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Spectral inclusion theorems for commuting subnormal pair

Abstract. Let (S_1, S_2) be a subnormal pair of commuting operators with a minimal normal extension (N_1, N_2). The paper gives some spectral inclusion theorems for such pairs. They are analogous of the well-known spectral inclusion theorem of Halmos for one subnormal operator and also resemble Hartman–Wintner spectral inclusion for Toeplitz operators. The work depends essentially on the ideas of Keough [5].

I. Let H be a complex Hilbert space. Let us recall the definition of commuting subnormal pair S_1, S_2. We say that a commuting pair $S_1, S_2 \in L(H)$ of operators is *subnormal*, if there is a Hilbert space $K \supset H$ and a commuting pair of normal operators $N_1, N_2 \in L(K)$ such that for every $k, l \geqslant 0$

$$S_1^k S_2^l f = N_1^k N_2^l f, \quad f \in H.$$

The pair N_1, N_2 is called *minimal normal extension*, if

$$K = \bigvee_{k,l \geqslant 0} N_1^{*k} N_2^{*l} H$$

(where $\bigvee_{\alpha} M_\alpha$ stands for the closed linear span of subspaces M_α).

This is a natural extension of the notion of subnormal operator. If $S \in L(H)$ is a subnormal operator with a minimal normal extension N, then $\sigma(N) \subseteq \sigma(S)$ (¹), as was proved by Halmos [3]. A similar result was proved in [4] for a commuting subnormal pair of operators, namely

$$\sigma(N_1, N_2) \subseteq \sigma_{\mathscr{A}}(S_1, S_2),$$

here $\sigma_{\mathscr{A}}(S_1, S_2)$ is a joint spectrum defined with respect to the maximal, commutative Banach algebra $\mathscr{A}$ containing S_1, S_2, J. (J stands for the identity of $L(H)$.) The above inclusion is not quite satisfactory because it depends on the special choice of the joint spectrum of (S_1, S_2). There is a canonical notion of joint spectrum introduced by Taylor in [7]. Does the

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(¹) If $T \in L(H)$, then $\sigma(T)$ denotes the spectrum of T.

spectral inclusion theorem holds for this joint spectrum? We do not know the answer in general, but in special cases the answer is: *yes*. In fact we will prove in these special cases much more general spectral inclusion theorems for some generalized Toeplitz operators. In order to formulate our results let us recall the definition of joint approximate point spectrum of a commuting pair of operators.

Let T_1, T_2 be a commuting pair of operators in H.

DEFINITION 1. We say that $(\lambda_1, \lambda_2) \in \sigma_\pi(T_1, T_2)$ (joint approximate point spectrum of T_1, T_2) if and only if there exists a sequence $(h_n) \subset H$, $\|h_n\| = 1$ such that $\|(\lambda_i - T_i) h_n\| \xrightarrow[n\to\infty]{} 0$, $i = 1, 2$.

Remark 1. If $h_n = h$ and $\lambda_i h = T_i h$, $i = 1, 2$, then we say that (λ_1, λ_2) belongs to the joint point spectrum of (T_1, T_2). We denote the joint point spectrum of (T_1, T_2) by $\sigma_p(T_1, T_2)$.

We begin with a few simple lemmas.

LEMMA 1. *Let* $S_i \in L(H)$ $(i = 1, 2)$ *be a commuting subnormal pair with a minimal normal extension* $N_i \in L(K)$ $(i = 1, 2)$. *We have*

$$\sigma_p(N_1, N_2) \subseteq \overline{\sigma_p(S_1^*, S_2^*)},$$

where the bar denotes the complex conjugation.

