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On the equivalence of logarithmic methods of summability of orthonormal series

1. Let

$$(1)\qquad \sum_{k=0}^{\infty} c_k\varphi_k(x) \qquad (x\in\langle 0, 1\rangle)$$

be a real orthonormal series with coefficients c_k such that

$$\sum_{k=0}^{\infty} c_k^2 < \infty,$$

i.e. the orthonormal series of L^2. Denote by $S_n(x)$ the nth partial sum of (1). Consider the following matrices $X^{(q)} = \|X_k(n, q)\|$ and the sequences $Y^{(q)} = \{Y_k(r, q)\}_{k=0}^{\infty}$ $(q = 1, 2, 3)$:

$$X_0(n, q) \equiv 1 \quad (q = 1, 2, 3),$$

$$X_k(n, q) = 1 - \begin{cases} \left(\sum_{p=0}^{n} \frac{1}{p+1}\right)^{-1} \sum_{p=0}^{k-1} \frac{1}{p+1} & \text{if } q = 1, \\ \left(\sum_{p=0}^{n}\sum_{m=0}^{p} \frac{1}{m+1}\right)^{-1} \sum_{p=0}^{k-1}\sum_{m=0}^{p} \frac{1}{m+1} & \text{if } q = 2, \\ \frac{k}{n+1} & \text{if } q = 3 \end{cases}$$

$(1\leqslant k\leqslant n;\ n = 0, 1, \ldots;\ X_k(n, q) = 0 \quad \text{if} \quad k > n)$;

$$Y_0(r, q) \equiv 1 \quad (q = 1, 2, 3),$$

$$Y_k(r,q) = 1+\begin{cases} \dfrac{1}{\log(1-r)}\displaystyle\sum_{p=0}^{k-1}\frac{r^{p+1}}{p+1} & \text{if } q=1,\\ \dfrac{1-r}{\log(1-r)}\displaystyle\sum_{n=0}^{k-1} r^{n+1}\sum_{p=0}^{n}\frac{1}{p+1} & \text{if } q=2,\\ \dfrac{(1-r)^2}{\log(1-r)}\displaystyle\sum_{n=0}^{k-1} r^{n+1}\sum_{p=0}^{n}\sum_{m=0}^{p}\frac{1}{m+1} & \text{if } q=3 \end{cases}$$

$(k = 1, 2, \ldots;\ r \in (0, 1))$.

We shall say that the series (1) is *summable by the method* $X^{(q)}$ ($X^{(q)}$-*summable*) *at* $x_0 \in \langle 0, 1\rangle$ to s if the sequence

$$L_n(x, X^{(q)}) = \sum_{k=0}^{n} X_k(n, q) c_k \varphi_k(x) \tag{2}$$

is convergent at x_0 to s. We shall say that the series (1) is *summable by the method* $Y^{(q)}$ *at* x_0 to s if

$$P(x_0, r, Y^{(q)}) \to s \quad \text{as} \quad r \to 1-, \tag{3}$$

where

$$P(x_0, r, Y^{(q)}) = \sum_{k=0}^{\infty} Y_k(r, q) c_k \varphi_k(x_0) \quad \text{for} \quad r \in (0, 1). \tag{4}$$

The methods $X^{(1)}$ and $Y^{(1)}$ are called the *logarithmic methods of summability* ([1], [3]); the method $X^{(3)}$ is the *Cesàro method* $(C, 1)$.

Two methods of summability are called the *equivalent methods in* L^2 if the summability of the series (1) in the set E of positive measure by one of these methods implies the summability of (1) to the same sum almost everywhere in E by the other method ([4]).

In this paper we shall prove that the methods $X^{(q)}$, $Y^{(q)}$ are equivalent in L^2 for $q = 1$ and $q = 2$.

2. First we shall give six lemmas on the summability of numerical series. Let S_n be the nth partial sum of the numerical series

$$\sum_{k=0}^{\infty} u_k \tag{5}$$

(u_k are real numbers). The means $L_n(X^{(q)})$, $P(r, Y^{(q)})$ and the summability we define as above. Clearly, the methods $X^{(q)}$ and $Y^{(q)}$ $(q = 1, 2, 3)$ are the regular methods of summability of the series.

