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Autodistributive n-groups

1. Introduction. In some recent papers several authors have extended the study of ordinary groups and semigroups to the case where the group operation is not a binary, but an n-ary one. The existence of such theories has motivated us to extend the study of ordinary rings to the case where the ring operations are respectively m-ary and n-ary.

The purpose of the present paper is to give a description of n-groups satisfying a natural generalization of the distributive law for operations. These n-groups are the special case of (m, n)-rings.

In the sequel we shall use the terminology and notation of [8] and [10]. The symbol $\mathscr{R}$ denotes the general algebra $\langle R; g, f\rangle$. The algebra $\langle G, f\rangle$ is denoted by $\mathscr{G}$.

A general algebra $\mathscr{R}$ is called an (m, n)-*semiring* if $\langle R, g\rangle$ is an m-semigroup and $\langle R, f\rangle$ is an n-semigroup, and the operation f is distributive with respect to g, i.e.

$$f\big(x_1^{i-1}, g(y_1^m), x_{i+1}^n\big) = g\big(\{f(x_1^{i-1}, y_j, x_{i+1}^n)\}_{j=1}^m\big) \tag{1}$$

for each $i = 1, 2, \ldots, n$.

If $\langle R, g\rangle$ is a commutative m-group, then an (m, n)-semiring $\mathscr{R}$ is called an (m, n)-*ring*. An ordinary ring is a special case of (m, n)-ring, namely for $m = n = 2$.

We say that $\mathscr{R}$ is a *commutative* (m, n)-*semiring* if the operation f is commutative. A neutral element of $\langle R, f\rangle$ is called an *identity* of $\mathscr{R}$. An idempotent of $\langle R, g\rangle$ is called an *additive idempotent*.

The concept of an (m, n)-ring was introduced by Čupona [5]. Paper [2] is concerned with homomorphism theorems and some ideal-theoretic aspects. (m, n)-quotient rings are studied in [4]. In [4] are also examined cancellative (m, n)-rings and it is proved that an integral domain with a zero may be embedded into a unique minimal (m, n)-field. We generalized the theory of divisibility to the case of (m, n)-rings in [7]. The construction of the covering ring for the (m, n)-ring (see [1] and [3]) is analogous to the construction of the covering group for the n-group (see [19], §3).

In the theory of n-semigroups the following identity

$$f\big(\{f(x_{i1}^{in})\}_{i=1}^{n}\big) = f\big(\{f(x_{1i}^{ni})\}_{i=1}^{n}\big) \tag{2}$$

plays an important role ([11], [13]; see also [16], p. 87). An n-semigroup $\mathcal{G}$ with property (2) is called *Abelian.*

2. Autodistributive n-semigroups. We say that an n-semigroup $\mathcal{G}$ is *autodistributive* if f is distributive with respect to itself.

It is clear that every autodistributive n-semigroup $\langle G, f\rangle$ is an (n, n)-semiring over itself, i.e., $\langle G; f, f\rangle$ is an (n, n)-semiring. It is easily verified that an idempotent Abelian n-semigroup is autodistributive. Moreover, a commutative idempotent n-group is a commutative (n, n)-ring and every element of G is an identity in this (n, n)-ring.

Note that for all natural n there exists an idempotent commutative n-semigroup (an n-group also). Hence, for every n, there exists an autodistributive n-semigroup and an (n, n)-ring such that every element is an identity.

Observe that if the operation f is autodistributive, then the long product

$$f_{(k)}(x_1^{k(n-1)+1}) = \underbrace{f\Big(f\big(\ldots f}_{k}\big(f(x_1^n), x_{n+1}^{2n-1}\big), \ldots\big), x_{(k-1)(n-1)+2}^{k(n-1)+1}\Big)$$

and the operation g defined as

$$g(x_1^n) = f(x_n, x_{n-1}, \ldots, x_2, x_1) \tag{3}$$

are autodistributive, too. Moreover, $\langle G; g, f\rangle$ is an (n, n)-semiring.

LEMMA 1. *An n-semigroup $\langle G, g\rangle$ (also $\langle G, f\rangle$) is autodistributive if and only if the algebra $\langle G; g, f\rangle$ is an (n, n)-semiring with f satisfying* (3).

Let now $\mathfrak{B} = \langle B, f\rangle$ be an autodistributive n-semigroup with an idempotent element b. Then, for all $x_1, x_2, \ldots, x_n \in B$ and $i = 1, 2, \ldots, n$, $f(x_1^{i-1}, b, x_{i+1}^n)$ is also an idempotent. This implies that the set of all idempotents in $\mathfrak{B}$ forms an ideal.

