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On the range of atomless vector measures

Abstract. Liapunov convexity theorem and its infinite dimension generalization by Knowles are applied to the study of atomless measures with values in a non-locally convex space. Conditions are given under which the range of such a measure is contained in a convex bounded set, or is itself convex and bounded. The paper gives also a type of Uniform Boundedness Principle. It ends with a list of a few counter-examples.

1. Introduction. We consider measures defined on a σ-field $\mathscr{A}$ of sets and with values in a topological vector space E, generally non-locally convex.

It is important to know whether the range $\mu(\mathscr{A})$ of a measure $\mu: \mathscr{A} \to E$ is included in a convex bounded subset of E ([21], [24], [27]). When this is the case we say that μ is *convexly bounded.* After [22] this problem has been investigated in several papers (see, for example, [4], [5], [14], [24]–[28] and their bibliographies).

In the present work we are mainly interested with atomless measures. The starting point was the following result, a consequence of the Liapunov theorem. If E is separated by its topological dual and is weakly polar, in other words if it has a basis of weakly closed neighbourhoods of zero, then every atomless countably additive measure $\mu: \mathscr{A} \to E$ verifying the so-called *countable chain condition* (Definition 2.3) is convexly bounded (Theorem 7.3).

This may be false for atomic measures (Example 8.5), or for atomless measures with values in non-weakly polar spaces with separating dual (Examples 8.1 and 8.4).

Such pathologies are supplied by measures μ with injective associated integral mapping $f \to \int f d\mu$. A similar phenomenon occurs with measures for which the Liapunov theorem does not apply.

And we get some positive results using the Liapunov theorem generalizations of [8] and [10] in the following way. We introduce the notion of a *convexly Liapunov measure*, i.e. a measure factorizing through a Liapunov measure (in the sense of [9], [10]) with values in some locally convex space (a finite rank atomless countably additive measure is a particular case). This notion is investigated.

A "non-injectivity" condition is given which implies (under suitable assumptions) that a measure is convexly Liapunov (Theorem 4.4). A measure will easily be convexly Liapunov (whence convexly bounded) for a given topology if it is so for some other Hausdorff topology (Theorem 5.3). We have also a uniform boundedness principle for convexly Liapunov measures (Theorem 6.1) which implies that a pointwise limit of a sequence of convexly Liapunov measures is convexly bounded (see a related counter-example in 8.2 and 8.3). A characterization of the convexly Liapunov measures is given in 5.5.

Main results being given in Sections 4, 5, 6, the particular case of measures with values in spaces separated by their topological dual is examined in Section 7, and counter-examples are gathered in the last section.

2. Preliminaries.

2.1. In this paper, $\mathscr{A}$ will be a σ-field of subsets of a set T.

If $A \in \mathscr{A}$, we let $A \cap \mathscr{A} = \{B \in \mathscr{A} \mid B \subset A\}$.

If E is a *topological vector space* (t.v.s.) a measure $\mu\colon \mathscr{A} \to E$ is an additive set function from $\mathscr{A}$ to E. When $\mu(A) = \sum\limits_{0}^{\infty} \mu(A_n)$ whenever A is the union of a disjoint sequence (A_n) of $\mathscr{A}$, we say that μ is *countably additive.*

All topological vector spaces will be assumed to be real.

The convex hull of a subset X of a vector space is denoted by $\operatorname{co} X$.

2.2. If μ is a measure on $\mathscr{A}$ with values in a Hausdorff t.v.s., a set $A \in \mathscr{A}$ is said to be *μ-negligible* when $\mu(B) = 0$ for every $B \in A \cap \mathscr{A}$.

If M is a set of measures on $\mathscr{A}$, a set $A \in \mathscr{A}$ is *M-negligible* when it is μ-negligible for every $\mu \in M$.

2.3. We say that a measure μ on $\mathscr{A}$ (resp. a set M of measures on $\mathscr{A}$) verifies the *countable chain condition* (cf. [3], [17], [18]) when every family of pairwise disjoint non-μ-negligible (resp. non-M-negligible) elements of $\mathscr{A}$ is finite or countable.

For example, μ verifies the countable chain condition when it is countably additive for some metrizable linear topology on the space of values.

2.4. Let $\lambda\colon \mathscr{A} \to D$ and $\mu\colon \mathscr{A} \to E$ be measures with values in Hausdorff t.v.s.'s D and E. Then μ is said to be *λ-continuous* when, for every zero-neighbourhood V in E, there exists a zero-neighbourhood U in D such that $\mu(A) \in V$ whenever $A \in \mathscr{A}$ and $\lambda(A \cap \mathscr{A}) \subset U$.

We say that λ and μ are *equivalent* when they are mutually continuous.

For the following statement see [3], 2.1 and [1], 8.5.

THEOREM. *Let us assume that λ and μ are countably additive and that λ verifies the countable chain condition.*

Then μ is λ-continuous if (and only if) every λ-negligible set $A \in \mathscr{A}$ is μ-negligible.

2.5. Clearly, a measure μ verifies the countable chain condition when it is a-continuous for some positive finite countably additive measure a. Conversely, we have the following generalization of a theorem of Bartle, Dunford and Schwartz.

THEOREM (Drewnowski [3], Musiał [18]). *If E is a locally convex Hausdorff t.v.s., every E-valued countably additive measure on $\mathscr{A}$ verifying the countable chain condition is equivalent to some positive finite countably additive measure a on $\mathscr{A}$.*

It is not known whether this remains true when E is an arbitrary Hausdorff t.v.s. This is equivalent to a problem of Maharam (cf. [3]).

COROLLARY. *Let $\mu_i\colon \mathscr{A}\to E_i$, $i\in I$, be a family of measures with values in Hausdorff t.v.s.'s E_i. We assume that $(\mu_i)_{i\in I}$ verifies the countable chain condition and that $\mu_i = u_i\circ\lambda_i$ for each $i\in I$, where λ_i is a countably additive measure on $\mathscr{A}$ with values in some locally convex space D_i and $u_i\colon D_i\to E_i$ is linear and continuous.*

Then there exists a probability measure a on $\mathscr{A}$ such that every μ_i, $i\in I$, is a-continuous.

