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A generalization of Reich's fixed point theorem

Abstract. In the paper we give a generalization of Browder's fixed point theorem for inward maps which enables us to obtain results concerning generalized condensing maps.

DEFINITION 1 [4]. Let a set X and a function S be given that satisfy the following conditions:

(1) $$S: X\times I\times X \ni (x, t, y)\mapsto S_x(t, y)\in X;$$

(2) $$S_x(0, y) = y, \quad S_x(1, y) = x \quad \text{for any } x, y\in X.$$

Then for any non-empty set $A\subset X$, let $\operatorname{co} SA = \inf\{D\subset X: A\subset D$ and for any $x\in A, t\in I\ S_x(t, D)\subset D\}$. For $A=\emptyset$, let $\operatorname{co} SA = \emptyset$. If $\operatorname{co} SA = A$ then A is S-convex.

DEFINITION 2 (cf. [4]). A space X is *S-contractible* if S satisfies conditions (1), (2) and for any $x\in X$ $\{S_x(t, \cdot)\}$ is a homotopy joining the identity with a constant map ([1], p. 22).

DEFINITION 3. A space X is *$\bar{S}$-contractible* if it is S-contractible for S such that for any $A\subset X$ $\operatorname{co} S\overline{\operatorname{co}} SA = \overline{\operatorname{co}} SA$.

DEFINITION 4 (cf. [4]). A space X is of *type* I (type $\bar{\mathrm{I}}$) if there exists S such that X is S-contractible ($\bar{S}$-contractible) and

(3) for any neighbourhood N of any $x\in X$ there exists a neighbourhood U such that $\operatorname{co} SU\subset N$.

Let (M, d) be a metric space. For the non-empty sets $A, D\subset M$ and $r>0$, let us write $d(A, D) = \inf\{d(x, y): x\in A, y\in D\}$, $B(A, r) = \{x\in M: d(A, x) < r\}$ and $\operatorname{dia} A = \sup\{d(x, y): x, y\in A\}$.

The following theorem was proved in [4].

THEOREM 1. *Let A be a compact type* I *subset of a metric space (M,d) and let $f: A\to M$ be a map. Let us write for $z\in M, s>0$ $A(z, s) = \overline{\operatorname{co}} S\{B(z, d(z, A)+s)\cap A\}$ and $A_z = \bigcap_{s>0} A(z, s)$. Then the condition*

(4) $$\text{for every } z\in M\setminus A,\ f^{-1}(z)\cap A_z = \emptyset$$

implies $\operatorname{Fix} f\neq\emptyset$.

Ky Fan [2] has proved the following:

THEOREM. *Let A be a non-empty compact convex set in a locally convex linear Hausdorff topological space X. Then if a continuous mapping $f\colon A\to X$ satisfies*

(5) *for every $y\in A$ there exists $c\in C$, $|c|<1$, such that $cy+(1-c)f(y)\in A$,*

then $\operatorname{Fix} f\neq\emptyset$.

PROPOSITION. *Let A be a convex subset of a linear normed space X. If $z\notin A$ and (5) holds with z in place of $f(y)$, then $y\notin A_z$.*

Proof. Suppose $y\in A_z$. Then $\|z-y\|=d(z,A)$, because for $S_x(t,y):=tx+(1-t)y$ we have $A_z=\bar{B}(z,d(z,A))\cap A$. On the other hand, $cy+(1-c)z\in A$ and thus $\|z-cy-(1-c)z\|\leqslant|c|\,\|z-y\|+(1-c)\,\|z-z\|<d(z,A)$ is a contradiction.

The Proposition shows that Theorem 1 is more general than Ky Fan's theorem in the case where X is a linear normed space (simple examples show that these theorems are not equivalent).