LEMMA 1. *The series* (5) *is* $X^{(2)}$-*summable to* s *if and only if it is* $X^{(3)}$-*summable to* s.

Proof. Using the Abel transformation, we have

$$L_n(X^{(2)}) = \Big(\sum_{k=0}^{n}\sum_{m=0}^{k}\frac{1}{m+1}\Big)^{-1}\Big\{-\sum_{k=0}^{n-1}\frac{k+1}{k+2}L_k(X^{(3)})+$$

$$+(n+1)\sum_{k=0}^{n}\frac{1}{k+1}L_n(X^{(3)})\Big\} = A_n+B_n,$$

$$L_n(X^{(3)}) = \frac{1}{n+1}\Big\{\sum_{k=0}^{n-1}\sum_{m=0}^{k}\sum_{p=0}^{m}\frac{1}{p+1}\Big((k+2)\sum_{p=0}^{k}\frac{1}{p+1}\sum_{p=0}^{k+1}\frac{1}{p+1}\Big)^{-1}L_k(X^{(2)})+$$

$$+\sum_{k=0}^{n}\sum_{p=0}^{k}\frac{1}{p+1}\Big(\sum_{k=0}^{n}\frac{1}{k+1}\Big)^{-1}L_n(X^{(2)})\Big\} = C_n+D_n$$

$(n = 1, 2, \ldots)$. If the sequence $L_n(X^{(3)})$ is convergent to s, then

$$B_n \to s \quad \text{as} \quad n\to\infty,$$

and, by the Toeplitz theorem ([2], p. 427),

$$A_n\to 0 \quad \text{as} \quad n\to\infty.$$

If the sequence $\{L_n(X^{(2)})\}$ is convergent to s, then $C_n\to 0$ and $D_n\to s$ as $n\to\infty$. Hence the lemma is proved.

LEMMA 2. *Suppose that the series* (5) *is* $X^{(q)}$*-summable to* s. *Then*

$$S_n = \begin{cases} o(n\log n) & \textit{if} \quad q = 1, \\ o(n) & \textit{if} \quad q = 2, 3;\ n\to\infty. \end{cases} \tag{6}$$

Proof. We prove (6) for $q = 1$. We have

$$L_n(X^{(1)}) = \Big(\sum_{k=0}^{n}\frac{1}{k+1}\Big)^{-1}\sum_{k=0}^{n}\frac{1}{k+1}S_k.$$

Write

$$\mathcal{L}_n = \Big(\sum_{k=0}^{n+1}\frac{1}{k+1}\Big)^{-1}\sum_{k=0}^{n}\frac{1}{k+1}S_k.$$

Clearly,

$$L_{n+1}(X^{(1)})-\mathcal{L}_n = \Big(\sum_{k=0}^{n+1}\frac{1}{k+1}\Big)^{-1}\frac{1}{n+1}S_{n+1}$$

and the sequence $\{\mathcal{L}_n\}$ is convergent to s if $\{L_n(X^{(1)})\}$ is convergent to s. Hence $(n\log n)^{-1}S_n = o(1)$ and we have (6). Lemma 2 in the cases $q = 2, 3$ follows by Theorem 3 of [2], p. 522, and Lemma 1.

LEMMA 3. *If the series* (5) *is summable to* s *by the method* $X^{(1)}$ ($X^{(2)}$), *then it is summable to* s *by the method* $Y^{(1)}$ ($Y^{(2)}$).