Proof. The product D of the quotient spaces $D_i/\ker u_i$ is a Hausdorff locally convex space and the measure $\mu\colon A\to\big(\mu_i(A)\big)_{i\in I}$ from $\mathscr{A}$ to the product space $E = \prod_{i\in I} E_i$ verifies $\mu = u\circ\lambda$ for a countably additive measure $\lambda\colon\mathscr{A}\to D$ and a linear continuous injection $u\colon D\to E$. The countable chain condition is verified by μ, and therefore by λ since u is injective. We can take for a a probability measure equivalent to λ.

2.6. A measure μ on $\mathscr{A}$ is said to be *atomless* when every non μ-negligible set $A\in\mathscr{A}$ contains two disjoint non μ-negligible elements of $\mathscr{A}$.

PROPOSITION. *If λ and μ are measures on $\mathscr{A}$ with values in Hausdorff t.v.s.'s D and E, if λ is atomless, countably additive and fulfils the countable chain condition and if μ is λ-continuous, then μ is atomless.*

Proof. If $A\in\mathscr{A}$ is not μ-negligible, then $\mu(A_0)\neq 0$ for some $A_0\in A\cap\mathscr{A}$. Let U and V be zero-neighbourhoods in D and E verifying $\mu(A_0)\notin V$ and $\lambda(B\cap\mathscr{A})\subset U\Rightarrow\mu(B)\in V$ for $B\in\mathscr{A}$. By [17], Proposition 2, one can find a finite sequence of pairwise disjont sets $A_i\in\mathscr{A}$, $1\leqslant i\leqslant j$, verifying $\lambda(A_i\cap\mathscr{A})\subset U$ (whence $\mu(A_i)\neq\mu(A_0)$) and $A_0 = \bigcup_{i=1}^{j} A_i$. This shows that at least two sets A_i are not μ-negligible.

2.7. We shall use a result of Kalton (Theorem 2.3 of [6]) given in Corollary 2.8 below. When the spaces are not metrizable the proof of [6] seems to be too much elliptic and we complete it. First we deduce the following result from Lemma 2.1 of [6]. The (topological) dual of a space D is denoted by D'.

THEOREM. *Let S be a weakly compact subset of a locally convex space D, let E be a Hausdorff t.v.s. and let $u\colon D\to E$ be a continuous linear map.*

Then the graph of the restriction $u_S\colon S\to E$ is closed in $S\times E$ if S is endowed with the weak topology $\sigma(S, D')$.

Proof. Let us assume first that D and E are metrizable.

Let (x, y) belong to the closure in $S\times E$ of the graph of u_S and let $(V_n)_{n\geqslant 1}$ be a fundamental decreasing sequence of neighbourhoods of y in E. It is enough to prove that $u(x)\in V_1$. First we use Kaplansky's theorem ([11]) with a slight refinement. We consider a covering of D' by an increasing sequence of $\sigma(D', D)$-compact sets K_n, $n\geqslant 1$. Since the sets K_n are compact, we can find finite sets $H_n\subset S$, $n\geqslant 1$, verifying the following condition: $u(H_n)\subset V_n$ and for each n-tuple $(f_1, \ldots, f_n)$ of elements of K_n there exists $z\in H_n$ such that $|f_i(x)-f_i(z)|<1/n$ for $1\leqslant i\leqslant n$. Clearly, x belongs to the weak closure of the at most countable subset $H = \bigcup_{n\geqslant 1} H_n$ of S. By a Šmulian–Dieudonné–Schwartz argument ([11], p. 312) x is the weak limit of a sequence (x_h) of points of H. If $x\in H$, then $u(x)\in V_1$. If not, $\{h\geqslant 0\mid x_h\in H_n\}$ is finite for every n, so $u(x_h)$ tends to y in E, whence $u(x) = y$ by [6].

In the general case, E is embedded in a product of complete metrizable t.v.s.'s E_i, $i\in I$. If $q_i\colon E\to E_i$ is the canonical projection, $q_i\circ u = u_i\circ p_i$ where p_i is a continuous linear map $D\to D_i$ with values in some complete metrizable locally convex space D_i and $u_i\colon D_i\to E_i$ is linear and continuous. Each $S_i = p_i(S)$, $i\in I$, is $\sigma(S_i, D_i')$-compact, so the graph of the restriction of u_i to $\big(S_i, \sigma(S_i, D_i')\big)$ is closed and we easily conclude that u_S has a closed graph.

2.8. COROLLARY (Kalton). *If u is a continuous linear mapping of a locally convex space D into a Hausdorff t.v.s. E, then, for every weakly compact subset S of D, $u(S)$ is a complete subset of E.*

Moreover, u is still continuous for some Hausdorff locally convex topology s on E which induces on every set $u(S)$, S weakly compact subset of D, a compact topology coarser than the initial topology of E.

Proof. S being weakly compact in D, let φ be a filter of $u(S)$ tending to a point y of the completion $E^\wedge$ of E and let $x\in S$ be a $\sigma(S, D')$-cluster point of the filter basis $u_S^{-1}(\varphi)$ of S. Clearly, (x, y) is adherent to the (closed) graph of $u_S\colon \big(S, \sigma(S, D')\big)\to E^\wedge$. So $y = u(x)$ belongs to $u(S)$ and $u(S)$ is complete.

Furthermore, let t be the finest locally convex topology on E for which u is continuous. Since $\ker u$ is closed, t is Hausdorff. If $S\subset D$ is weakly compact and $M\subset u(S)$ is weakly closed in (E, t), then M is closed in E since it is the image of the weakly compact subset $S\cap u^{-1}(M)$ of D. So we can take for s the weak topology of (E, t).

3. Convexly bounded measures.

3.1. DEFINITION. *If E is a t.v.s. a measure $\mu\colon \mathscr{A}\to E$ is said to be convexly bounded when the convex hull* $\operatorname{co}\mu(\mathscr{A})$ *of the range $\mu(\mathscr{A})$ of μ is bounded in E (i.e. absorbed by every zero-neighbourhood).*

Every countably additive measure with values in a locally convex space in convexly bounded, but this is false in general for non-locally convex spaces ([22] and Section 8 infra).

3.2. If $\mu\colon \mathscr{A}\to E$ is countably additive (with E Hausdorff), we denote by $L^\infty(\mu)$ the Banach space of μ-classes of bounded $\mathscr{A}$-measurable real functions on T, the norm being the μ-essential supremum $\|\cdot\|_\infty$ (this is a norm since a countable union of μ-negligible sets is μ-negligible).