Proof. Let P: $K \to H$ be orthogonal projection. If $(\lambda_1, \lambda_2) \in \sigma_p(N_1, N_2)$, then there exists $k \in K, k \neq 0$ s.t. $N_i k = \lambda_i k$, $i = 1, 2$. Denote by Q: $K \to H^\perp$ (²) the orthogonal projection. Since $N_i^* k = \bar{\lambda}_i k$, we can write

$$(N_i^* - \bar{\lambda}_i) k = 0 \Leftrightarrow (N_i^* - \bar{\lambda}_i) Pk = -(N_i^* - \bar{\lambda}_i) Qk.$$

There are two cases:

(a) $Pk \neq 0$. Then we have (note that $N_i^* H^\perp \subseteq H^\perp$)

$$P(N_i^* - \bar{\lambda}_i) Pk = -P(N_i^* - \bar{\lambda}_i) Qk = 0,$$

and so

$$S_i^* Pk = \bar{\lambda}_i Pk, \quad i = 1, 2,$$

i.e.

$$(\lambda_1, \lambda_2) \in \overline{\sigma_p(S_1^*, S_2^*)}.$$

(b) $Pk = 0$. This contradics the minimality of (N_1, N_2). Indeed, the subspace $[k]$ (spanned by the vector k) reduces N_1, N_2 and $k \perp H$. Therefore Pk must be different from zero.

The above cases complete the proof.

(²) $H^\perp$ stands for the orthogonal complement.

Remark 2. One would like to extend Lemma 1 (using the Berberian trick [1]) and prove the following inclusion

$$\sigma_\pi(N_1, N_2) \subseteq \overline{\sigma_\pi(S_1^*, S_2^*)}.$$

Unfortunately the Berberian trick does not preserves the minimality of normal extension.

Now let us consider the Taylor spectrum $\sigma(S_1, S_2)$ of (S_1, S_2), see [7] (3).

It is well known that for a given subnormal operator S whose spectrum $\sigma(S)$ does not separate the plane

$$\sigma(S) = \sigma(N) \cup (\text{all holes of } \sigma(N)),$$

where N is a minimal normal extension of S. Since $\sigma(N) \cup (\text{all holes of } \sigma(N))$ = polynomial convex hull of $\sigma(N) = \hat{\sigma}(N)$, we can write $\sigma(S) = \hat{\sigma}(N)$.

Now we prove the same equality for the Taylor spectrum.

PROPOSITION 1. *Let (S_1, S_2) and (N_1, N_2) be the same as in Lemma 1. We have the inclusion*

$$\sigma(S_1, S_2) \subseteq \hat{\sigma}(N_1, N_2),$$

where $\hat{\sigma}(N_1, N_2)$ denotes the polynomial convex hull of $\sigma(N_1, N_2)$.

Proof. Let $z \in \sigma(S_1, S_2) = R$ and let p be an arbitrary polynomial. By the spectral mapping for Taylor spectrum see [6] we have

$$\begin{aligned} |p(z)| &\leqslant \sup_{\zeta \in R} |p(\zeta)| = \sup\{|\lambda|,\ \lambda \in \sigma(p(S_1, S_2))\} \\ &= r(p(S_1, S_2)) = \|p(S_1, S_2)\| \leqslant \|p(N_1, N_2)\| \\ &= r(p(N_1; N_2)) = \sup_{\mathfrak{z} \in \sigma(N_1, N_2)} |p(\mathfrak{z})|, \end{aligned}$$

where $r(p(S_1, S_2))$ denotes the spectral radius of $p(S_1, S_2)$. Since p is arbitrary, the proof is complete.

COROLLARY 1. *If $\sigma(S_1, S_2)$ is polynomially convex set, then $\sigma(S_1, S_2) = \hat{\sigma}(N_1, N_2)$.*

Proof. Let $\mathcal{A}$ be the maximal, commutative Banach algebra containing S_1, S_2, I. By the work of Taylor [7] we known that for polynomially convex $\sigma(S_1, S_2)$ we have

$$\sigma(S_1, S_2) = \sigma_{\mathcal{A}}(S_1, S_2).$$

Applying Proposition 1 and Theorem 5 of [4] we obtain the desired equality.

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(3) See the Appendix.

The above corollary seems to confirm conjecture that for any subnormal pair (S_1, S_2)

$$\sigma(N_1, N_2) \subseteq \sigma(S_1, S_2).$$

Using the recent result of Vasilescu [8] we also prove the next remark.