Proof. We shall give the proof for $q = 2$. From the definition of $Y_k(r, 2)$ we have

$$Y_k(r,2) = O\left(r^k \log k \frac{1-r}{|\log(1-r)|}\right) \tag{7}$$

for any fixed $r \in (0, 1)$. Hence the function $P(r, Y^{(2)})$ is defined in $(0, 1)$ and

$$\begin{aligned} P(r, Y^{(2)}) &= \sum_{k=0}^{\infty} Y_k(r,2)u_k \equiv \frac{-(1-r)}{\log(1-r)} \sum_{n=0}^{\infty} r^{n+1} \sum_{k=0}^{n} \frac{1}{k+1} S_n \\ &\equiv \frac{-(1-r)^2}{\log(1-r)} \sum_{n=0}^{\infty} r^{n+1} \sum_{k=0}^{n} \sum_{p=0}^{k} \frac{1}{p+1} L_n(X^{(2)}) \\ &\equiv \sum_{n=0}^{\infty} \big(Y_n(r,3) - Y_{n+1}(r,3)\big) L_n(X^{(2)}). \end{aligned}$$

The convergence of sequence $\{L_n(X^{(2)})\}$ to s and the properties of $Y_k(r, 3)$ imply the existence of $\lim_{r\to 1-} P(r, Y^{(2)}) = s$ ([2], p. 513). The proof is completed.

LEMMA 4. *Suppose that* q ($q = 1, 2$) *is a fixed number. If the series* (5) *is* $Y^{(q)}$*-summable to* s, *then the series*

$$\sum_{k=0}^{\infty} \big(L_k(X^{(q)}) - L_{k-1}(X^{(q)})\big) \quad (L_{-1}(X^{(q)}) = 0) \tag{8}$$

is $Y^{(q+1)}$*-summable to* s.

Proof. Let $q = 1$. Using the Abel transformation we have

$$\begin{aligned} P(r, Y^{(1)}) &= \sum_{k=0}^{\infty} Y_k(r,1) u_k \\ &\equiv \frac{-(1-r)}{\log(1-r)} \sum_{k=0}^{\infty} r^{k+1} \sum_{p=0}^{k} \frac{1}{p+1} L_k(X^{(1)}) \\ &\equiv \sum_{k=0}^{\infty} \big(Y_k(r,2) - Y_{k+1}(r,2)\big) L_k(X^{(1)}) \end{aligned} \tag{9}$$

for $r \in (0, 1)$. By (9) and (7),

$$Y_n(r,2) L_n(X^{(1)}) \to 0 \quad \text{as} \quad n\to\infty,\ r \in (0,1),$$

and

$$P(r, Y^{(1)}) = \sum_{k=0}^{\infty} Y_k(r, 2)\big(L_k(X^{(1)}) - L_{k-1}(X^{(1)})\big) \tag{10}$$

$(L_{-1}(X^{(1)}) = 0)$. The $Y^{(1)}$-summability of (5) to s and (10) imply the $Y^{(2)}$-summability of the series (8) to s. The proof for $q = 2$ is analogous.

LEMMA 5. *If the series* (5) *is* $Y^{(2)}$*-summable to* s *and if*

$$\sum_{k=1}^{\infty} u_k^2 k \log(k+1) < \infty, \tag{11}$$

then (5) *is convergent to* s.

Proof. Write

$$w_n = \sum_{k=n}^{\infty} u_k^2 k \log(k+1), \quad r_n = 1 - \frac{w_n}{n} \quad (n = 1, 2, \ldots).$$

If $r_n \in (0, 1)$, then

$$P(r_n, Y^{(2)}) - S_n = \sum_{k=0}^{n} \big(Y_k(r_n, 2) - 1\big) u_k + \sum_{k=n+1}^{\infty} Y_k(r_n, 2) u_k \equiv V_n + Z_n.$$

By (11),

$$|V_n| \leqslant \frac{1-r_n}{|\log(1-r_n)|} \sum_{k=1}^{n} |u_k| k \log(k+1)$$

$$\leqslant \frac{1-r_n}{|\log(1-r_n)|} \Big\{\sum_{k=1}^{n} u_k^2 k \log(k+1)\Big\}^{1/2} \Big\{\sum_{k=1}^{n} k \log(k+1)\Big\}^{1/2},$$

$$|Z_n| \leqslant \Big\{\sum_{k=n}^{\infty} Y_k(r_n, 2) \frac{1}{k \log k}\Big\}^{1/2} \Big\{\sum_{k=n}^{\infty} u_k^2 k \log(k+1)\Big\}^{1/2}$$

