||f|||_{m,p,\lambda}. \tag{4.6}$$

(b) *If in addition we suppose that* Ω_i *is convex, then the space* $L_m^{p,\lambda}(\Omega)$ *is isomorphic to the space* $C^{s,\alpha}(\bar{\Omega})$.

Proof (b). Let $f \in C^{s,\alpha}(\bar{\Omega})$, $x_i^0 \in \bar{\Omega}_i$, $0 < \varrho_i \leqslant d(\Omega_i)$, $x_i \in I(x_i^0, \varrho_i) \cap \Omega_i$. Taking into account that the derivatives of order $|s|$ of the function f satisfy Hölder condition we can use the Taylor formula

$$\begin{aligned} (4.7)\quad & \Big|f(x) - \sum_{|l_i| \leqslant m_i - 1} D^l f(x_0)\,(|l|!)^{-1} \prod_{i=1}^{n} (x_i - x_i^0)^{l_i}\Big| \\ &= \Big|\sum_{|l_i| = s_i} [D^l f(\xi) - D^l f(x_0)]\,(|l|!)^{-1} \prod_{i=1}^{n} (x_i - x_i^0)^{l_i}\Big| \\ &\leqslant [f]_{s,\alpha} \cdot \sum_{|l_i| = s_i} (|l|!)^{-1} \prod_{i=1}^{n} |x_i - x_i^0|^{\frac{\lambda_i - k_i}{p_i}}, \end{aligned}$$

where ξ is the suitable point of the segment $[x_0, x]$. Integrating both sides of the above inequality with respect to all variables x_i on S_i, taking a suitable power p_{i+1}/p_i we obtain

$$\inf_{P \in P_S} \int_{{}_b S} |f(x) - P(x)|^p\, dx \leqslant c_{11} [f]_{s,\alpha}^{p_n} \prod_{i=1}^{n} \varrho_i^{\lambda_i p_n/p_i}. \tag{4.8}$$

By (4.8) and (6) [7] we obtain

$$|||f|||_{m,p,\lambda}^{p_n} \leqslant c_{11} [f]_{s,\alpha}. \tag{4.9}$$

We can replace (see (1.1), (1.2)) the expression $|f|_S = \sum_{|l_i| \leqslant s_i} \sup |D^l f|$ by $\sum_{|l_i| \leqslant s_i} \|D^l f\|_{L^p(\Omega)}$ and thus, by (4.9) we deduce that

$$\|f\|_{m,p,\lambda} \leqslant c_{12} |f|_{s,\alpha}.$$

Hence the thesis follows.

Note that we obtain some characterization of the space $C^{s,\alpha}(\Omega)$ using the properties of $L_m^{p,\lambda}(\Omega)$-spaces. Application of this fact, to some generalization of the Marcinkiewicz theorem on interpolation, we can find in [3] and [11].

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