Remark 3. Let $S_i \in L(H_i)$, $i = 1, 2$, be a subnormal operator. Denote by $N_i \in L(K_i)$ a minimal normal extension of S_i. We put

$$\tilde{S}_1 = \tilde{S}_1 \otimes I_{H_2}, \quad \tilde{S}_2 = I_{H_1} \otimes S_2,$$
$$\tilde{N}_1 = N_1 \otimes I_{K_2}, \quad \tilde{N}_2 = I_{K_1} \otimes N_2.$$

Then we have

$$\sigma(\tilde{N}_1, \tilde{N}_2) \subseteq \sigma(\tilde{S}_1, \tilde{S}_2).$$

Proof. Since $\sigma(N_i) \subseteq \sigma(S_i)$, $i = 1, 2$, applying Theorem 3.1 of [8] we get

$$\sigma(\tilde{N}_1, \tilde{N}_2) = \sigma(N_1) \times \sigma(N_2) \subseteq \sigma(S_1) \times \sigma(S_2) = \sigma(\tilde{S}_1, \tilde{S}_2).$$

Note that $(\tilde{N}_1, \tilde{N}_2)$ is a minimal normal extension of $(\tilde{S}_1, \tilde{S}_2)$.

Now we, are going to prove spectral inclusion theorems for some generalized Toeplitz operators. First of all note that the definition of approximate point spectrum has also the meaning for not necessary commuting operators T_1, T_2. But it is not longer true that $\sigma_\pi(T_1, T_2) \neq \emptyset$. Nevertheless in what follows we will consider $\sigma_\pi(T_1, T_2)$ for not necessary commuting T_1, T_2. We begin following the ideas of Keough [5] (after suitable modifications) with some necessary propositions.

Let S_1, S_2 be a subnormal commuting pair of quasi-normal operators (i.e., $S_i(S_i^* S_i) = (S_i^* S_i) S_i$, $i = 1, 2$) with a minimal normal extension N_1, N_2. Let $L_i \in \{N_1, N_2\}'$, $i = 1, 2$, where $\{N_1, N_2\}'$ stands for the commutant of N_1, N_2. For a given pair (k, l) of integer numbers we define

$$m_{k,l}(L_1, L_2) = \inf \{\|L_1 k\| + \|L_2 k\|,\ \|k\| = 1,\ k \in [N_1^{*k} N_2^{*l}]\}.$$

PROPOSITION 2. *For any* $k \geq 1, l \geq 0$ *we have*

$$m_{k,l}(L_1, L_2) \geq m_{k-1,l}(L_1, L_2).$$

Proof. Take $h \in H$ for which $\|N_1^{*k} N_2^{*l} h\| = 1$. We have

$$\|L_1 N_1^{*k} N_2^{*l} h\| + \|L_2 N_1^{*k} N_2^{*l} h\|$$
$$= \|L_1 N_1^{*(k-1)} N_2^{*l} N_1 h\| + \|L_2 N_1^{*(k-1)} N_2^{*l} N_1 h\|$$
$$\geq m_{k-1,l}(L_1, L_2) \|N_1^{*(k-1)} N_2^{*l} N_1 h\| = m_{k-1,l}(L_1, L_2).$$

Since h is arbitrary the inequality follows.

PROPOSITION 3. *Let* (S_1, S_2) *be a subnormal commuting pair with a* *minimal normal extension* (N_1, N_2). *Assume that* S_1, S_2 *are quasi-normal and* $\operatorname{Ker} S_1 = \operatorname{Ker} S_2 = \{0\}$.

Then $[N_i^* H] \supseteq H$, $i = 1, 2$.

Proof. The proof is the same as Lemma II.3.5 of [5] and is omitted.

