ANNALES SOCIETATIS MATHEMATICAE POLONAE
Series I: COMMENTATIONES MATHEMATICAE XXIII (1983)
ROCZNIKI POLSKIEGO TOWARZYSTWA MATEMATYCZNEGO
Seria I: PRACE MATEMATYCZNE XXIII (1983)

M. NOWAK (Poznań)

On two linear topologies on Orlicz spaces $L^{*\varphi}$. I

Abstract. In this paper we consider two linear topologies on Orlicz spaces $L_E^{*\varphi}$, where φ is a φ-function and E is a finite-dimensional Euclidean space with the usual Lebesgue measure. We denote these topologies by $\mathcal{T}^{\triangleleft\varphi}$ and $\mathcal{T}^{\ll\varphi}$. In § 1 we prove that if φ is an N-function, then these topologies are locally convex. In § 2 we compare the topologies $\mathcal{T}^{\triangleleft\varphi}$ and $\mathcal{T}^{\ll\varphi}$ with the topology generated by the F-norm $\|\cdot\|_\varphi$ in $L_E^{*\varphi}$, and in § 3 compare the φ-modular convergence of sequences (x_n) in $L_E^{*\varphi}$ with convergence in the topologies $\mathcal{T}^{\triangleleft\varphi}$ and $\mathcal{T}^{\ll\varphi}$. Finally, in § 4 and § 5 we prove that the spaces $(L_E^{*\varphi}, \mathcal{T}^{\triangleleft\varphi})$ and $(L_E^{*\varphi}, \mathcal{T}^{\ll\varphi})$ are separable and complete.

In the case where φ is the Orlicz function which satisfies the Δ_2-condition and E is a finite non-atomic measure space, then the topology $\mathcal{T}^{\ll\varphi}$ was considered by Ph. Turpin ([8], Chapter I, Theorem 1.2.2).

§ 0. Introduction.

0.1. φ-*functions.*

0.1.1. It is said that a function $\varphi: [0, \infty)\to[0, \infty)$ is a φ-*function* if it is continuous, non-decreasing and such that $\varphi(0) = 0$, $\varphi(u) > 0$ for $u > 0$ and $\varphi(u)\to\infty$ for $u\to\infty$.

0.1.2. A φ-function φ is called an N-*function* if it admits the representation

$$\varphi(u) = \int_0^u p(t)\,dt,$$

where $p(t)$ is right-continuous for $t \geqslant 0$, $p(0) = 0$, $p(t) > 0$ for $t > 0$, non-decreasing and $p(t)\to\infty$ for $t\to\infty$ ([2], p. 6).

It is sometimes useful to use the following definition of an N-function: A φ-function φ is called an N-*function* if φ is convex and satisfies the conditions

$$(0_1) \qquad \lim_{u\to 0}\frac{\varphi(u)}{u} = 0,$$

$$(\infty_1) \qquad \lim_{u\to\infty}\frac{\varphi(u)}{u} = \infty \qquad ([2],\ \text{p. } 9).$$

0.1.3. Let φ and ψ be φ-functions.

(a) It is wrote $\psi \prec \varphi$ if there exist constants $a > 0$ and $b > 0$ such that

$$\psi(u) \leqslant a\varphi(bu) \quad \text{for} \quad u \geqslant 0.$$

(b) We shall say that a φ-function φ *increases essentially more rapidly than a φ-function* ψ and shall write $\psi \ll \varphi$ if, for an arbitrary $c > 0$,

$$\lim_{u\to 0}\frac{\psi(cu)}{\varphi(u)} = 0 \quad \text{and} \quad \lim_{u\to\infty}\frac{\psi(cu)}{\varphi(u)} = 0 \quad ([2], \text{ p. } 144; [5]).$$

(c) We shall write $\psi \lhd \varphi$ if, for an arbitrary $c > 0$, there exists a constant $d > 0$ such that

$$\psi(cu) \leqslant d\varphi(u) \quad \text{for} \quad u \geqslant 0.$$

It is easy to check that if $\psi \ll \varphi$, then $\psi \lhd \varphi$ ([5]).

(d) It is said that φ satisfies the *Δ_2-condition* if there exists a constant $d > 1$ such that

$$\varphi(2u) \leqslant d\varphi(u) \quad \text{for} \quad u \geqslant 0 \quad ([2], \text{ p. } 23).$$

(e) It is said that φ satisfies the *V_2-condition* for large (small) values of u if there exist constants $d > 1$ and $u_0 > 0$ such that

$$2\varphi(u) \leqslant \varphi(du) \quad \text{for} \quad u \geqslant u_0 \quad (u \leqslant u_0).$$

0.2. *Orlicz spaces.*

0.2.1. Let E be a finite-dimensional Euclidean space with the usual Lebesgue measure. For a φ-function φ and a real valued function x defined and measurable on E we write

$$\varrho_\varphi(x) = \int_E \varphi(|x(t)|)\,dt.$$

By $L^{*\varphi}$ we denote the linear space of those functions x for which $\varrho_\varphi(\lambda x) < \infty$ for some $\lambda > 0$, and by $L^{0\varphi}$ the linear space of those functions x for which $\varrho_\varphi(\lambda x) < \infty$ for all $\lambda > 0$ ([3]). It is well known that if $\psi \prec \varphi$, then $L^{*\varphi} \subset L^{*\psi}$ ([3]). It is easy to verify that if $\psi \lhd \varphi$, then $L^{*\varphi} \subset L^{0\psi}$ ([5], proof of Theorem 1).

0.2.2. Let φ be a φ-function. A sequence (x_n) in $L^{*\varphi}$ is called *φ-modular convergent* to $x \in L^{*\varphi}$ and is denoted by

$$x_n \xrightarrow{\varphi} x$$

if there exists a constant $\lambda_0 > 0$ such that $\varrho_\varphi(\lambda_0(x_n - x)) \to 0$ ([3]).

0.2.3. In $L^{*\varphi}$ we may define an *F-norm*:

$$\|x\|_\varphi = \inf\{\lambda > 0 \colon \varrho_\varphi(x/\lambda) \leqslant \lambda\}.$$

The space $L^{*\varphi}$ is complete with respect to the F-norm $\| \|_\varphi$. Moreover, if φ is an N-function, then in $L^{*\varphi}$ we may define also a B-norm:

$$\|x\|_\varphi^0 = \inf\{\lambda > 0 : \varrho_\varphi(x/\lambda) \leqslant 1\};$$

this norm is equivalent to the F-norm $\| \|_\varphi$ ([4]).