Now, let us assume that E is Hausdorff and sequentially complete and that $\mu\colon \mathscr{A}\to E$ is convexly bounded and countably additive. Then, denoting by χ_A the μ-class of the characteristic functions of a set $A\in\mathscr{A}$, we have the following proposition ([24], Chap. VII, [27]; see also [19]),

PROPOSITION. (a) *There exists a unique continuous linear mapping $\mu^\sim\colon f\to\int f d\mu$ of $L^\infty(\mu)$ into E such that $\mu^\sim(\chi_A)=\mu(A)$ for every $A\in\mathscr{A}$.*

(b) *$\mu^\sim$ maps $\{f\in L^\infty(\mu)\mid 0\leqslant f\leqslant 1\}$ into the closed convex hull $\overline{\operatorname{co}}\,\mu(\mathscr{A})$ of $\mu(\mathscr{A})$.*

(c) *If (f_n) is a bounded sequence of $L^\infty(\mu)$ converging μ-a.e. to a function f then $\mu^\sim(f_n)\to\mu^\sim(f)$ (dominated convergence property).*

Let us observe that, using Kalton's arguments (Corollary 2.8 and proof of Theorem 3.3 below), we can replace "into" by "onto" in (b) if μ is equivalent to some probability measure on $\mathscr{A}$.

3.3. THEOREM (Kalton [6]). *Let E be a Hausdorff t.v.s. and $\mu\colon \mathscr{A}\to E$ a convexly bounded measure, a-continuous for some positive finite countably additive measure a on $\mathscr{A}$.*

Then $\mu=u\circ\lambda$, where $\lambda\colon \mathscr{A}\to D$ is an a-continuous (hence countably additive) measure with values in some locally convex t.v.s. D and $u\colon D\to E$ is a continuous linear injection.

If E is complete, D may be taken complete.

Proof. We follow Kalton (only completeness of D is added). Let t be the finest linear topology on $L^\infty(a)$ agreeing with the topology of $L^1(a)$ on the closed unit ball B of $L^\infty(a)$.

Let us assume first that E is complete. By the dominated convergence property the linear map $I\colon f\to\int f d\mu$ from $L^\infty(a)$ to E is t-continuous. If D is the quotient t.v.s. $\big(L^\infty(a),t\big)/\ker I$, then $I=u\circ q$ where $q\colon L^\infty(a)\to D$ is the quotient mapping and $u\colon D\to E$ is a continuous linear injection. The measure $\lambda\colon \mathscr{A}\to D$ defined by $\lambda(A)=q(\chi_A)$ is clearly a-continuous. The spaces $\big(L^\infty(a),t\big)$ and therefore D are locally convex. Furthermore, D is complete. Indeed, if $B_n=\{f\in L^\infty(a)\mid \|f\|_\infty\leqslant n\}$ for n integer and if s_n is the topology induced on $q(B_n)$ by the topology s of D, then s is the finest linear topology on D inducing the s_n's on the $q(B_n)$'s. But the B_n's are weakly compact in $(L^\infty(\alpha), t)$, because the dual of $(L^\infty(\alpha), t)$ is $L^1(\alpha)$. So $q(B_n)$ is weakly compact and consequently, complete in D. By [20] or [24], Chap. I, D is therefore complete.

If E is not complete, the above construction gives a factorization of μ through a linear injection v of some locally convex space D_1 into the completion $E^\wedge$ of E and we take $D = v^{-1}(E)$ and $u = v_D$.

4. Convexly Liapunov measures.

4.1. In [9] and [10], a measure $\lambda: \mathscr{A} \to D$, where D is a Hausdorff locally convex t.v.s., is called a *Liapunov measure* when it is countably additive and $\lambda(A \cap \mathscr{A})$ is convex and weakly compact for every $A \in \mathscr{A}$.

DEFINITION. When E is a Hausdorff t.v.s., we say that $\mu: \mathscr{A} \to E$ is *convexly Liapunov* when there exists a locally convex space D, a Liapunov measure $\lambda: \mathscr{A} \to D$ and a continuous linear mapping $u: D \to E$ verifying $\mu = u \circ \lambda$.

For example, by the classical Liapunov theorem ([15], [9]), $\mu: \mathscr{A} \to E$ is convexly Liapunov if μ is countably additive, atomless and of finite rank (i.e. if the linear span of $\mu(\mathscr{A})$ is of finite dimension).

Of course, every convexly Liapunov measure is convexly bounded and countably additive.

4.2. PROPOSITION. *If $\mu: \mathscr{A} \to E$ is convexly Liapunov then, for every $A \in \mathscr{A}$, $\mu(A \cap \mathscr{A})$ is a complete bounded convex subset of E.*

This is a consequence of Corollary 2.8.

4.3. Let $\mu: \mathscr{A} \to E$ be a convexly bounded countably additive measure with values in a Hausdorff t.v.s. E with completion $E^\wedge$.

For every $A \in \mathscr{A}$, we denote by μ_A the measure $B \to \mu(A \cap B)$ on $\mathscr{A}$.

DEFINITION. We say that *μ has a large kernel* when, for every non-μ-negligible set $A \in \mathscr{A}$, the map $\tilde{\mu}_A: f \to \int f d\mu_A$ of $L^\infty(\mu_A)$ into $E^\wedge$ is not injective.

The measure μ is atomless if it has a large kernel (if A is an atom of μ, $\tilde{\mu}_A$ is injective). Vector measures of the form $A \to \chi_A$ show that the converse is false in general, when E is of infinite dimension.

And μ has a large kernel if it is convexly Liapunov or, more generally, if for example $\frac{1}{2}\mu(A) \in \mu(A \cap \mathscr{A})$ for every $A \in \mathscr{A}$.

Thanks to Theorem 2.5 the following result is a particular case of a theorem of Knowles ([10], [9], p. 82); see also [8].

THEOREM. *If E is a Hausdorff locally convex space, a countably additive measure $\mu: \mathscr{A} \to E$ verifying the countable chain condition is Liapunov if it has a large kernel.*

4.4. This can be generalized in following way.

THEOREM. *Let E be a Hausdorff t.v.s. and let $\mu\colon \mathscr{A}\to E$ be a convexly bounded measure, a-continuous for some positive finite countably additive measure a on $\mathscr{A}$.*

Then μ is convexly Liapunov if it has a large kernel.