As the corollary of the above proposition we obtain

COROLLARY 2. *For any* $k \geqslant 1$, $l \geqslant 0$,

$$m_{kl}(L_1, L_2) \leqslant m_{k-1,l}(L_1, L_2).$$

Proof. Since $[N_i^* H] \supseteq H$ the proof is immediate by the inclusion

$$[N_1^{*(k-1)} N_2^{*l} H] \subseteq [N_1^{*k} N_2^{*l} H].$$

Now we are able to prove (promised before) spectral inclusion theorem for generalized Toeplitz operators. Let $P\colon K \to H$ be the orthogonal projection. For $L \in \{N_1, N_2\}'$ we define generalized Toeplitz operator T_L on H by

$$T_L f = P\,L f, \quad f \in H.$$

THEOREM 1. *If* S_1, S_2 *is a commuting subnormal pair of quasi-normal operators, then for any* $L_i \in \{N_1, N_2\}'$, $i = 1, 2$,

$$\sigma_\pi(L_1, L_2) \subseteq \sigma_\pi(T_{L_1}, T_{L_2}).$$

Proof. (a) Assume first that $\operatorname{Ker} S_1 = \operatorname{Ker} S_2 = \{0\}$. By Corollary 2 and Proposition 2 we have

$$m_{ij}(L_1, L_2) = m_{00}(L_1, L_2), \quad i, j \geqslant 0.$$

If $(0, 0) \notin \sigma_\pi(T_{L_1}, T_{L_2})$, then $m_{00}(L_1, L_2) > 0$ (by the definition of $\sigma_\pi(T_{L_1}, T_{L_2})$). Since

$$K = \overline{\bigcup_{i,j \geqslant 0} [N_1^{*i} N_2^{*j} H]} \quad \text{(the closure)}.$$

Applying Proposition 3 we see that $(0, 0) \notin \sigma_\pi(L_1, L_2)$ (by the definition of $m_{00}(L_1, L_2)$).

(b) Let $\operatorname{Ker} S_i = M_i$. Writing the matrix of

$$N_i = \begin{pmatrix} S_i & A_i \\ 0 & B_i \end{pmatrix}, \quad i = 1, 2,$$

with respect to the decomposition $K = H \oplus H^\perp$, we obtain by the quasi-normality of S_i and a straightforward matrix computation the equality

$$\operatorname{Ker} N_i = \operatorname{Ker} S_i, \quad i = 1, 2.$$

Thus M_i $(i = 1, 2)$ reduces both S_1 and S_2 and we have the following decompositions

$$S_1 = \bigoplus_{k=0}^{3} S_1^{(k)} = 0 \oplus 0 \oplus S_1^{(2)} \oplus S_1^{(3)}, \qquad S_2 = \bigoplus_{k=0}^{3} S_2^{(k)} = 0 \oplus S_2^{(1)} \oplus 0 \oplus S_2^{(3)},$$

where $S_1^{(2)}$, $S_1^{(3)}$, $S_2^{(1)}$, $S_2^{(3)}$ are injective operators. Since (N_1, N_2) is minimal we have

$$N_1 = \bigoplus_{k=0}^{3} N_1^{(k)} = 0 \oplus 0 \oplus N_1^{(2)} \oplus N_1^{(3)}, \qquad N_2 = \bigoplus_{k=0}^{3} N_2^{(k)} = 0 \oplus N_2^{(1)} \oplus 0 \oplus N_2^{(3)},$$

where $(N_1^{(k)}, N_2^{(k)})$ is a minimal normal extension of $(S_1^{(k)}, S_2^{(k)})$. Now it is easy to see that L and T_L are of the form

$$L = \bigoplus_{k=0}^{3} L^{(k)}, \qquad T_L = \bigoplus_{k=0}^{3} T_{L^{(k)}},$$

where $L^{(k)} = (L_1^{(k)}, L_2^{(k)})$, $L_p^{(k)} \in \{N_1^{(k)}, N_2^{(k)}\}'$. Thus we have

$$\sigma_\pi(L) = \bigcup_{k=0}^{3} \sigma_\pi(L^{(k)}), \qquad \sigma_\pi(T_L) = \bigcup_{k=0}^{3} \sigma_\pi(T_{L^{(k)}}).$$

To prove the theorem it is sufficient to show that the inclusion $\sigma_\pi(L) \subseteq \sigma_\pi(T_L)$ holds true, or equivalently, to resume the hypotesis of the theorem to the following three cases:

1° $S_1 = S_2 = 0$, 2° $S_1 = 0$, $\operatorname{Ker} S_2 = \{0\}$, 3° $\operatorname{Ker} S_1 = \operatorname{Ker} S_2 = \{0\}$.