Throughout the paper we shall denote by $\mathcal{T}_\varphi$ the topology on $L^{*\varphi}$ generated by the F-norm $\| \|_\varphi$.

0.2.4. THEOREM. *Let* (x_n) *be a sequence in* $L^{*\varphi}$. *Then*

(a) $\|x_n\|_\varphi \to 0$ *implies* $x_n \xrightarrow{\varphi} 0$,

(b) $x_n \xrightarrow{\varphi} 0$ *implies* $\|x_n\|_\varphi \to 0$ *if and only if* φ *satisfies the* Δ_2*-condition* ([3]).

0.3. *Some equalities for Orlicz spaces.* Let φ be a φ-function. Denote by $\Psi^{\lhd\varphi}$ ($\Psi^{\ll\varphi}$) the set of all φ-functions ψ such that $\psi \lhd \varphi$ ($\psi \ll \varphi$). It is shown in [5], Theorem 2, that

$$L^{*\varphi} = \bigcap_{\psi \in \Psi^{\ll\varphi}} L^{\circ\psi}. \tag{1}$$

In the case where φ is an Orlicz function which satisfies the Δ_2-condition and E is a finite non-atomic measure space, then the above equality was obtained by Ph. Turpin ([8], Chapter I, Theorem 1.2.2). Since $L^{*\varphi} \subset L^{\circ\psi}$ for each $\psi \in \Psi^{\lhd\varphi}$ and $\Psi^{\ll\varphi} \subset \Psi^{\lhd\varphi}$, it follows from equality (1) that

$$L^{*\varphi} = \bigcap_{\psi \in \Psi^{\lhd\varphi}} L^{\circ\psi}. \tag{2}$$

0.4. *Topology of convergence in measure.*

0.4.1. Let S be the linear space of real-valued functions defined and measurable on E. Then in S a pseudo-modular may be defined as follows:

$$\varrho_0(x) = \left|\{t \in E : |x(t)| > 1\}\right|.$$

Let S_0 be the subspace of S consisting of all functions which are almost everywhere finite valued. Then in S_0 an F-norm $\| \|_0$ may be defined as follows:

$$\|x\|_0 = \inf\left\{\varepsilon > 0 : \left|\{t \in E : |x(t)| > \varepsilon\}\right| \leqslant \varepsilon\right\}.$$

Throughout this paper we shall denote by $\mathcal{T}_0$ the topology on S_0 generated by the F-norm $\| \|_0$.

It is seen that a sequence (x_n) in S_0 is convergent to $x \in S_0$ in the topology $\mathcal{T}_0$ if and only if a sequence (x_n) is convergent to x in measure. Moreover, for every φ-function φ we have that $L^{*\varphi} \subset S_0$ and that the topology $\mathcal{T}_\varphi$ is strictly finer than the topology $\mathcal{T}_0$ restricted to $L^{*\varphi}$ ([8], p. 30).

0.4.2. THEOREM. *The space* $(S_0, \mathcal{T}_0)$ *is complete* ([8], p. 30).

0.4.3. THEOREM. *Let φ be a φ-function. Then the balls*

$$\bar{K}_\varphi(r) = \{x \in L^{*\varphi}\colon \|x\|_\varphi \leqslant r\}, \quad \textit{where } r > 0$$

are closed in the space $(S_0, \mathscr{T}_0)$ ([8], p. 30).

§ 1. Definitions of linear topologies $\mathscr{T}^{\lhd\varphi}$ and $\mathscr{T}^{\ll\varphi}$ on Orlicz spaces.

1.1. DEFINITION. Let φ be a φ-function. Since $L^{*\varphi} \subset L^{\circ\psi}$ for every $\psi \in \Psi^{\lhd\varphi}$, we have two linear projective systems:

$$(1) \qquad j_\psi\colon L^{*\varphi} \hookrightarrow (L^{\circ\psi}, \mathscr{T}'_\psi), \quad \text{where} \quad \psi \in \Psi^{\lhd\varphi},$$

$$(2) \qquad j_\psi\colon L^{*\varphi} \hookrightarrow (L^{\circ\psi}, \mathscr{T}'_\psi), \quad \text{where} \quad \psi \in \Psi^{\ll\varphi},$$

where $\mathscr{T}'_\psi$ denotes, for every $\psi \in \Psi^{\lhd\varphi}$ ($\psi \in \Psi^{\ll\varphi}$), the usual topology $\mathscr{T}_\psi$ on $L^{*\psi}$ restricted to $L^{\circ\psi}$.

We shall denote by $\mathscr{T}^{\lhd\varphi}$ and $\mathscr{T}^{\ll\varphi}$ the linear topologies of the projective systems (1) and (2).

The topology $\mathscr{T}^{\lhd\varphi}$ ($\mathscr{T}^{\ll\varphi}$) has a base of neighbourhoods of 0 consisting of all sets

$$\bigcap_{i=1}^{k} j_{\psi_i}^{-1}(U_{\psi_i}),$$

where $\psi_i \in \Psi^{\lhd\varphi}$ ($\psi_i \in \Psi^{\ll\varphi}$) and U_{ψ_i} is a neighbourhood of 0 for the topology $\mathscr{T}'_\psi$ on $L^{\circ\psi_i}$ ([7]). Since $j_{\psi_i}^{-1}(U_{\psi_i}) = L^{*\varphi} \cap U_{\psi_i}$ and $U_{\psi_i} = K_{\psi_i}(r_i) \cap L^{\circ\psi_i}$, where $r_i > 0$ and $K_{\psi_i}(r_i) = \{x \in L^{*\psi_i}\colon \|x\|_{\psi_i} < r_i\}$, this base of neighbourhoods of 0 for $\mathscr{T}^{\lhd\varphi}$ ($\mathscr{T}^{\ll\varphi}$) consists of all sets of the form

$$\bigcap_{i=1}^{k} K_{\psi_i}(r_i) \cap L^{*\varphi}, \quad \text{where} \quad \psi_i \in \Psi^{\lhd\varphi}\ (\psi_i \in \Psi^{\ll\varphi}) \text{ and } r_i > 0.$$