Proof. We can assume that E is complete. Let us use the factorization $\mu = u\circ\lambda$ of Kalton's theorem 3.3. The measure $\lambda\colon \mathscr{A}\to D$ is a-continuous by 2.4, D is complete, so the mappings $\tilde{\lambda}_A$, $A\in\mathscr{A}$, take their values in D and since $\tilde{\mu}_A = u\circ\tilde{\lambda}_A$ and u is injective, λ has a large kernel and is a Liapunov measure by Knowles' theorem above.

Remark. Under hypotheses of the above theorem, but without large kernel assumption and taking E complete, we conclude that there exists a σ-field $\mathscr{A}'$ containing $\mathscr{A}$ as a subalgebra and a convexly Liapunov extension $\mu'\colon \mathscr{A}'\to E$ of μ whose range is equal to the closed convex hull $\overline{\mathrm{co}}\,\mu(\mathscr{A})$ of $\mu(\mathscr{A})$.

Indeed, $\mu = u\circ\lambda$ as above and by [9], p. 93, λ has a Liapunov extension $\lambda'\colon \mathscr{A}'\to D$ with $\lambda'(\mathscr{A}') = \overline{\mathrm{co}}\,\lambda(\mathscr{A})$. We can take $\mu' = u\circ\lambda'$. Since $\lambda'(\mathscr{A}')$ is weakly compact in D, Corollary 2.8 gives $\mu'(\mathscr{A}') = \overline{\mathrm{co}}\,\mu(\mathscr{A})$.

4.5. LEMMA. *Let E be a Hausdorff t.v.s. and $\mu\colon \mathscr{A}\to E$ a convexly Liapunov measure, a-continuous for some probability measure a on $\mathscr{A}$.*

Then, for every $\varepsilon\in(0,1)$ and every $A\in\mathscr{A}$, there exists $B\in A\cap\mathscr{A}$ verifying $\mu(B) = \varepsilon\mu(A)$ and $a(B)\leqslant\varepsilon$ (in fact, $a(B) = \varepsilon a(A)$ when a is atomless).

When E has finite dimension and a is atomless, let us observe that this lemma, or something like it, appears in induction proofs of the classical Liapunov theorem (references of [9]).

Proof. The atoms of a are μ-negligible since μ is atomless. This allows us to assume that a is atomless.

The measure $(a,\mu)\colon \mathscr{A}\to \boldsymbol{R}\times E$ defined by $(a,\mu)(A) = \big(a(A),\mu(A)\big)$ is clearly convexly bounded and equivalent to a ($\boldsymbol{R}$ denotes the real field). Furthermore, it has a large kernel. Indeed, let $A\in\mathscr{A}$ be non-(a,μ)-negligible. If A is μ-negligible, $\ker(a,\mu)\tilde{}_A = \ker\tilde{a}_A$ is non-null. If not, let q be the canonical mapping of $L^\infty\big((a,\mu)_A\big) = L^\infty(a_A)$ onto $L^\infty(\mu_A)$. The kernels of $\tilde{\mu}_A$ and hence of $\tilde{\mu}_A\circ q$ have infinite dimension since μ is atomless and $\ker\tilde{\mu}_B$ is non-null for every non-μ-negligible set $B\in A\cap\mathscr{A}$. But $\ker\tilde{a}_A$ is of codimension 1. So $\ker(a,\mu)\tilde{}_A = \ker\tilde{a}_A\cap\ker\tilde{\mu}_A\circ q$ is non-null.

Then, by 4.4, (a,μ) is convexly Liapunov. Consequently, $A\in\mathscr{A}$ and $\varepsilon\in(0,1)$ being given, $B\in A\cap\mathscr{A}$ can be found such that $(a,\mu)(B) = \varepsilon(a,\mu)(A)$, which completes the proof.

5. Generation of convexly bounded and convexly Liapunov measures.

5.1. Let E be a vector space, let $p_i\colon E\to E_i$, $i\in I$, be a family of linear maps into Hausdorff t.v.s.'s E_i. Let s be the projective limit topology on E

for the p_i's. We assume that the sets $p_i^{-1}(V)$, $i \in I$, V zero-neighbourhood in E_i, constitute a basis of zero-neighbourhoods for s.

Moreover, let t be a Hausdorff linear topology on E.

THEOREM. *Let us consider a measure* $\mu\colon \mathscr{A} \to E$. *We assume that the following conditions are fulfilled.*

(a) $\mu\colon \mathscr{A} \to (E, t)$ *is countably additive and verifies the countable chain condition.*

(b) $p_i \circ \mu\colon \mathscr{A} \to E_i$ *is convexly Liapunov for every* $i \in I$.

(c) t *is s-polar* (*i.e.* t *admits a basis of s-closed zero-neighbourhoods*).

Then μ *is convexly bounded for* t *and* (*under the sole assumption* (b)) *the s-closure of* $\mu(\mathscr{A})$ *is convex.*

Proof. Let V be a balanced zero-neighbourhood in some E_i, $i \in I$; if x and y are points of $\mu(\mathscr{A}) + p_i^{-1}(V)$ and $0 \leqslant r \leqslant 1$ then, $p_i\big(\mu(\mathscr{A})\big)$ being convex, $p_i\big(rx + (1-r)y\big) \in p_i\big(\mu(\mathscr{A})\big) + V + V$. This proves the last statement.

By (c) it suffices now to prove that $\mu(\mathscr{A})$ is t-bounded. The set of convexly Liapunov measures $p_i \circ \mu$, $i \in I$, verifies the countable chain condition: by Corollary 2.5 they are α-continuous for a probability measure α on $\mathscr{A}$. The topology s is Hausdorff since t is Hausdorff and s-polar. So, by 2.4, μ is α-continuous. Given a zero-neighbourhood V in (E, t), there exists a non-null $\varepsilon \in (0, 1)$ such that $\mu(B) \in V$ as soon as $B \in \mathscr{A}$ and $\alpha(B) \leqslant \varepsilon$. Let $A \in \mathscr{A}$. If $i \in I$ Lemma 4.5 gives some $B_i \in \mathscr{A}$ verifying $p_i\big(\varepsilon\mu(A)\big) = p_i\big(\mu(B_i)\big)$ with $\alpha(B_i) \leqslant \varepsilon$, whence $\mu(B_i) \in V$. This shows that $\varepsilon\mu(A)$ belongs to the s-closure of V and completes the proof.