A *zero* of $\mathcal{G}$ is an element $\theta \in G$ such that $f(\theta, x_2^n) = f(x_2, \theta, x_3^n) = \ldots$ $\ldots = f(x_2^n, \theta) = \theta$ for all elements $x_2, x_3, \ldots, x_n \in G$. G^* denotes the set $G \setminus \{\theta\}$, if θ exists, and G otherwise. From the above remarks it follows that if an autodistributive n-semigroup has only one idempotent, then this element is a zero.

By a zero of an (m, n)-semiring $\mathcal{R}$ is meant a zero of $\langle R, f\rangle$.

Notice that if the condition: $x\varrho y$ if and only if

$$f(a_1^{i-1}, x, a_{i+1}^n) = f(a_1^{i-1}, y, a_{i+1}^n) \tag{4}$$

holds for some i and all $a_k \in G$, then it defines an equivalence relation in G. It is obvious that $\mathcal{G}$ is autodistributive if and only if this relation is a congruence of $\mathcal{G}$ for all $i = 1, 2, \ldots, n$.

We say that the *i-cancellative law* holds in an n-groupoid $\mathcal{G}$ if the condition (4) implies $x = y$ for each $x, y \in G$ and all $a_1, a_2, \ldots, a_n \in G^*$. If $\mathcal{G}$ is i-cancellative for all $i = 1, 2, \ldots, n$, then it is called a *cancellative n-groupoid*. An (m, n)-semiring $\mathcal{R}$ is cancellative if the n-semigroup $\langle R, f\rangle$ is cancellative.

LEMMA 2. *If $\mathcal{G}$ is an n-semigroup, then the following conditions are equivalent*:

(i) *$\mathcal{G}$ is cancellative,*

(ii) *$\mathcal{G}$ is i-cancellative for $i = 1$ and $i = n$,*

(iii) *$\mathcal{G}$ is i-cancellative for some $i = 2, 3, \ldots, n-1$.*

Proof. (iii)$\Rightarrow$(i). Let $\mathcal{G}$ be i-cancellative for some $1 < i < n$. If (4) holds for $j > i$, then we have

$$f\big(b_1^k, f(a_1^{j-1}, x, a_{j+1}^n), b_{k+2}^n\big) = f\big(b_1^k, f(a_1^{j-1}, y, a_{j+1}^n), b_{k+2}^n\big),$$

where $k = n-1+i-j$. Hence

$$f\big(b_1^{i-2}, f(b_{i-1}^k, a_1^{j-1}), x, a_{j+1}^n, b_{k+2}^n\big) = f\big(b_1^{i-2}, f(b_{i-1}^k, a_1^{j-1}), y, a_{j+1}^n, b_{k+2}^n\big)$$

implies $x = y$, i.e., $\mathcal{G}$ is j-cancellative for all $j \geqslant i$.

If (4) holds for $j < i$, then putting $k = i-j-1$ and proceeding as before, one get $x = y$.

The remaining part of the proof easily follows (see also [4]).

THEOREM 1. *If $\mathcal{R}$ is a cancellative (m, n)-semiring, then*

(i) *$\mathcal{R}$ has not additive idempotents, or*

(ii) *only zero (if it exists) is an additive idempotent, or*

(iii) *$\langle R, g\rangle$ is an idempotent m-semigroup.*

Proof. Assume that $b = g(\overset{(m)}{b})$ and b is not a zero of $\mathcal{R}$. Then by distributivity we have

$$f(x_2^n, b) = f\big(x_2^n, g(\overset{(m)}{b})\big) = g\big(f(x_2^n, b), \ldots, f(x_2^n, b)\big) = f\big(g(\overset{(m)}{x_2}), x_3^n, b\big)$$

for all $x_2, x_3, \ldots, x_n \in R^*$.

Cancellativity implies $x_2 = g(\overset{(m)}{x_2})$ for each $x_2 \in R^*$.

COROLLARY 1. *If $\mathcal{G}$ is an autodistributive and cancellative n-semigroup, then*

(i) *$\mathcal{G}$ has not an idempotent element, or*

(ii) *only zero (if it exists) is an idempotent, or*

(iii) *every element is an idempotent.*

Following E. L. Post ([19], p. 282), we define:

$$a^{\langle k\rangle}=\begin{cases} f(a^{\langle k-1\rangle},\overset{(n-1)}{a}) & \text{for} \quad k>0,\\ a & \text{for} \quad k=0,\\ x\colon f(x,a^{\langle -k-1\rangle},\overset{(n-2)}{a})=a & \text{for} \quad k<0.\end{cases}$$

The exponential laws given below are easily verified:

$$(a^{\langle r\rangle})^{\langle s\rangle}=a^{\langle rs(n-1)+s+r\rangle}, \tag{5}$$

$$f(a^{\langle s_1\rangle},a^{\langle s_2\rangle},\dots,a^{\langle s_n\rangle})=a^{\langle s_1+s_2+\dots+s_n+1\rangle}. \tag{6}$$

A minimal natural number p such that $a^{\langle p\rangle}=f_{(p)}(\overset{p(n-1)+1}{a})=a$ is called an *n-ary order* of a and is denoted by $\operatorname{ord}_n(a)$.