1.2. THEOREM. *The topology $\mathscr{T}^{\lhd\varphi}$ ($\mathscr{T}^{\ll\varphi}$) has a base of neighbourhoods of 0 consisting of all sets of the form*

$$K_\psi(r) \cap L^{*\varphi}, \quad \textit{where} \quad \psi \in \Psi^{\lhd\varphi}\ (\psi \in \Psi^{\ll\varphi}) \textit{ and } r > 0.$$

Proof. It suffices to show that, if $\psi_1, \ldots, \psi_k \in \Psi^{\lhd\varphi}$ ($\psi_1, \ldots, \psi_k \in \Psi^{\ll\varphi}$) and $r_1, \ldots, r_k > 0$, then there exists $\psi \in \Psi^{\lhd\varphi}$ ($\psi \in \Psi^{\ll\varphi}$) and a number $r > 0$ such that

$$K_\psi(r) \subset \bigcap_{i=1}^{k} K_{\psi_i}(r_i).$$

In fact, let us set $\psi(u) = \max\big(\psi_1(u), \ldots, \psi_k(u)\big)$ for $u \geqslant 0$. Then $\psi \in \Psi^{\lhd\varphi}$ ($\psi \in \Psi^{\ll\varphi}$). Since $\psi_i(u) \leqslant \psi(u)$ for $u \geqslant 0$, it follows from 0.2.1 that

$$L^{*\psi} \subset L^{*\psi_i} \quad \text{and} \quad \|x\|_{\psi_i} \leqslant \|x\|_\psi \quad \text{for every } x \in L^{*\psi}.$$

Hence, for $r = \min(r_1, \ldots, r_k)$, we have

$$K_\psi(r) \subset K_{\psi_i}(r_i) \quad \text{for} \quad i = 1, \ldots, k.$$

1.3. Corollary. *A sequence* (x_n) *in* $L^{*\varphi}$ *is convergent to* $x \in L^{*\varphi}$ *in the topology* $\mathscr{T}^{\lhd\varphi}$ $(\mathscr{T}^{\ll\varphi})$ *if and only if*

$$\|x_n - x\|_\psi \to 0 \quad \text{for every } \psi \in \Psi^{\lhd\varphi} \ (\psi \in \Psi^{\ll\varphi}).$$

Now, we shall show that if φ is an N-function, then the topologies $\mathscr{T}^{\lhd\varphi}$ and $\mathscr{T}^{\ll\varphi}$ are locally convex. The proof is based on the following lemma.

1.4. Lemma. *Let* φ *be an* N*-function and* ψ *a* φ*-function such that* $\psi \ll \varphi$ $(\psi \lhd \varphi)$. *Then, there exists an* N*-function* ψ_0 *such that*

$$\psi_0 \ll \varphi \ (\psi_0 \lhd \varphi) \quad \text{and} \quad \psi(u) \leqslant \psi_0(2u) \ \text{for} \quad u \geqslant 0.$$

Proof. Take an arbitrary N-function ψ_1 such that $\psi_1 \ll \varphi$ $(\psi_1 \lhd \varphi)$. Let us set $\psi_2(u) = \max\big(\psi(u), \psi_1(u)\big)$ for $u \geqslant 0$. We see that ψ_2 satisfies conditions (0_1) and (∞_1), i.e.

$$\lim_{u\to 0} \frac{\psi_2(u)}{u} = 0 \quad \text{and} \quad \lim_{u\to\infty} \frac{\psi_2(u)}{u} = \infty.$$

Indeed, we have

$$\lim_{u\to 0} \frac{\psi_2(u)}{u} \leqslant \lim_{u\to 0} \frac{\psi(u)+\psi_1(u)}{u} \leqslant \lim_{u\to 0} \frac{\varphi(u)}{u} + \lim_{u\to 0} \frac{\psi_1(u)}{u} = 0,$$

$$\lim_{u\to\infty} \frac{\psi_2(u)}{u} \geqslant \lim_{u\to\infty} \frac{\psi_1(u)}{u} = \infty.$$

Next, let us put

$$p(s) = \begin{cases} 0 & \text{for} \quad s = 0, \\ \sup\limits_{0<t\leqslant s} \big(\psi_2(t)/t\big) & \text{for} \quad s > 0. \end{cases}$$

At last, we define a function ψ_0 by the equality

$$\psi_0(u) = \int_0^u p(s)\,ds.$$

In virtue of the conditions (0_1) and (∞_1), the function ψ_0 is an N-function. It is easy to verify that

$$\psi(u) \leqslant \psi_0(2u) \quad \text{for} \quad u \geqslant 0 \quad \text{if} \quad \psi \ll \varphi \ (\psi \lhd \varphi).$$

Indeed, since

$$\psi_0(2u) = \int_0^{2u} p(s)\,ds \geqslant p(u)u,$$

and

$$p(u) = \sup_{0<t\leqslant u} \frac{\psi_2(t)}{t} \geqslant \sup_{0<t\leqslant u} \frac{\psi(t)}{t} \geqslant \frac{\psi(u)}{u},$$

we have

$$\psi_0(2u) \geqslant \frac{\psi(u)}{u} u = \psi(u) \quad \text{for} \quad u \geqslant 0.$$

Now, we shall show that if $\psi \ll \varphi$ and $\psi_1 \ll \varphi$ then $\psi_0 \ll \varphi$, i.e. for arbitrary $c > 0$, $\varepsilon > 0$ there exist constants $u_0^1 > 0$ and $u_0^2 > 0$ such that

$$\psi_0(cu) \leqslant \varepsilon\varphi(u) \quad \text{for} \quad u \leqslant u_0^1 \text{ and } u \geqslant u_0^2.$$

Since $\psi_0(cu) \leqslant p(cu)cu$, it suffices to show that

$$p(cu) \leqslant \varepsilon\varphi(u)/cu \quad \text{for} \quad u \leqslant u_0^1 \text{ and } u \geqslant u_0^2.$$