5.2. THEOREM. *Let D be a Hausdorff t.v.s., let E be a complete metrizable t.v.s., let* $\lambda\colon \mathscr{A} \to D$ *be a convexly Liapunov measure and let* $u\colon D \to E$ *be a linear mapping with closed graph.*

Then the measure $\mu = u \circ \lambda\colon \mathscr{A} \to E$ *is convexly bounded.*

Proof. Since λ is convexly Liapunov, Proposition 3.2, (b) shows that $\tilde{\lambda}\colon L^\infty(\lambda) \to \hat{D}$ takes its values in D (one can use Proposition 4.2 or a more direct argument). So, $u \circ \tilde{\lambda}\colon L^\infty(\lambda) \to E$ has a closed graph and is therefore continuous. This implies that $\mu(\mathscr{A})$ is a bounded convex set.

5.3. THEOREM. *Let E be a Hausdorff t.v.s. and let* $\mu\colon \mathscr{A} \to E$ *be a countably additive measure. We assume that* $\mu = u \circ \lambda$, *where* $\lambda\colon \mathscr{A} \to D$ *is a convexly Liapunov measure verifying the countable chain condition with values in a Hausdorff t.v.s.* D *and* $u\colon D \to E$ *is an arbitrary linear mapping.*

Then μ *is convexly Liapunov* (*whence, in particular, convexly bounded*).

Proof. By Corollary 2.5, λ is equivalent to some probability measure α on $\mathscr{A}$. By 2.4, μ is α-continuous. Let V be a zero-neighbourhood in E. For some non null $\varepsilon \in (0, 1)$, $\mu(B) \in V$ if $B \in \mathscr{A}$ and $\alpha(B) \leqslant \varepsilon$. By Lemma 4.5, for every $A \in \mathscr{A}$, there exists $B \in \mathscr{A}$ verifying $\varepsilon\lambda(A) = \lambda(B)$ and $\alpha(B) \leqslant \varepsilon$, whence $\varepsilon\mu(A) = \mu(B) \in V$. This shows that $\mu(\mathscr{A})$ is bounded in E. Since $\mu(A \cap \mathscr{A})$ is clearly convex for every $A \in \mathscr{A}$, μ is convexly Liapunov (Theorem 4.4).

5.4. As a consequence of the above theorem, a measure $\mu: \mathscr{A} \to E$ (E Hausdorff t.v.s.) is convexly Liapunov (and therefore convexly bounded) if it is countably additive, verifies the countable chain condition, and if the linear span D of $\mu(\mathscr{A})$ can be separated by some set L of linear forms (not necessarily continuous) $D \to \mathbf{R}$ verifying following conditions: (1) $x' \circ \mu: \mathscr{A} \to \mathbf{R}$ is countably additive for every $x' \in L$ and (2) for every non-μ-negligible set $A \in \mathscr{A}$, there exists a non-null $f \in L^\infty(\mu_A)$ verifying $\int_A f d(x' \circ \mu) = 0$ for every $x' \in L$.

Indeed, μ is convexly Liapunov for the topology $\sigma(D, L)$ by 4.3, and hence for the topology of E by 5.3.

From this we derive the following characterization of convexly Liapunov measures verifying the countable chain condition.

5.5. PROPOSITION. *E being a Hausdorff t.v.s., a countably additive measure $\mu: \mathscr{A} \to E$ verifying the countable chain condition is convexly Liapunov if and only if, for every $A \in \mathscr{A}$, $\mu(A \cap \mathscr{A})$ is convex and separated by the set of continuous affine maps of $\mu(A \cap \mathscr{A})$ into the real field $\mathbf{R}$.*

Necessity comes from Kalton's theorem 2.8.

6. Uniform boundedness properties. It is known ([16], [2]) that if M is a pointwise bounded set of countably additive measures $m: \mathscr{A} \to E$, E t.v.s., then $\sup\{r(m(A)) \mid m \in M, A \in \mathscr{A}\}$ is finite for every continuous F-semi-norm r on E. However, $\{m(A) \mid m \in M, A \in \mathscr{A}\}$ may be unbounded, even if each $m \in M$ is bounded (example 8.2 infra). We have nevertheless the following results.

6.1. THEOREM. *Let M be a set of convexly Liapunov measures $m: \mathscr{A} \to E$, E being a Hausdorff t.v.s. Then M is uniformly bounded (i.e. $\{m(A) \mid m \in M, A \in \mathscr{A}\}$ is bounded in E) if at least one of the following conditions is fulfilled.*

(i) *M is equi-countably additive (i.e. $m(A_n)$ tends to zero uniformly for $m \in M$ if (A_n) is a decreasing sequence of $\mathscr{A}$ with void intersection).*

(ii) *M is pointwise bounded (i.e. $\{m(A) \mid m \in M\}$ is bounded in E for every $A \in \mathscr{A}$).*

Proof. Let us first assume that (i) is verified. It is enough to prove the theorem when E is metrizable and M countable. In this case, M verifies the countable chain condition and by 2.5 there exists a probability measure α on $\mathscr{A}$ such that each $m \in M$ is α-continuous. By [1], Theorem 8.5, M is α-equicontinuous: if V is a zero-neighbourhood in E there exists a number ε, with $0 < \varepsilon \leqslant 1$, such that $\{m(B) \mid m \in M\} \subset V$ whenever $B \in \mathscr{A}$ and $a(B) \leqslant \varepsilon$. By Lemma 4.5 we see as in 5.1 that $\varepsilon m(A) \in V$ for every $A \in \mathscr{A}$ and every $m \in M$, so M is uniformly bounded.

Now let us assume that (ii) is fulfilled. Let $(m_n)_{n \geqslant 1}$ be a sequence of elements of M. By (ii), $n^{-1}m_n(A)$ tends to zero for every $A \in \mathscr{A}$. By the Nikodym theorem ([1], [12]) $\{n^{-1}m_n \mid n \geqslant 1\}$ is equi-countably additive and consequently equibounded. So $n^{-2}m_n$ tends to zero uniformly on $\mathscr{A}$. This proves that M is uniformly bounded.