$$\leqslant \Big\{\frac{w_n(1-r_n)}{|\log(1-r_n)| n \log(n+1)}\Big\}^{1/2} \Big\{\sum_{k=n}^{\infty} \sum_{m=k}^{\infty} r_n^{m+1} \sum_{p=0}^{m} \frac{1}{p+1}\Big\}^{1/2},$$

$$\sum_{k=n}^{\infty} \sum_{m=k}^{\infty} r_n^{m+1} \sum_{p=0}^{m} \frac{1}{p+1} \leqslant \sum_{m=n}^{\infty} r_n^m \sum_{p=0}^{m} \frac{1}{p+1} \sum_{k=n}^{m} 1$$

$$\leqslant \sum_{k=0}^{\infty} (k+1) r_n^k \sum_{p=0}^{k} \frac{1}{p+1} = \left(\frac{d}{dr} \frac{-\log(1-r)}{1-r}\right)_{r=r_n}$$

$$= O\left(\frac{|\log(1-r_n)|}{(1-r_n)^2}\right).$$

Hence

$$|V_n| = O(w_n), \quad |Z_n| = O(\log^{-1/2} n)$$

and

(12) $$|P(r_n, Y^{(2)}) - S_n| \to 0 \quad \text{as} \quad n \to \infty.$$

The $Y^{(2)}$-summability of the series (5) to s and (12) imply the convergence of the sequence $\{S_n\}$ to s. Thus the proof is completed.

Analogously we obtain

LEMMA 6. *If the series* (5) *is* $Y^{(3)}$*-summable to* s *and if*

$$\sum_{n=1}^{\infty} n u_n^2 < \infty,$$

then (5) *is convergent to* s.

3. Now, we consider the summability of the series (1). We have

LEMMA 7. *The series*

(13) $$\sum_{n=1}^{\infty} n \log(n+1) \big(L_n(x, X^{(1)}) - L_{n-1}(x, X^{(1)})\big)^2,$$

(14) $$\sum_{n=1}^{\infty} n \big(L_n(x, X^{(2)}) - L_{n-1}(x, X^{(2)})\big)^2$$

are convergent almost everywhere in $\langle 0, 1 \rangle$.

Proof. The convergence of (13) is proved in [3]. The proof of the convergence of (14) is similar.

Next, we shall present two theorems on the equivalence of methods of summability.

THEOREM 1. *The methods* $X^{(1)}$ *and* $Y^{(1)}$ *of summability of orthonormal series* (1) *are equivalent in* L^2.

Proof. If the series (1) is $X^{(1)}$-summable in a set E of positive measure to $S(x)$, then it is $Y^{(1)}$-summable in E to $S(x)$ by Lemma 3.

Suppose that the series (1) is $Y^{(1)}$-summable in E to $S(x)$. Then, by Lemma 4, the series

(15) $$\sum_{n=0}^{\infty} \big(L_n(x, X^{(1)}) - L_{n-1}(x, X^{(1)})\big)$$

$(L_{-1}(x, X^{(1)}) = 0)$ is $Y^{(2)}$-summable in E to $S(x)$. Lemmas 7 and 5 imply that the series (15) is convergent almost everywhere in E to $S(x)$, which proves that the sequence $L_n(x, X^{(1)})$ is convergent almost everywhere in E to $S(x)$. The proof is completed.

Applying Lemmas 1, 3, 4, 6, and 7, we can prove

THEOREM 2. *The Cesàro method* $(C, 1)$ (*i.e.* $X^{(3)}$) *and the method* $Y^{(2)}$ *of summability of orthonormal series* (1) *are equivalent in* L^2.

References

[1] B. L. Kaufman, *On Taub'er type methods for logarithmic methods of summability,* Izv. vuzov., Matem. 1 (1967), 57–62 (in Russian).

[2] K. Knopp, *Infinite series,* Warszawa 1956 (in Polish).

[3] J. Meder, *On the summability almost everywhere of orthonormal series by the method of first logarithmic means,* Rozprawy Matematyczne XVII, Warszawa 1959.

[4] O. A. Ziza, *On the summability of orthogonal series by the Euler methods,* Mat. Sbornik 6 (1965), 354–377 (in Russian).