LEMMA. *Let X be a $\bar{S}$-contractible space and $g\colon X\to 2^X$ a set valued mapping. If there is a compact set $D\subset X$ for which $\hat{g}(D):=\bigcup_{x\in D}g(x)\subset D$, then there also exists a set $C=\overline{\mathrm{co}}SC$ with $\overline{\mathrm{co}}S\,\hat{g}(C)=C$.*

Proof. Let $\mathscr{F}:=\{F=\overline{\mathrm{co}}SF\subset X\colon \mathrm{co}S\,\hat{g}(F)\subset F\}$. Obviously, $\mathscr{F}\neq\emptyset$ as $X\in\mathscr{F}$. There exists a maximal chain $\mathscr{G}\subset\mathscr{F}$ consisting of sets $G\in\mathscr{F}$ with $G\cap D\neq\emptyset$. Then $C:=\bigcap_{G\in\mathscr{G}}G$ is non-empty; moreover, $C=\overline{\mathrm{co}}SC$ (cf. [4]) and $\overline{\mathrm{co}}S\,\hat{g}(C)\subset C$. Besides we have $\overline{\mathrm{co}}S\hat{g}(C)\in\mathscr{G}$ as $\emptyset\neq\hat{g}(D\cap C)\subset D$. Therefore $\overline{\mathrm{co}}S\hat{g}(C)=C$ as $\overline{\mathrm{co}}S\;\overline{\mathrm{co}}S\hat{g}(C)=\overline{\mathrm{co}}\,S\hat{g}(C)$.

THEOREM 2. *Let $A=\bar{A}$ be a type $\bar{\mathrm{I}}$ subset of a metric space (M,d) and let a map $f\colon A\to M$ satisfy (4). If for any $C\subset A$ for which $A_{f(C)}\subset C\big(A_{f(C)}:=\bigcup_{x\in f(C)}A_x\big)$, it follows from the compactness of $\overline{C\setminus\overline{\mathrm{co}}SA_{f(C)}}$ that $\bar{C}$ is compact,* $\operatorname{Fix} f\neq\emptyset$.

Proof. Let $g(x):=A_{f(x)}$ for $x\in A$ and let $D\subset A$ be any compact set. We take a minimal set $\bar{E}=E$ containing D and such that $\hat{g}(E)\subset E$. We can see that $\overline{E\setminus\hat{g}(E)}\subset D$, because $\big(E\setminus\overline{\hat{g}(E)}\big)\cap(A\setminus D)$ is relatively open in E and would be rejected when nonempty. Therefore, E must be compact. According to Lemma there exists a set C such that $\overline{\mathrm{co}}S\,\hat{g}(C)=C$ and so C must be compact. It can be seen that if $f(x)\notin C$, $f(x)\notin A$ and $x\notin C_{f(x)}$, and hence f has a fixed point in C.

THEOREM 3. *Let (M,d) be a S-contractible metric space and let $A=\overline{\mathrm{co}}S\,A$ be a type $\bar{\mathrm{I}}$ subset of M for S. Let $f\colon A\to M$ be a map. Suppose that for some $z\in A$ and for any $C\subset A$ satisfying $\hat{g}(C):=S_z(I,f(C))\cap A\subset C$ it follows* *from the compactness of* $\overline{C \setminus \overline{\mathrm{co}} S\, \hat{g}(C)}$ *that* C *is compact. If, in addition,*

(6) *for any* $C = \overline{\mathrm{co}} S\, \hat{g}(C)$ *and* $z \in f(C) \setminus A$, $A_z \cap C \cap f^{-1}(z) = \emptyset$

holds, f *has a fixed point.*

Proof. Let $g(x) := S_z(I, f(x)) \cap A$. Similarly as in the proof of Theorem 2 we obtain a compact set $C = \overline{\mathrm{co}} S\, \hat{g}(C)$, and hence, by Theorem 1, $\mathrm{Fix} f \neq \emptyset$.

Remark 1. Theorem 3 is more general than Reich's theorem ([5], Theorem C), because if a map is condensing for Kuratowski's measure of non-compactness in a Banach space, it is condensing for a class of measures [3] and is generalized condensing too. Besides, by [5], Lemma 3, it follows from (6) that the inwardness condition holds.

Remark 2. We can take condition (2) [4] (Leray–Schauder boundary condition) with $w = z$ instead of (6) in Theorem 3 for any open convex subset A of a Banach space where $S_x(t, y) = tx + (1-t)y$.

References

[1] K. Borsuk, *Theory of retracts*, PWN, Warszawa 1967.
[2] K. Fan, *Extensions of two fixed point theorems of F. E. Browder*, Math. Z. 112 (1969), 234–240.
[3] L. Pasicki, *On the measures of noncompactness*, Comment. Math. 21 (1980), 203–205.
[4] —, *Retracts in metric spaces*, Proc. Amer. Math. Soc. (to appear).
[5] S. Reich, *Fixed points of condensing functions*, J. Math. Anal. Appl. 41 (1973), 460–467.