6.2. THEOREM. *Let E be a Hausdorff t.v.s. and M a pointwise bounded set of countably additive measures $\mathscr{A} \to E$. We assume that each $m \in M$ verifies the countable chain condition and is purely atomic* (cf. [17]).

Then M is uniformly bounded.

Proof. It is enough to prove that $(r_n m_n)_{n \geqslant 0}$ is uniformly bounded on $\mathscr{A}$ if $(m_n)_{n \geqslant 0}$ is a sequence of M and if $(r_n)_{n \geqslant 0}$ is a null real sequence. Then $\mu = (r_n m_n)_{n \geqslant 0}$ is a countably additive measure with values in the space $c_0(E)$ of null sequences of E endowed with the linear topology of uniform convergence (the Nikodym Theorem: [1], [12]) and it fulfils the countable chain condition. If $A \in \mathscr{A}$ contains no μ-atom, then, for every n, A contains no m_n-atom by 2.6 and is therefore m_n-negligible, whence μ-negligible. So μ is purely atomic and therefore bounded by [17], Theorem 3.

6.3. THEOREM. *E being a Hausdorff t.v.s., every pointwise bounded set M of finite rank countably additive measures $m\colon \mathscr{A} \to E$ is uniformly bounded.*

Proof. We may assume that M is countable and E metrizable. Then the result can be reduced to Theorems 6.1 and 6.2 for we can find $A \in \mathscr{A}$, with complement B, such that for every $m \in M$ the induced measures m_A and m_B are respectively purely atomic and atomless (whence convexly Liapunov). Indeed, the product space E^M is metrizable. So, if $\mu\colon \mathscr{A} \to E^M$ is the countably additive measure canonically associated to M, μ_A and μ_B are respectively purely atomic and atomless for some $A \in \mathscr{A}$ by [17], Theorem 1 and the same holds for m_A and m_B since they are μ_A- and μ_B-continuous.

6.4. As a consequence of Theorems 6.1 and 6.3 we have the following statement.

THEOREM. *Let E be a Hausdorff t.v.s. and let μ be a measure $\mathscr{A} \to E$.*

If μ is pointwise adherent to a pointwise bounded set M of convexly Liapunov (*resp. finite rank and countably additive*) *measures $m\colon \mathscr{A} \to E$, then μ is convexly bounded* (*resp. bounded*).

Proof. In both cases $\mu(\mathscr{A})$ is bounded since it is included in the closure of $\{m(A) \mid m \in M,\ A \in \mathscr{A}\}$. When each $m \in M$ is convexly Liapunov, this is also true for $\operatorname{co}\mu(\mathscr{A})$. Indeed, let $x = \sum_1^N r_i \mu(A_i)$ with $A_i \in \mathscr{A}$, $r_i > 0$ and $\sum_1^N r_i = 1$. If V is a zero-neighbourhood in E we can find $m \in M$ such that $\sum_1^N r_i\big(\mu(A_i)-m(A_i)\big) \in V$ and $A \in \mathscr{A}$ such that $m(A) = \sum_1^N r_i m(A_i)$, whence $x-m(A) \in V$.

COROLLARY. *If μ is the pointwise limit on $\mathscr{A}$ of a sequence of convexly Liapunov (resp. finite rank and countably additive) measures $\mu_n\colon \mathscr{A}\to E$, then μ is a convexly bounded (resp. bounded) countably additive measure.*

As above, countable additivity of μ is given by [1] or [12].

7. Measures with values in spaces with separating dual. As above, E' denotes the topological dual of a Hausdorff t.v.s. E.

7.1. PROPOSITION. *If $\mu\colon \mathscr{A}\to E$ is atomless, countably additive and fulfils the countable chain condition, then the closure of $\mu(\mathscr{A})$ for the weak topology $\sigma(E, E')$ is convex.*

Via 2.6 this is an immediate consequence of the Liapunov theorem (see [9], p. 95 or Theorem 5.1 above).

Let us observe that E' separates E if and only if $\sigma(E, E')$ is Hausdorff.

COROLLARY. *Under above assumptions, μ is convexly bounded if it is bounded and any bounded subset of E has a bounded weak closure, or if E' separates E and $\mu(\mathscr{A})$ is relatively compact. In the last case, the closure of $\mu(\mathscr{A})$ is convex (and compact).*

See related counter-examples in 8.1 and 8.4 below.

7.2. Let us consider the following conditions for a Hausdorff t.v.s. E.

(a) E is weakly polar (i.e. E has a basis of weakly closed zero-neighbourhoods).

(b) There exists a Hausdorff t.v.s. F containing E as a topological vector subspace and there exists a pointwise bounded set H of finite rank and continuous linear mappings $E\to F$ such that the canonical injection of E into F is in the closure of H for topology of pointwise convergence on E.

Condition (b) implies condition (a) if E is complete and metrizable and the converse is true if E is also separable ([7], Theorem 7.4).

For example, E is weakly polar if it is complete metrizable and has a Schauder basis.

7.3. THEOREM. *If a Hausdorff t.v.s. E fulfils one of the above conditions* (a), (b), *then every atomless countably additive measure $\mu\colon \mathscr{A}\to E$ verifying the countable chain condition is convexly bounded.*

Proof. Under condition (a), for every finite sequence $x' = (x_i')_{1\leqslant i\leqslant n}$ of elements of E', the measure $x'\circ\mu\colon \mathscr{A}\to \boldsymbol{R}^n$ is Liapunov by 2.6 and the Liapunov theorem and we can apply Theorem 5.1.

Under condition (b), μ is pointwise adherent to the pointwise bounded set M of measures $u\circ\mu\colon \mathscr{A}\to F$, $u \in H$, which are convexly Liapunov as above, and Theorem 6.4 gives the result.

Remark. When E is a non-locally convex space with a Schauder basis it can often be shown that bounded closed convex subsets of E are compact (see [23] and Lemma of [24], p. 98). In these cases the range of above measure μ has a compact convex closure (as observed in [4] when $E = l^r$, $0 < r < 1$).

7.4. THEOREM. *If a Hausdorff t.v.s. E fulfils above condition* (b) (*resp.* (a)) *then every countably additive measure* (*resp. every countably additive measure verifying the countable chain condition*) $\mu: \mathscr{A} \to E$ *is bounded.*

Proof. Under condition (b) we have to apply 6.4 to the measures $u \circ \mu$, $u \in H$.