THEOREM 2. *Let $\mathcal{G}$ be an autodistributive and cancellative n-semigroup. Then*

(i) *for each $x\in G$ there exists a unique $y\in G$ such that*

$$f(\overset{(n-i)}{x},y,\overset{(i-1)}{x})=x \quad \textit{and} \quad y=x^{\langle n-2\rangle} \quad \textit{for all } i=1,2,\dots,n,$$

(ii) $x=x^{\langle n-1\rangle}$ *for all* $x\in G$,

(iii) $x=(x^{\langle n-2\rangle})^{\langle 1\rangle}$ *for all* $x\in G$,

(iv) *if* $\operatorname{ord}_n(x)=p$, *then* $x=(x^{\langle p-1\rangle})^{\langle 1\rangle}$.

Proof. (i) and (ii). We have

$$f(\overset{(n-1)}{x},f(\overset{(n)}{x}))=f(f(\overset{(n)}{x}),\dots,f(\overset{(n)}{x}))=f(\overset{(n-1)}{x},f(\overset{(n-1)}{x},x^{\langle n-2\rangle}))$$

and

$$x=f(\overset{(n-1)}{x},x^{\langle n-2\rangle})=x^{\langle n-1\rangle}$$

by virtue of cancellativity.

$$\begin{aligned}\text{(iii)}\quad (x^{\langle n-2\rangle})^{\langle 1\rangle}&=f(x^{\langle n-2\rangle},\dots,x^{\langle n-2\rangle},f(x^{\langle n-3\rangle},\overset{(n-1)}{x}))\\ &=f(f(x^{\langle n-2\rangle},\dots,x^{\langle n-2\rangle},x^{\langle n-3\rangle}),\overset{(n-1)}{x})\\ &=f(f(x^{\langle n-2\rangle},\overset{(n-1)}{x}),\dots,f(x^{\langle n-2\rangle},\overset{(n-1)}{x}),f(x^{\langle n-3\rangle},\overset{(n-1)}{x}))\\ &=f(x^{\langle n-1\rangle},\dots,x^{\langle n-1\rangle},x^{\langle n-2\rangle})=f(\overset{(n-1)}{x},x^{\langle n-2\rangle})=x^{\langle n-1\rangle}=x.\end{aligned}$$

(iv) Similarly as (iii).

COROLLARY 2. *Every autodistributive and cancellative n-semigroup $\mathcal{G}$ is regular, i.e., for every $y\in G$ there exist $x_2,x_3,\dots,x_{n-1}\in G$ such that $y=f(y,x_2^{n-1},y)$. Moreover, each element of this n-semigroup has a finite n-ary order and $\operatorname{ord}_n(x)$ is a divisor of $n-1$.*

O. W. Kolesnikov proved in [15] that an n-semigroup $\mathcal{G}$ is inverse if and only if all regular conjugated elements with a given sequence of elements of G are identical ones. Hence each regular and cancellative n-semigroup is inverse. From Theorem 2 follows that each autodistributive and cancellative n-semigroup is inverse.

COROLLARY 3. *Each autodistributive and cancellative n-semigroup $\mathcal{G}$ is a set-theoretic union of disjoint cyclic autodistributive n-groups without proper subgroups.*

Proof. Since every element of this n-semigroup has a finite n-ary order which is a divisor of $n-1$, we have $G = \bigcup_{x\in G} G(x)$, where $G(x)$ is a cyclic n-group generated by an element $x \in G$. Moreover, $G(x)$ has not any proper subgroups. Indeed, if d is the greast common divisor of $\mathrm{ord}_n(x)$ and $k(n-1)+1$, then $\mathrm{ord}_n(x) = d\cdot \mathrm{ord}_n(x^{\langle k\rangle})$ (see [19], p. 283). Because every divisor of $\mathrm{ord}_n(x)$ divides $n-1$ therefore $\mathrm{GCD}\{k(n-1)+1, \mathrm{ord}_n(x)\} = 1$, i.e., $\mathrm{ord}_n(x) = \mathrm{ord}_n(x^{\langle k\rangle})$ for every natural k. This implies that $G(x)$ has any proper subgroups and G is a set-theoretic union of disjoint cyclic n-groups.

Theorem 2 (i) implies that, in autodistributive and cancellative n-semigroup, there exists a uniquely defined a unary operation $x \to x^{\langle n-2\rangle}$. An element $x^{\langle n-2\rangle}$ is a skew element to x.