In fact, since $\psi \ll \varphi$, $\psi_1 \ll \varphi$, there exist constants $u_1 > 0$ and $u_2 > 0$ such that

$$\psi(u) \leqslant \varepsilon\varphi(u/c), \quad \psi_1(u) \leqslant \varepsilon\varphi(u/c) \quad \text{for} \quad u \leqslant u_1 \text{ and } u \geqslant u_2.$$

But $\psi_2(u) = \max\big(\psi(u), \psi_1(u)\big)$, hence there holds

$$\psi_2(u) \leqslant \varepsilon\varphi(u/c) \quad \text{for} \quad u \leqslant u_1 \text{ and } u \geqslant u_2.$$

First, let $u_0^1 = u_1/c$. Since $cu \leqslant cu_0^1 \leqslant u_1$ for $u \leqslant u_0^1$, we obtain

$$p(cu) = \sup_{0<t\leqslant cu} \frac{\psi_2(t)}{t} \leqslant \sup_{0<t\leqslant cu} \frac{\varepsilon\varphi(t/c)}{t} = \frac{\varepsilon\varphi(u)}{cu}.$$

Next, let $u_2' > 0$ be a number such that $\varepsilon\varphi(u)/cu \geqslant K = \sup_{0<t\leqslant u_2} \big(\psi_2(t)/t\big)$ for $u \geqslant u_2'$. Then, for $u \geqslant u_0^2 = \max\,(u_2, u_2')$, we obtain

$$\begin{aligned} p(cu) &= \sup_{0<t\leqslant cu} \frac{\psi_2(t)}{t} = \max\left(\sup_{0<t\leqslant u_2} \frac{\psi_2(t)}{t}, \sup_{u_2\leqslant t\leqslant cu} \frac{\psi_2(t)}{t}\right) \\ &\leqslant \max\left(K, \sup_{u_2\leqslant t\leqslant cu} \frac{\varepsilon\varphi(t/c)}{t}\right) = \max\left(K, \frac{\varepsilon\varphi(u)}{cu}\right) = \frac{\varepsilon\varphi(u)}{cu}. \end{aligned}$$

Thus, we obtain $\psi_0 \ll \varphi$.

Finally, we shall show that if $\psi \lhd \varphi$ and $\psi_1 \lhd \varphi$, then $\psi_0 \lhd \varphi$, i.e. for an arbitrary $c > 0$ there exists a constant $d > 0$ such that

$$\psi_0(cu) \leqslant d\varphi(u) \quad \text{for} \quad u \geqslant 0.$$

In fact, since $\psi \lhd \varphi$ and $\psi_1 \lhd \varphi$, it follows that for an arbitrary $c > 0$ there exists a constant $d > 0$ such that

$$\psi(u) \leqslant d\varphi(u/c) \quad \text{and} \quad \psi_1(u) \leqslant d\varphi(u/c) \quad \text{for} \quad u \geqslant 0,$$

from which it follows that

$$\psi_2(u) \leqslant d\varphi(u/c) \quad \text{for} \quad u \geqslant 0.$$

Therefore

$$p(cu) = \sup_{0<t\leqslant cu} \frac{\psi_2(t)}{t} \leqslant \sup_{0<t\leqslant cu} \frac{d\varphi(u/c)}{t} = \frac{d\varphi(u)}{cu};$$

hence $\psi_0(cu) \leqslant p(cu)cu \leqslant d\varphi(u)$ for $u \geqslant 0$. Thus, we obtain $\psi_0 \lhd \varphi$.

1.5. THEOREM. *If φ is an N-function, then the topologies $\mathscr{T}^{\lhd\varphi}$ and $\mathscr{T}^{\ll\varphi}$ are locally convex.*

Proof. From Theorem 1.2 we know that the system of all sets

$$K_\psi(r) \cap L^{*\varphi}, \quad \text{where} \quad \psi \in \Psi^{\lhd\varphi}\ (\psi \in \Psi^{\ll\varphi}),\ r > 0,$$

constitutes a base of neighbourhoods of 0 for $\mathscr{T}^{\lhd\varphi}(\mathscr{T}^{\ll\varphi})$. For an N-function ψ_0 we define

$$K^\circ_{\psi_0}(r) = \{x \in L^{*\psi_0}\colon \|x\|^\circ_{\psi_0} < r\},$$

where $\|\ \|^\circ_{\psi_0}$ is the B-norm in $L^{*\psi_0}$ which is equivalent to the F-norm $\|\ \|_{\psi_0}$. We shall prove that the system of all sets

$$K^\circ_{\psi_0}(r) \cap L^{*\varphi}, \quad \text{where } \psi_0 \text{ is an } N\text{-function}, \ \psi_0 \lhd \varphi\ (\psi_0 \ll \varphi),\ r > 0,$$

constitutes a base of neighbourhoods of 0 for $\mathscr{T}^{\lhd\varphi}(\mathscr{T}^{\ll\varphi})$. Since for an arbitrary N-function ψ_0 the F-norms $\|\ \|_{\psi_0}$ and $\|\ \|^\circ_{\psi_0}$ are equivalent in $L^{*\psi_0}$, it suffices to show that for every neighbourhood of 0 for $\mathscr{T}^{\lhd\varphi}(\mathscr{T}^{\ll\varphi})$ of the form

$$K_\psi(r) \cap L^{*\varphi}, \quad \text{where} \quad \psi \in \Psi^{\lhd\varphi}\ (\psi \in \Psi^{\ll\varphi}),\ r > 0,$$

there exist an N-function $\psi_0 \in \Psi^{\lhd\varphi}$ ($\psi_0 \in \Psi^{\ll\varphi}$) and $r_0 > 0$ such that

$$K^\circ_{\psi_0}(r) \cap L^{*\varphi} \subset K_\psi(r) \cap L^{*\varphi}.$$

Indeed, let ψ be an arbitrary φ-function such that $\psi \in \Psi^{\lhd\varphi}$ ($\psi \in \Psi^{\ll\varphi}$). Let r be an arbitrary positive number. Then, from Lemma 1.4 it follows that we can find an N-function $\psi_0 \in \Psi^{\lhd\varphi}$ ($\psi_0 \in \Psi^{\ll\varphi}$) such that

$$\psi(u) \leqslant \psi_0(2u) \quad \text{for} \quad u \geqslant 0.$$

Hence

$$(*) \qquad \|x\|_\psi \leqslant \|2x\|_{\psi_0} \quad \text{for every } x \in L^{*\psi_0}.$$

On the other hand, since the F-norms $\|\ \|_{\psi_0}$ and $\|\ \|^\circ_{\psi_0}$ are equivalent in $L^{*\psi_0}$, it follows that there exists $r_1 > 0$ such that

$$(**) \qquad K^\circ_{\psi_0}(r_1) \subset K_{\psi_0}(r).$$

Now, let $r_0 = r_1/2$ and let $x \in K^\circ_{\psi_0}(r_0) \cap L^{*\varphi}$.