Cases 1° and 2° are simple. Indeed, we have respectively:

1° $N_1 = N_2 = 0$ and 2° $N_1 = 0$ and N_2 is a minimal normal extension of S_2.

Hence in case 1° there is nothing to prove and in case 2° the proof is an easy adaptation of the above proof of case (a). Case 3° was considered above as case (a).

The proof is complete.

Note that for commuting L_1, L_2 we have $\sigma_\pi(L_1, L_2) \neq \emptyset$. Thus from the above theorem we derive

COROLLARY 3. *For any commuting pair*

$$L_i \in \{N_1, N_2\}', \qquad \sigma_\pi(T_{L_1}, T_{L_2}) \neq \emptyset.$$

This is not obvious, because T_{L_1}, T_{L_2} do not commute in general. In particular case for $L_i = N_i$, $i = 1, 2$, we have

$$\sigma_\pi(N_1, N_2) \subseteq \sigma_\pi(S_1, S_2).$$

Since

$$\overline{\sigma_\pi(N_1^*, N_2^*)} = \sigma_\pi(N_1, N_2) \subseteq \sigma_\pi(S_1, S_2) \subseteq \overline{\sigma_\pi(S_1^*, S_2^*)}$$

and

$$\sigma_\pi(N_1, N_2) = \sigma(N_1, N_2)$$

we also obtain

COROLLARY 4. *If (S_1, S_2) is a quasi-normal commuting subnormal pair with a minimal normal extension (N_1, N_2), then*

$$\sigma(N_1, N_2) \subseteq \sigma(S_1, S_2).$$

Proof. By the result of [8] (see also [2]) we know that

$$\sigma_\pi(S_1, S_2) \cup \overline{\sigma_\pi(S_1^*, S_2^*)} \subseteq \sigma(S_1, S_2)$$

and the proof is immediate by the preceding remarks.

In what follows we shall give a simple relation between spectral inclusion theorem for subnormal pair and the above theorem. Let (S_1, S_2) be a commuting subnormal pair with a minimal normal extension (N_1, N_2). Assume that $\sigma(N_1, N_2) \subseteq \sigma(S_1, S_2)$. We have the following consequences:

(i) *For any* $L_i \in \{N_1, N_2\}'$ *such that* $L_1 L_2 = L_2 L_1$, L_1, L_2 *are normal and* $L_i H \subseteq H$ $(i = 1, 2)$, $\sigma(L_1, L_2) \subseteq \sigma(T_{L_1}, T_{L_2})$;

(ii) *For any polynomial* $p(z_1, z_2)$, $\|p(T_{L_1}, T_{L_2})\| = \|p(L_1, L_2)\|$;

(iii) $\sigma(T_{L_1}, T_{L_2}) \subseteq \hat{\sigma}(L_1, L_2)$.

Proof. (i) The proof is a simple modification of the proof of Proposition II.4.4, of [5].

(ii) This is immediate because $p(T_{L_1}, T_{L_2}) = T_p(L_1, L_2)$.

(iii) A similar reasoning as in the proof of Proposition 1.

II. Now we shall consider (following Keough) spectral inclusion property for S_1, S_2 of different type. Namely we have the following

DEFINITION 2. We say that a given commuting subnormal pair (S_1, S_2) with a minimal normal extension (N_1, N_2) has *C^*-Spectral Inclusion Property* (SIP), if for any $L_1, L_2 \in C^*(N_1, N_2)$ (C^* algebra generated by N_1, N_2, J)

$$\sigma(L_1, L_2) \subseteq \sigma_\pi(T_{L_1}, T_{L_2}).$$

We are going to give some equivalent conditions to SIP. First let us recall the notion of the angle $\alpha(M_1, M_2)$ between two closed subspaces $M_1, M_2 \subset H$, $M_1 \cap M_2 = \{0\}$. Namely

$$\cos\alpha(M_1, M_2) = \sup_{\substack{m_i \in M \\ \|m_i\| = 1}} |(m_1, m_2)|.$$