Under condition (a), the countable chain condition yields (by [17]) a decomposition $\mu = \mu_0 + \mu_1$ into a purely atomic part μ_0 with finite or countable set of atoms, which will be compact ([17]), and an atomless part μ_1 which will be convexly bounded by 7.3.

7.5. As a particular case of 5.2 and 5.3 we have the following statement.

THEOREM. *Let s and t be Hausdorff linear topologies on a t.v.s. E, s being locally convex and coarser than t, and let $\mu: \mathscr{A} \to (E, s)$ be a countably additive measure with large kernel verifying the countable chain condition.*

(1) *If (E, t) is metrizable and complete, then $\mu: \mathscr{A} \to (E, t)$ is convexly bounded.*

(2) *If μ is countably additive for t, then it is convexly Liapunov* (*whence convexly bounded*) *for t.*

Let us observe that, if (E, t) is locally convex metrizable and complete, conclusion of (1) remains valid assuming only that μ is bounded for s. This is a well-known consequence of barrelledness of the space of $\mathscr{A}$-simple functions for the supremum norm. For the general case, see Example 8.1 below.

8. Counter-examples.

Examples 8.1 and 8.4 below are to be compared to results of Section 7 and Examples 8.2 and 8.3 to those of Section 6.

Let I be the closed unit intervall $[0, 1]$ endowed with its Borel σ-field $\mathscr{B}$ and the Lebesgue measure $\alpha: \mathscr{B} \to \boldsymbol{R}_+$.

8.1. *An unbounded countably additive atomless measure on the σ-field $\mathscr{B}$ with values in a complete metrizable space F with separating dual.*

One can even choose F with a basis of zero-neighbourhoods closed for some normable topology $\mathscr{S}$ on F coarser than the given topology of F.

The construction looks like that of [25], with some complications due to the closed neighbourhoods condition. We need it in 8.2 below. Furthermore, thanks to this condition, our example contrasts with Theorem 7.3 above and also with a result of Labuda ([13], Theorem 3).

For every Borel function $f: I \to \boldsymbol{R}$, let $\operatorname{var}(f)$ be the variation of f and $s(f) = \alpha(\{t \in I \mid f(t) \neq 0\})$. If f is an α-class of Borel functions, let

$\|f\|_v = \inf\{|\varphi(0)| + \mathrm{var}(\varphi) \mid \varphi \in f\}$ and $s(f) = s(\varphi)$ for $\varphi \in f$. $\|\cdot\|_1$, $\|\cdot\|_\infty$ are the usual norms of $L^1(a)$, $L^\infty(a)$.

For every integer $j \geqslant 1$, let us define subadditive functionals p_j and $p\colon L^1(a) \to [0, \infty]$ by

$$p_j(f) = \inf\{\|g\|_v + s(h) + \|h\|_1 + j\|u\|_\infty \mid f = g + h + u\}$$

with g, h, u in $L^1(a)$, and

$$p(f) = \sup\{p_j(f) \mid j \geqslant 1\}.$$

We have for every $j \geqslant 1$

$$\|\cdot\|_1 \leqslant p_j \leqslant 2j^{1/2}\|\cdot\|_1^{1/2} + \|\cdot\|_1. \tag{1}$$

Indeed, for every decomposition $f = g + h + u$ we can write

$$\|f\|_1 \leqslant \|g\|_1 + \|h\|_1 + \|u\|_1 \leqslant \|g\|_v + \|h\|_1 + j\|u\|_\infty,$$

whence $\|f\|_1 \leqslant p_j(f)$. On the other hand, if $h = f\chi_{(|f|>c)}$ with $c = \|f/j\|_1^{1/2}$ we get the second inequality (1) writing

$$p_j(f) \leqslant s(h) + \|h\|_1 + j\|f - h\|_\infty \leqslant c^{-1}\|f\|_1 + \|f\|_1 + jc.$$

So, p_j is an F-norm defining the usual topology of $L^1(a)$, p is lower semi-continuous on $L^1(a)$ and verifies $\|\cdot\|_1 \leqslant p$. Consequently,

$$F = \{f \in L^1(a) \mid \lim_{r \to 0} p(rf) = 0\}$$

is a complete F-normed vector space for the F-norm induced by p, the inclusion $F \subset L^1(a)$ is continuous and F admits a basis of zero-neighbourhoods closed for the topology induced on F by $L^1(a)$.

F contains the space of functions $f \in L^1(a)$ which admit for every $\varepsilon > 0$ a decomposition $f = g + h$ with $\|g\|_v < \infty$ and $s(h) < \varepsilon$. So, as in [26], we see that the set-function $\chi\colon A \to \chi_A$ (characteristic function of A) is a measure $\mathscr{B} \to F$ and is a-continuous (we have $p(\chi_A) \leqslant 2a(A)$), whence countably additive and atomless.

Let us show that $\chi(\mathscr{B})$ is unbounded in F. It is enough to prove that $p(f_n) \geqslant 1/2$ if, for every integer $n \geqslant 1$, f_n is the a-class of φ_n, with

$$\varphi_n = n^{-1}\sum_{i=0}^{n-1} \chi(A_{n,2i}), \quad A_{n,k} = \left]\frac{k}{2n}, \frac{k+1}{2n}\right], \quad 0 \leqslant k < 2n.$$

If n is given let us assume that $\varphi_n = g + h + u$ almost everywhere for some Borel functions g, h, u. Let J' be the set of integers $i \in [0, n-1]$ such that at least one of the sets $h^{-1}(0) \cap A_{n,2i}$, $h^{-1}(0) \cap A_{n,2i+1}$ is negligible. Then $(2n)^{-1}\mathrm{Card}(J') \leqslant s(h)$ so $\mathrm{Card}(J) \geqslant n - 2ns(h)$ if $J = \{0, \ldots, n-1\} \setminus J'$. For every $i \in J$ we have $\varphi_n(t) = g(t) + u(t)$ for some points

$t = t_{2i} \in A_{n,2i}$ and $t = t_{2i+1} \in A_{n,2i+1}$, whence

$$\begin{aligned}\operatorname{var}(g) &\geqslant \sum_{i \in J} |g(t_{2i+1}) - g(t_{2i})| \\ &\geqslant \sum_{i\in J} |\varphi_n(t_{2i+1}) - \varphi_n(t_{2i})| - |u(t_{2i})| - |u(t_{2i+1})| \\ &\geqslant n^{-1}\operatorname{Card}(J) - 2n \sup|u| \geqslant 1 - 2s(h) - 2n\sup|u|\end{aligned}$$

and $\operatorname{var}(g) + s(h) + n\sup|u| \geqslant 1/2$, which shows that $p_n(f_n) \geqslant 1/2$, whence $p(f_n) \geqslant 1/2$.