Then $\|2x\|^{\circ}_{\varphi_0} < r_1$, and using (**) we have $\|2x\|_{\varphi_0} < r$. Hence by (*) we obtain

$$\|x\|_{\psi} \leqslant \|2x\|_{\varphi_0} < r.$$

Thus $x \in K_{\psi}(r) \cap L^{*\varphi}$.

§2. Comparison of the topologies $\mathscr{T}^{\lhd\varphi}$, $\mathscr{T}^{\ll\varphi}$ and $\mathscr{T}_{\varphi}$.

2.1. THEOREM. *Let φ be a φ-function. Then*

$$\mathscr{T}^{\ll\varphi} \subset \mathscr{T}^{\lhd\varphi}.$$

Proof. It follows from this that $\Psi^{\ll\varphi} \subset \Psi^{\lhd\varphi}$.

2.2. THEOREM. *Let φ be a φ-function. Then*

$$\mathscr{T}^{\lhd\varphi} \subset \mathscr{T}_{\varphi}.$$

Proof. Since the system of all sets

$$K_{\psi}(r) \cap L^{*\varphi}, \quad \text{where} \quad \psi \in \Psi^{\lhd\varphi},$$

constitutes a base of neighbourhoods of 0 for $\mathscr{T}^{\lhd\varphi}$, it suffices to show that for every $\psi \in \Psi^{\lhd\varphi}$ and $r > 0$ there exists $r_1 > 0$ such that

$$K_{\varphi}(r_1) \subset K_{\psi}(r).$$

Indeed, let ψ be an arbitrary φ-function such that $\psi \in \Psi^{\lhd\varphi}$. Let r be an arbitrary positive number. Then there exists a constant $d > 1$ such that

$$\psi\left(\frac{2u}{r}\right) \leqslant d\varphi(u) \quad \text{for} \quad u \geqslant 0.$$

Let $r_1 = \min(2u/r, 1)$. Then $\|x\|_{\varphi} < r_1$ implies $\|x\|_{\psi} < r$. Indeed, if $\|x\|_{\varphi} < r_1 \leqslant 1$, then $\varrho_{\varphi}(x) < r/2d$, and hence

$$\varrho_{\psi}\left(\frac{2}{r}|x|\right) = \int_E \psi\left(\frac{2}{r}|x(t)|\right) dt \leqslant d \int_E \varphi\big(|x(t)|\big) dt = d\varrho_{\varphi}(x) < \frac{r}{2}.$$

Hence $\|x\|_{\psi} \leqslant r/2 < r$.

From Theorems 2.1 and 2.2 we have

$$\mathscr{T}^{\ll\varphi} \subset \mathscr{T}^{\lhd\varphi} \subset \mathscr{T}_{\varphi}.$$

2.3. THEOREM. *Suppose that a φ-function φ satisfies the Δ_2-condition. Then*

$$\mathscr{T}^{\lhd\varphi} = \mathscr{T}_{\varphi}.$$

Proof. It suffices to notice that if φ satisfies the Δ_2-condition, then $\varphi \in \Psi^{\lhd\varphi}$.

Now we shall show that if φ does not satisfy the Δ_2-condition, then $\mathcal{T}^{\lhd\varphi}$ is strictly weaker than $\mathcal{T}_\varphi$. Namely, we have the following theorem.

2.4. THEOREM. *Suppose that a φ-function φ does not satisfy the Δ_2-condition. Then there exists a sequence (x_n) in $L^{*\varphi}$ such that*

$$x_n \xrightarrow{\mathcal{T}^{\lhd\varphi}} 0 \quad \textit{and} \quad \varrho_\varphi(x_n) > 1 \quad \textit{for every integer } n > 0.$$

Proof. Since φ does not satisfy the Δ_2-condition, it follows that for an integer $n > 0$ there exists a number $u_n > 0$ such that

$$(*) \qquad \varphi(2u_n) > n\varphi(u_n).$$

Let (E_n) be a sequence of measurable sets in E such that

$$|E_n| = \frac{1}{n\varphi(u_n)},$$

and let us put

$$x_n(t) = \begin{cases} 2u_n & \text{for } t \in E_n, \\ 0 & \text{for } t \notin E_n. \end{cases}$$

Then $\|x_n\|_\psi \to 0$ for every $\psi \in \Psi^{\lhd\varphi}$, i.e. $x_n \xrightarrow{\mathcal{T}^{\lhd\varphi}} 0$.

Indeed, let ψ be an arbitrary φ-function such that $\psi \in \Psi^{\lhd\varphi}$. Let ε be an arbitrary positive number. Then there exists a constant $d > 0$ such that

$$\psi\left(\frac{2u}{\varepsilon}\right) \leqslant d\varphi(u) \quad \text{for} \quad u \geqslant 0.$$

Let N be a natural number such that $N \geqslant d/\varepsilon$. Hence, for $n \geqslant N$ we obtain

$$\varrho_\psi\left(\frac{x_n}{\varepsilon}\right) = \psi\left(\frac{2u_n}{\varepsilon}\right)\cdot\frac{1}{n\varphi(u_n)} \leqslant \frac{d}{n} \leqslant \varepsilon,$$

so we have $\|x_n\|_\psi \leqslant \varepsilon$. On the other hand, by $(*)$ we obtain

$$\varrho_\varphi(x_n) = \frac{\varphi(2u_n)}{n\varphi(u_n)} > 1 \quad \text{for every positive integer } n;$$

hence $\|x_n\|_\varphi \not\to 0$.