Before stating the promised equivalent conditions to SIP we prove the following

PROPOSITION 4. *Let* $\mathscr{C} = \{(L_1, L_2), L_i \in C^*(N_1, N_2)$, s.t. $\sigma(L_1, L_2) \subseteq \sigma_\pi(T_{L_1}, T_{L_2})\}$. $\mathscr{C}$ *is normed closed.*

Proof. Let $L_i^{(n)} = f_i^{(n)}(N_1, N_2) \in \mathscr{C}$, where $f_i^{(m)} \in C(\sigma(N_1, N_2))$. Assume that

$$L_i^{(n)} \xrightarrow[n \to \infty]{} L_i = f_i(N_1, N_2), \qquad i = 1, 2.$$

We have to show that $(L_1, L_2) \in \mathscr{C}$. Take $\varepsilon > 0$. Assume that dist $((0, 0), \sigma_\pi(T_{f_1}, T_{f_2})) > 3\varepsilon$, where dist $((0, 0), (\lambda_1, \lambda_2)) = |\lambda_1| + |\lambda_2|$.

Let $0 = \{z = (z_1, z_2) \in C^2, \text{dist}\,(z, \sigma_\pi(T_{f_1}, T_{f_2})) < \varepsilon\}$. By the semi-continuity of the mapping $(T_1, T_2) \to \sigma_\pi(T_1, T_2)$ (which can be easly proved by the definition of $\sigma_\pi(T_1, T_2)$) there exists $\delta > 0$ s.t.

$$\sigma_\pi(A_1, A_2) \subseteq 0, \quad \text{when} \quad \|A_i - T_{f_i}\| < \delta, \ i = 1, 2.$$

Thus

$$\sigma_\pi(T_{f_1^{(n)}}, T_{f_2^{(n)}}) \subseteq 0, \quad \text{when} \quad n \geqslant n_0$$

(because $\|f_i^{(n)} - f_i\| < \min(\varepsilon, \delta)$ for $n \geqslant n_0$). By our assumption we obtain (for $n \geqslant n_0$) $\sigma(L_1^{(n)}, L_2^{(n)}) \subseteq 0$. It follows that

$$(*) \qquad |\lambda_1| + |\lambda_2| > 2\varepsilon \quad \text{for } (\lambda_1, \lambda_2) \in \sigma(L_1^{(n)}, L_2^{(n)}) \text{ and } n \geqslant n_0.$$

Assume for contrary that

$$\begin{aligned}(0, 0) \in \sigma(L_1, L_2) &= \sigma(f_1(N_1, N_2), f_2(N_1, N_2)) \\ &= \{(f_1(z), f_2(z)), z \in \sigma(N_1, N_2)\}.\end{aligned}$$

Let μ be a scalar spectral measure for (N_1, N_2). Then for

$$\Delta = \{\lambda \in \sigma(N_1, N_2), |f_1(\lambda)| + |f_2(\lambda)| < \varepsilon\}$$

we have $\mu(\Delta) > 0$.

Pick up n_1 s.t.

$$|f_i| \geqslant |f_i^{(n)}| - \tfrac{1}{2}\varepsilon \quad \text{for } n \geqslant n_1, \ i = 1, 2.$$

Now for $n \geqslant \max(n_0, n_1)$ we have (by $(*)$)

$$\begin{aligned}\varepsilon \cdot \mu(\Delta) > \int_\Delta (|f_1| + |f_2|)\, d\mu &\geqslant \int_\Delta (|f_1^{(n)}| \pm |f_2^{(n)}| - \varepsilon)\, d\mu \\ &\geqslant \int_\Delta (2\varepsilon - \varepsilon)\, d\mu = \varepsilon \cdot \mu(\Delta).\end{aligned}$$

This contradiction proves the proposition.

Now we are ready to formulate the next theorem (this is a result analogous to III, Theorem 5.1 of [5]).