8.2. *An unbounded atomless (countably additive) vector measure on the σ-field $\mathscr{B}$ which is the pointwise limit on $\mathscr{B}$ of a sequence of convexly bounded countably additive atomless vector measures.*

We use the notations of 8.1.

Let G be the vector space of sequences $(x_j)_{j\geqslant 1}$ of $L^1(a)$ verifying $\lim_j p_j(x_j) = 0$, endowed with the F-norm $\|x\| = \sup_j p_j(x_j)$.

Let $\mu: \mathscr{B} \to G$ be defined by $\mu(A) = (j^{-1/2}\chi_A)_{j\geqslant 1}$. The values of μ belong to G since the space F of 8.1 contains χ_A when $A \in \mathscr{B}$. Clearly, μ is the pointwise limit of the sequence of measures $\mu_n: \mathscr{B} \to G$, $n \geqslant 1$, where the jth coordinate of μ_n is $j^{-1/2}\chi_A$ if $j \leqslant n$ and 0 if $j > n$.

Since $\|\mu_n(A)\| \leqslant \|\mu(A)\| \leqslant 2a(A)$, μ and the μ_n's are a-continuous (whence countably additive and atomless). By (1) the μ_n's are convexly bounded (μ_n takes its values in a normable subspace of G, in fact a copy of $(L^1(a))^n_\bullet$). But μ is not bounded since $\|n^{-1/2}\mu(A_n)\| \geqslant p_n(f_n) \geqslant 1/2$ when $A_n = \bigcup_{i=0}^{n-1} A_{n,2i}$ (see 8.1 above).

8.3. *A pointwise bounded sequence of atomless convexly bounded countably additive measures on the σ-field $\mathscr{B}$ which is not uniformly bounded.*

The above sequence of vector measures μ_n, $n \geqslant 1$, fulfils these conditions (we have $\|n^{-1/2}\mu_n(A_n)\| \geqslant 1/2$).

8.4. *A bounded but not convexly bounded atomless countably additive measure with values in a complete metrizable space with separating dual.*

Such an example is given in [25] and in Section 7.4 of [24]. As above, the measure assigns to every Borel set its characteristic function and takes its values in some complete F-normed space continuously included in $L^1(a)$.

8.5. *A complete metrizable space F without the bounded multiplier property such that every atomless F-valued countably additive measure defined on a σ-field is convexly bounded.*

A complete metrizable space E is said to have the *bounded multiplier property* when every E-valued bounded measure is convexly bounded or, equivalently, when a series $\sum_0^\infty r_n x_n$, $x_n \in E$, converges in E for every bounded real sequence (r_n) as soon as it converges when each r_n is equal to 0 or 1.

The first example of a complete metrizable space without the bounded multiplier property, given in [22] by S. Rolewicz and C. Ryll-Nardzewski, is weakly polar: by Theorem 7.3 above it furnishes the example.

This shows that the convex boundedness problems for arbitrary bounded measures and for those which are atomless are not equivalent, answering negatively a question of [21], p. 91.

8.6. *A countably additive measure with values in a complete metrizable space without countably additive atomless extension.*

Let us first observe that if E is a complete metrizable t.v.s., then every convexly bounded countably additive E-valued measure μ on a σ-field $\mathscr{A}$ admits an atomless countably additive convexly bounded extension μ' on some σ-field $\mathscr{A}'$ containing $\mathscr{A}$ as a subalgebra, with $\mu'(\mathscr{A}')$ included in the closed convex hull of $\mu(\mathscr{A})$.

Indeed, by [17] we can decompose μ into an atomless and atomic part, with a finite or countable set of atoms, and we apply Remark 4.4 to this atomic part.

Conversely, if E is moreover weakly polar and if $\mu\colon \mathscr{A} \to E$ has an atomless countably additive E-valued extension on some σ-field $\mathscr{A}' \supset \mathscr{A}$, then μ is by 7.3 necessarily convexly bounded.

So, if E is weakly polar, complete metrizable and does not verify the bounded multiplier property (as in 8.5 above), there exists a countably additive measure $2^N \to E$ without E-valued atomless countably additive extension to any σ-field $\mathscr{A}' \supset \mathscr{A}$.

8.7. *Concluding remarks.* All pathological measures presented in this section are "injective" (the corresponding integral on simple functions is injective) and this suggests that Knowles generalization of the Liapunov theorem still holds for measures with values in an arbitrary t.v.s.

We have partial results of this type (see 4.4 and 5.4) but under conditions which enable to reduce the problem to the locally convex case; it would be good to avoid these conditions.

More precisely, let $\mu\colon \mathscr{A} \to E$ be a countably additive measure, E being a Hausdorff t.v.s., and let $S(\mu)$ be the vector space generated by the μ-classes of characteristic functions χ_A, $A \in \mathscr{A}$ (the space of μ-simple functions).

An integral mapping $f \to \int f d\mu$ of $S(\mu)$ into E is clearly defined and we say that μ has a *large algebraic kernel* when condition of Definition 4.3 is fulfilled for this mapping (μ is not assumed to be convexly bounded, so bounded measurable functions may not be integrable a priori).

Let us assume that μ has a large algebraic kernel. We take E metrizable for sake of simplicity.

Then, is μ bounded? convexly bounded? and is $\mu(\mathscr{A})$ convex?

Let a be a probability measure on $\mathscr{A}$.

If μ is a-continuous, and if we can show that every a-continuous measure on $\mathscr{A}$ with large algebraic kernel has a convex range, then μ will be convexly bounded (and even convexly Liapunov): see the proofs of 4.5 and 5.3.

The above questions have also positive answers if μ is a-continuous and convexly bounded for some Hausdorff topology on E coarser than initial one, by 4.4 and 5.3.

In particular, they have positive answers if E is separated by its dual (Theorem 7.5), since this hypothesis yields a probability a equivalent to μ.

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