The topology $\mathcal{T}^{\ll\varphi}$ is always strictly weaker than the topology $\mathcal{T}_\varphi$.

2.5. THEOREM. *Let φ be a φ-function. Then there exists a sequence (x_n) in $L^{*\varphi}$ such that*

$$x_n \xrightarrow{\mathcal{T}^{\ll\varphi}} 0 \quad \textit{and} \quad \varrho_\varphi(x_n) = 1.$$

Proof. Let (u_n) be an arbitrary sequence of positive numbers such that $u_n \to \infty$. Then, let (E_n) be a sequence of measurable sets in E such that

$$|E_n| = 1/\varphi(u_n).$$

Define

$$x_n(t) = \begin{cases} u_n & \text{for} \quad t \in E_n, \\ 0 & \text{for} \quad t \notin E_n. \end{cases}$$

Then

$$\|x_n\|_\psi \to 0 \quad \text{for every } \psi \in \Psi^{\ll\varphi}, \quad \text{i.e.} \quad x_n \xrightarrow{\mathcal{T}^{\ll\varphi}} 0.$$

In fact, let ψ be an arbitrary φ-function such that $\psi \in \Psi^{\ll\varphi}$. Let ε be an arbitrary positive number. Then there exists a number $u_0 > 0$ such that

$$\psi(u/\varepsilon) \leqslant \varepsilon\varphi(u) \quad \text{for} \quad u \geqslant u_0.$$

Let N be a natural number such that $u_n \geqslant u_0$ for $n \geqslant N$. Then

$$\varrho_\psi(x_n/\varepsilon) = \psi(u_n/\varepsilon) \cdot \frac{1}{\varphi(u_n)} \leqslant \varepsilon \quad \text{for} \quad n \geqslant N;$$

hence $\|x_n\|_\psi \to 0$. Thus, we obtain $x_n \xrightarrow{\mathcal{T}^{\ll\varphi}} 0$.

On the other hand, $\varrho_\varphi(x_n) = \varphi(u_n)/\varphi(u_n) = 1$; hence $x_n \overset{\mathcal{T}_\varphi}{\not\to} 0$.

§ 3. Comparison of convergence of sequences (x_n) in $L^{*\varphi}$ in the topologies $\mathcal{T}^{\lhd\varphi}$ and $\mathcal{T}^{\ll\varphi}$ with the φ-modular convergence.

3.1. Lemma. *Let φ and ψ be φ-functions such that $\psi \lhd \varphi$. Then $x_n \xrightarrow{\varphi} 0$ implies $\|x_n\|_\psi \to 0$ for a sequence (x_n) in $L^{*\varphi}$.*

Proof. Let λ_0 be a positive number such that $\varrho_\varphi(\lambda_0 x_n) \to 0$. Since $\psi \lhd \varphi$, for an arbitrary number $\varepsilon > 0$, there exists a constant $d > 0$ such that

$$(*) \qquad \psi(u/\varepsilon) \leqslant d\varphi(\lambda_0 u) \quad \text{for} \quad u \geqslant 0.$$

We have

$$\int_E \varphi\big(\lambda_0 |x_n(t)|\big)\, dt \to 0;$$

hence there exists a natural number N such that

$$(**) \qquad \int_E \varphi\big(\lambda_0 |x_n(t)|\big)\, dt \leqslant \varepsilon/d \quad \text{for} \quad n \geqslant N.$$

From $(*)$ and $(**)$ it follows that

$$\int_E \psi\left(\frac{|x_n(t)|}{\varepsilon}\right) dt \leqslant d \int_E \varphi\big(\lambda_0 |x_n(t)|\big)\, dt \leqslant d\frac{\varepsilon}{d} = \varepsilon \quad \text{for} \quad n \geqslant N;$$

hence $\|x_n\|_\psi \leqslant \varepsilon$ for $n \geqslant N$.

This lemma implies the following theorem.

3.2. THEOREM. *Let φ be a φ-function. Then $x_n \xrightarrow{\varphi} 0$ implies $x_n \xrightarrow{\mathscr{T}^{\lhd\varphi}} 0$ for a sequence (x_n) in $L^{*\varphi}$.*

3.3. COROLLARY. *Suppose that φ satisfies the Δ_2-condition. Then $x_n \xrightarrow{\varphi} 0$ if and only if $x_n \xrightarrow{\mathscr{T}^{\lhd\varphi}} 0$ for any sequence (x_n) in $L^{*\varphi}$.*

Proof. It suffices to remark that if φ satisfies the Δ_2-condition, then by Theorem 2.3 we have that $\mathscr{T}_\varphi = \mathscr{T}^{\lhd\varphi}$.

3.4. THEOREM. *Suppose that a φ-function φ does not satisfy the V_2-condition for large values of u. Then there exists a sequence (x_n) in $L^{*\varphi}$ such that*

$$x_n \xrightarrow{\mathscr{T}^{\ll\varphi}} 0 \quad \text{and} \quad x_n \overset{\varphi}{\not\to} 0.$$

Proof. Since φ does not satisfy the V_2-condition (0.1.3.e) for large values u, it follows that for every positive integer n there exists a real number u_n with $u_n \geqslant n$ such that

$$(*) \qquad 2\varphi(u_n) > \varphi(nu_n).$$

Let (E_n) be a sequence of measurable sets in E such that

$$|E_n| = 1/\varphi(nu_n),$$

and let us put

$$x_n(t) = \begin{cases} nu_n & \text{for } t \in E_n, \\ 0 & \text{for } t \notin E_n. \end{cases}$$

First, we shall show that $x_n \xrightarrow{\mathscr{T}^{\ll\varphi}} 0$. Indeed, let ψ be an arbitrary φ-function such that $\psi \in \Psi^{\ll\varphi}$. Let ε be an arbitrary positive number. Then there exists a number $u_0 > 0$ such that

$$\psi\left(\frac{u}{\varepsilon}\right) \leqslant \varepsilon\varphi(u) \quad \text{for} \quad u \geqslant u_0.$$

Let N be a natural number such that $nu_n \geqslant u_0$ for $n \geqslant N$. Then

$$\varrho_\psi\left(\frac{x_n}{\varepsilon}\right) = \psi\left(\frac{nu_n}{\varepsilon}\right)\frac{1}{\varphi(nu_n)} \leqslant \varepsilon \quad \text{for} \quad n \geqslant N;$$

hence $\|x_n\|_\psi \leqslant \varepsilon$ for $n \geqslant N$. Thus, we obtain $x_n \xrightarrow{\mathscr{T}^{\ll\varphi}} 0$. On the other hand, for any real number $\lambda_0 > 0$, there holds by $(*)$

$$\varrho_\varphi(\lambda_0 x_n) \geqslant \varrho_\varphi\left(\frac{x_n}{n}\right) = \frac{\varphi(u_n)}{\varphi(nu_n)} > \frac{1}{2} \quad \text{for every integer } n \geqslant \frac{1}{\lambda_0}.$$

Thus $x_n \overset{\varphi}{\not\to} 0$.