THEOREM 2. *Let E be the spectral measure for (N_1, N_2). The following conditions are equivalent*:

(i) (S_1, S_2) *has SIP*,

(ii) $\alpha(E(\Delta)K, H) = 0$ *for every relatively open set* $\Delta \subseteq \sigma(N_1, N_2)$,

(iii) $\sigma(N_1, N_2) = \sigma_\pi(S_1, S_2)$.

Proof. (i) $\Rightarrow$ (ii) $\Rightarrow$ (iii).

The proofs of the above implications are an obvious adaptation of a suitable part of the proof of III, Theorem 5.1 of [5].

(iii) ⇒ (i).

By Proposition 4 it is enough to show that for any polynomials (in $z_1, \bar{z}_1, z_2, \bar{z}_2$) $p_i(z, \bar{z})$, $i = 1, 2$, we have

$$(p_1(N^*, N), p_2(N^*, N)) \subseteq \sigma_\pi(T_{p_1(N^*,N)}, T_{p_2(N^*,N)}).$$

Let $p_r(z, \bar{z}) = \hat{p}_r(z, \bar{z})$, where $\hat{p}_r(z, \bar{z})$ is the same polynomial written with the terms grouped as

$$\hat{p}_r(z, \bar{z}) = \sum_{k=0}^{m} \Big(\sum_{i+j+s+t=k} \alpha_{ijst}^{(r)} \bar{z}_1^i \bar{z}_2^j z_1^s z_2^t \Big), \qquad r = 1, 2.$$

A straightforward matrix calculation shows that

$$T_{p_r(N^*,N)} = T_{\hat{p}_r(N^*,N)} = \hat{p}_r(S^*, S), \qquad r = 1, 2.$$

By assumption we can write

$$\begin{aligned}\sigma(p_1(N^*, N), p_2(N^*, N)) &= \{(p_1(\bar{\lambda}, \lambda), p_2(\bar{\lambda}, \lambda)), \lambda \in \sigma(N_1, N_2)\} \\ &= \{(p_1(\bar{\lambda}, \lambda), p_2(\bar{\lambda}, \lambda)), \lambda \in \sigma_\pi(S_1, S_2)\}.\end{aligned}$$

Therefore it is sufficient to prove that

$$\{(p_1(\bar{\lambda}, \lambda), p_2(\bar{\lambda}, \lambda)), \lambda \in \sigma_\pi(S_1, S_2)\} \subseteq \sigma_\pi(\hat{p}_1(S^*, S), \hat{p}_2(S^*, S)).$$

Now we can write:

$$\begin{aligned}(+) \qquad & S_1^{*i} S_2^{*j} S_1^s S_2^t - \bar{\lambda}_1^i \bar{\lambda}_2^j \lambda_1^s \lambda_2^t \\ &= S_1^{*i} S_2^{*j} S_1^s (S_2^t - \lambda_2^t) + \lambda_2^t S_1^{*i} S_2^{*j} (S_1^s - \lambda_1^s) + \\ &\quad + \lambda_1^s \lambda_2^t S_1^{*i} (S_2^{*j} - \bar{\lambda}_2^j) + \lambda_1^s \lambda_2^t \bar{\lambda}_2^j (S_1^{*i} - \bar{\lambda}_1^i).\end{aligned}$$

Since S_i are subnormal so $\|(S_p^* - \bar{\lambda}_p) h\| \leqslant \|(S_p - \lambda_p) h\|$, $h \in H$. If $(\lambda_1, \lambda_2) \in \sigma_\pi(S_1, S_2)$, then for a certain sequence

$$(h_p) \subset H, \qquad \|h_p\| = 1, \qquad \|(\lambda_r - S_r) h_p\| \underset{p \to \infty}{\longrightarrow} 0, \qquad r = 1, 2.$$

By (+) and the trivial equality $\hat{p}_r(\bar{\lambda}, \lambda) = p_r(\bar{\lambda}, \lambda)$, $r = 1, 2$, we obtain

$$\|(\hat{p}_r(S^*, S) - \hat{p}_r(\bar{\lambda}, \lambda)) h_p\| \underset{p \to \infty}{\longrightarrow} 0, \qquad r = 1, 2.$$

Thus $(p_1(\bar{\lambda}, \lambda), p_2(\bar{\lambda}, \lambda)) \in \sigma_\pi(\hat{p}_1(S^*, S), \hat{p}_2(S^*, S))$ and the proof is complete.