§ 4. The separability of the spaces $(L^{*\varphi}, \mathscr{T}^{\ll\varphi})$ and $(L^{*\varphi}, \mathscr{T}^{\lhd\varphi})$. We shall prove that the space $(L^{*\varphi}, \mathscr{T}^{\lhd\varphi})$ is separable. Then, since $\mathscr{T}^{\ll\varphi} \subset \mathscr{T}^{\lhd\varphi}$, the space $(L^{*\varphi}, \mathscr{T}^{\ll\varphi})$ also will be separable.

4.1. Lemma. *Let $E = \mathbf{R}^k$ and let φ be a φ-function. Let $B_0 \subset L^{*\varphi}$ be the set of all measurable, bounded functions vanishing outside intervals $A_n = \underbrace{(-n,n)\times \ldots \times(-n,n)}_{k \text{ times}}$ for some $n > 0$. Then the set B_0 is dense in $L^{*\varphi}$ in the sense of φ-modular convergence.*

Proof. We must show that, if $x \in L^{*\varphi}$, then there exists a sequence (x_n) in B_0 such that $x_n \xrightarrow{\varphi} x$. Let $x \in L^{*\varphi}$. Then $\varrho_\varphi(\lambda_0 x) < \infty$ for some $\lambda_0 > 0$. Let us put

$$x_n(t) = \begin{cases} x(t) & \text{for} \quad |x(t)| \leqslant n \text{ and } t \in A_n, \\ 0 & \text{for} \quad |x(t)| > n \text{ or } t \notin A_n. \end{cases}$$

Then $x_n \in B_0$ and $\int\limits_E \varphi\big(\lambda_0|x(t)-x_n(t)|\big)\,dt \to 0$.

In fact, we have

$$\varphi\big(\lambda_0|x(t)-x_n(t)|\big) \leqslant \varphi\big(\lambda_0|x(t)|\big),$$

and

$$\varphi\big(\lambda_0|x(t)-x_n(t)|\big) \to 0 \quad \text{almost everywhere}.$$

Hence, by Lebesgue's bounded convergence theorem ([1], p. 110) we have

$$\int\limits_E \varphi\big(\lambda_0|x(t)-x_n(t)|\big)\,dt \to 0.$$

4.2. Corollary. *The set B_0 is dense in $(L^{*\varphi}, \mathscr{T}^{\lhd\varphi})$.*

Proof. This follows from Theorem 3.2.

4.3. Theorem. *The space $(L^{*\varphi}, \mathscr{T}^{\lhd\varphi})$ is separable.*

Proof. Let W_0 be the set of polynomials with rational coefficients, vanishing outside intervals $A_n = \underbrace{(-n,n)\times \ldots \times(-n,n)}_{k \text{ times}}$ for some $n > 0$. It is well known that W_0 is dense in $L^{\circ\varphi}$ in the topology $\mathscr{T}_\varphi$([3]). Since $\mathscr{T}^{\lhd\varphi} \subset \mathscr{T}_\varphi$, it follows that W_0 is dense in $L^{\circ\varphi}$ in the topology $\mathscr{T}^{\lhd\varphi}$, i.e. $\overline{W}_0 \supset L^{\circ\varphi}$, where a closure is taken in $\mathscr{T}^{\lhd\varphi}$. Then, by the Corollary 4.2, we obtain $\overline{W}_0 = \overline{(\overline{W}_0)} \supset \overline{L}^{\circ\varphi} \supset \overline{B}_0 = L^{*\varphi}$, where closures are taken in $\mathscr{T}^{\lhd\varphi}$.

§ 5. The completeness of the spaces $(L^{*\varphi}, \mathscr{T}^{\lhd\varphi})$ and $(L^{*\varphi}, \mathscr{T}^{\ll\varphi})$.

5.1. Theorem. *The spaces $(L^{*\varphi}, \mathscr{T}^{\lhd\varphi})$ and $(L^{*\varphi}, \mathscr{T}^{\ll\varphi})$ are complete.*

Proof. Let $\{x_\sigma : \sigma \in \Sigma\}$ be any Cauchy M-S sequence in $L^{*\varphi}$ in the topology $\mathscr{T}^{\lhd\varphi}$ $(\mathscr{T}^{\ll\varphi})$, i.e. for every $\psi \in \Psi^{\lhd\varphi}$ $(\psi \in \Psi^{\ll\varphi})$ and every number r

with $0 < r < 1$ there exists $\sigma_0 \in \Sigma$ such that

(1) $$\|x_\sigma - x_{\sigma'}\|_\psi < r \quad \text{for} \quad \sigma, \sigma' \geqslant \sigma_0;$$

hence

(2) $$\varrho_\psi(x_\sigma - x_{\sigma'}) < r \quad \text{for} \quad \sigma, \sigma' \geqslant \sigma_0.$$

We shall show that

$$x_\sigma \xrightarrow[\sigma]{\mathscr{T}^{\lhd\varphi}} x \quad (x_\sigma \xrightarrow[\sigma]{\mathscr{T}^{\ll\varphi}} x) \quad \text{for some } x \in L^{*\varphi}.$$

Define

$$A_{\sigma,\sigma'}(\varepsilon) = \{t \in E\colon |x_\sigma(t) - x_{\sigma'}(t)| \geqslant \varepsilon\} \quad \text{for an arbitrary } \varepsilon > 0.$$