We can also give a different condition which is equivalent to SIP.

Namely for $L_i \in C^*(N_1, N_2)$ $(i = 1, 2)$ we define

$$\sigma_H(L_1, L_2) = \{(\lambda_1, \lambda_2) \in C^2,\ \exists\ (h_p) \subset H,$$
$$\|h_p\| = 1, \text{ such that } \|(\lambda_i - L_i) h_p\| \xrightarrow[p \to \infty]{} 0,\ i = 1, 2\}.$$

We conclude our paper with the following

PROPOSITION 5. *For any* $L_1, L_2 \in C^*(N_1, N_2)$

$$\sigma(L_1, L_2) \subseteq \sigma_\pi(T_{L_1}, T_{L_2}) \Leftrightarrow \sigma(L_1, L_2) = \sigma_H(L_1, L_2).$$

Proof. The implication $\Leftarrow$ is clear. Suppose now that $L_i = \varphi_i(N_1, N_2) \in C^*(N_1, N_2)$, $\varphi_i \in C(\sigma(N_1, N_2))$. Let $\lambda \in \sigma(L_1, L_2)$. Take $\Delta_n \subseteq \sigma(L_1, L_2)$ relatively open sets s.t. $\lambda \in \bigcap_{n \geqslant 1} \Delta_n$ and $\operatorname{diam} \Delta_n \xrightarrow[n \to \infty]{} 0$. Put $\Phi = (\varphi_1, \varphi_2)$. We see that $\Phi^{-1}(\Delta_n)$ is relatively open set. Applying Theorem 2 there exists $(h_n) \subset H$, $\|h_n\| = 1$, s.t.

$$\|E(\Phi^{-1}(\Delta_n)) h_n\| > \sqrt{1 - 1/n}, \quad n = 1, 2, \ldots$$

Thus

$$\|(L_i - \lambda_i) h_n\|^2 = \int\limits_{\Phi^{-1}(\Delta_n)} |\varphi_i(z) - \lambda_i|^2 \, d(Eh_n, h_n) +$$
$$+ \int\limits_{\sigma(N_1, N_2) \setminus \Phi^{-1}(\Delta_n)} |\varphi_i(z) - \lambda_i|^2 \, d(Eh_n, h_n) \leqslant (\operatorname{diam} \Delta_n)^2 + \frac{4\|\varphi_i\|_\infty^2}{n} \xrightarrow[n \to \infty]{} 0,$$
$$i = 1, 2.$$

The proof is complete.

Appendix. For a convenience of the reader we shall recall now the definition of a joint spectrum of two commuting operators given by Taylor [7].

Let X be a Banach space. Suppose we are given two commuting operators $T_i \in L(X)$, $i = 1, 2$. Consider the sequence

$$(*) \qquad 0 \to X \xrightarrow{\delta_1} X \oplus X \xrightarrow{\delta_0} X \to 0,$$

where $\delta_1(x) = (-T_2 x, T_1 x)$ and $\delta_0(x_1, x_2) = T_1 x_1 + T_2 x_2$.

Since $T_1 T_2 = T_2 T_1$ it is obvious that $\delta_0 \circ \delta_1 = 0$ so that $(*)$ is a chain complex.

We say that (T_1, T_2) is *non-singular* if $(*)$ is exact. Now the definition of $\sigma(T_1, T_2)$ is clear. Namely $\sigma(T_1, T_2)$ is equal to the set of all $(\lambda_1, \lambda_2) \in C^*$ for which $(T_1 - \lambda_1, T_2 - \lambda_2)$ is singular.

Added in proof. The inclusion $\sigma(N_1, \ldots, N_k) \subset \sigma(S_1, \ldots, S_k)$ holds true for only commuting subnormal system $(S_1, \ldots, S_k)$, as has been proved by M. Putinar (preprint).

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