Let ψ be an arbitrary φ-function such that $\psi \in \Psi^{\lhd\varphi}$ ($\psi \in \Psi^{\ll\varphi}$), and let ε be an arbitrary positive number. Let $r = \varepsilon\psi(\varepsilon)$. Then from (2) it follows that

$$|A_{\sigma,\sigma'}(\varepsilon)|\psi(\varepsilon) = \int_{A_{\sigma,\sigma'}(\varepsilon)} \psi(\varepsilon)\,dt \leqslant \int_{A_{\sigma,\sigma'}(\varepsilon)} \psi\big(|x_\sigma(t) - x_{\sigma'}(t)|\big)\,dt$$
$$\leqslant \int_E \psi\big(|x_\sigma(t) - x_{\sigma'}(t)|\big)\,dt < r = \varepsilon\psi(\varepsilon) \quad \text{for} \quad \sigma, \sigma' \geqslant \sigma_0;$$

hence we have

$$|A_{\sigma,\sigma'}(\varepsilon)| \leqslant \varepsilon \quad \text{for} \quad \sigma, \sigma' \geqslant \sigma_0.$$

It means that this Cauchy M-S sequence $\{x_\sigma\colon \sigma \in \Sigma\}$ satisfies the Cauchy condition in measure. Since by 0.4.2 the space $(S_0, \mathscr{T}_0)$ is complete, it follows that there exists a function $x \in S_0$ such that $x_\sigma \xrightarrow[\sigma]{\mathscr{T}_0} x$. Hence

(3) $$(x_{\sigma'} - x_\sigma) \xrightarrow[\sigma']{\mathscr{T}_0} (x - x_\sigma).$$

Now, we shall prove that $x_\sigma \xrightarrow[\sigma]{\mathscr{T}^{\lhd\varphi}} x$ ($x_\sigma \xrightarrow[\sigma]{\mathscr{T}^{\ll\varphi}} x$), i.e. $\|x_\sigma - x\|_\psi \xrightarrow[\sigma]{} 0$ for every $\psi \in \Psi^{\lhd\varphi}$ ($\psi \in \Psi^{\ll\varphi}$).

Indeed, let ψ be an arbitrary φ-function such that $\psi \in \Psi^{\lhd\varphi}$ ($\psi \in \Psi^{\ll\varphi}$). Let r be an arbitrary number with $0 < r < 1$. Then from (1) there exists $\sigma_0 \in \Sigma$ such that

(4) $$x_{\sigma'} - x_\sigma \in \bar{K}_\psi(r) \quad \text{for} \quad \sigma, \sigma' \geqslant \sigma_0.$$

The balls $\bar{K}_\psi(r)$ are by 0.4.3 closed in $(S_0, \mathscr{T}_0)$. Therefore from (3) and (4) it follows that

(5) $$x - x_\sigma \in \bar{K}_\psi(r) \quad \text{for} \quad \sigma \geqslant \sigma_0.$$

Hence $\|x_\sigma - x\|_\psi \xrightarrow[\sigma]{} 0$. Thus, we proved that $x_\sigma \xrightarrow[\sigma]{\mathscr{T}^{\lhd\varphi}} x$ ($x_\sigma \xrightarrow[\sigma]{\mathscr{T}^{\ll\varphi}} x$).

For the completeness of the spaces $(L^{*\varphi}, \mathcal{T}^{\lhd\varphi})$ and $(L^{*\varphi}, \mathcal{T}^{\ll\varphi})$ it suffices to show that $x \in L^{*\varphi}$. Since by 0.3 we have the equalities $L^{*\varphi} = \bigcap_{\psi \in \Psi^{\lhd\varphi}} L^{\circ\psi} = \bigcap_{\psi \in \Psi^{\ll\varphi}} L^{\circ\psi}$, we shall show that for every $\psi \in \Psi^{\lhd\varphi}$ there holds $\varrho_\psi(\lambda x) < \infty$ for all $\lambda > 0$. In fact, let ψ be an arbitrary φ-function such that $\psi \in \Psi^{\lhd\varphi}$. Then, from (5) it follows that for any number $\lambda > 0$ there exists $\sigma_0 \in \Sigma$ such that $\|x_{\sigma_0} - x\|_\psi \leqslant 1/2\lambda$; hence

$$\varrho_\psi\big(2\lambda(x_{\sigma_0} - x)\big) \leqslant 1/2\lambda. \tag{6}$$

On the other hand, since $x_{\sigma_0} \in L^{*\varphi}$, we have

$$\varrho_\psi(2\lambda x_{\sigma_0}) < \infty. \tag{7}$$

Finally, from (6) and (7) it follows that

$$\varrho_\psi(\lambda x) = \varrho_\psi\Big(\tfrac{1}{2}(2\lambda x_{\sigma_0}) + \tfrac{1}{2}\big(2\lambda(x - x_{\sigma_0})\big)\Big) \leqslant \varrho_\psi(2\lambda x_{\sigma_0}) + \varrho_\psi\big(2\lambda(x_{\sigma_0} - x)\big) < \infty.$$

I would like to express my thanks to Doc. R. Leśniewicz for his help while working on this paper.

References

[1] P. Halmos, *Measure theory,* Springer-Verlag, New York 1974.
[2] M. A. Krasnosel'skii and Ya. B. Rutickii, *Convex functions and Orlicz spaces,* Groningen 1961.
[3] W. Matuszewska, *On generalized Orlicz spaces,* Bull. Acad. Polon. Sci. 8 (1960), 349–353.
[4] J. Musielak, W. Orlicz, *Some remarks on modular spaces,* ibidem 7 (1959), 661–668.
[5] M. Nowak, *On two equalities for Orlicz spaces. I,* Funct. et Approx. 10 (1980), 69–81.
[6] R. O'Neil, *Integral transforms and tensor products on Orlicz spaces and L(p, q) spaces,* J. Anal. Math. 21 (1968), 4–276.
[7] H. Schaefer, *Topological vector spaces,* New York–London 1967.
[8] Ph. Turpin, *Convexités dans les espaces vectoriels topologiques généraux,* Dissert. Math. 131 (1976).

INSTITUTE OF MATHEMATICS, A. MICKIEWICZ UNIVERSITY, Poznań
INSTYTUT MATEMATYCZNY, UNIWERSYTET im. A. MICKIEWICZA, Poznań