THEOREM 3. *If $\mathcal{G}$ is an autodistributive and cancellative n-semigroup, then*

$$\overline{f(x_1^n)} = f(x_1^{i-1}, \overline{x}_i, x_{i+1}^n) \tag{7}$$

for all $x_1, x_2, \ldots, x_n \in G$ and $i = 1, 2, \ldots, n$.

Proof. The proof follows from the equalities

$$f(x_1^n) = f\big(x_1^{i-1}, f(\overset{(n-1)}{x_i}, \overline{x}_i), x_{i+1}^n\big) = f\big(f(x_1^n), \ldots, f(x_1^n), f(x_1^{i-1}, \overline{x}_i, x_{i+1}^n)\big)$$

and Theorem 2 (i).

COROLLARY 4. *Each autodistributive 3-group is Abelian. Conversely, every idempotent 3-group is autodistributive and Abelian.*

Proof. W. Dörnte proved in [6] that $\overline{f(x, y, z)} = f(\overline{z}, \overline{y}, \overline{x})$. Hence

$$f(x, y, z) = f(\overline{\overline{x}}, \overline{\overline{y}}, z) = \overline{f(\overline{x}, \overline{y}, \overline{z})} = f(\overline{\overline{z}}, \overline{\overline{y}}, \overline{\overline{x}}) = f(z, y, x).$$

3. Autodistributive n-groups. Let now $\mathcal{G}$ be an n-group (not necessarily autodistributive). Let $x = \overline{x}^{(0)}$ and let $\overline{x}^{(s+1)}$ be the skew element to $\overline{x}^{(s)}$, where $s \geqslant 0$. In other words, $\overline{x} = \overline{x}^{(1)}$, $\overline{\overline{x}} = \overline{x}^{(2)}$, $\overline{\overline{\overline{x}}} = \overline{x}^{(3)}$, etc.

It is easily verified that the operation $x \to \overline{x}$ is one-to-one if there exists a natural number k such that $x = \overline{x}^{(k)}$ for all $x \in G$. Conversely, if $\mathcal{G}$ is a finite n-group and $x \to \overline{x}$ is one-to-one, then $x = \overline{x}^{(k)}$ for all $x \in G$

and some k. If $\mathcal{G}$ is also Abelian, then this operation is an automorphism. Obviously, if $\mathcal{G}$ is autodistributive then $x \to \bar{x}$ is one-to-one.

COROLLARY 5. *If $\mathcal{G}$ is an autodistributive n-group, then*

$$x^{\langle k\rangle} = \bar{x}^{(n-k-1)} \quad \text{and} \quad \bar{x}^{(k)} = (\bar{x}^{(k+1)})^{\langle 1\rangle} \tag{8}$$

is valid for all $x \in G$ and $k = 0, 1, \ldots, n-1$.

Proof. Applying (7) and Corollary 3, we obtain

$$x = f(x, \bar{x}, \bar{x}, \ldots, \bar{x}, \bar{\bar{x}}) = f(\overset{(n-2)}{x}, \bar{x}, \bar{x}^{(n-1)}) = \bar{x}^{(n-1)}$$

and

$$x^{\langle 1\rangle} = f(\overset{(n)}{x}) = f(\overset{(n-1)}{x}, \bar{x}^{(n-1)}) = f(\overset{(n-2)}{x}, \bar{x}, \bar{x}^{(n-2)}) = \bar{x}^{(n-2)}, \quad \text{etc.}$$

Analogously as Corollary 5, we prove

COROLLARY 6. *If an n-group is autodistributive and* $\mathrm{ord}_n(x) = p$ *for some element x, then $x = \bar{x}^{(p)}$.*

From Corollaries 1 and 5 we obtain

COROLLARY 7. *If $\mathcal{G}$ is an autodistributive n-group, then it is either an idempotent n-group or, for every $x \in G$, we have $x \neq \bar{x}$ and $x = \bar{x}^{\langle 1\rangle}$.*

THEOREM 4. *If Abelian n-group satisfies* (7), *then it is autodistributive.*

Proof. In the same manner as Corollary 5 we can prove that condition (7) implies $x = \bar{x}^{(n-1)}$ for all elements x. Now, because this n-group is also Abelian,

$$\begin{aligned} f\big(f(x_1^n), y_2^n\big) &= f\big(f(x_1^n), f(\overset{(n-1)}{y_2}, \bar{y}_2), \ldots, f(\overset{(n-1)}{y_n}, \bar{y}_n)\big) \\ &= f\big(f(x_1, y_2^n), \ldots, f(x_{n-1}, y_2^n), f(x_n, \bar{y}_2, \bar{y}_3, \ldots, \bar{y}_n)\big) \\ &= f\big(f(x_1, y_2^n), \ldots, f(x_{n-1}, y_2^n), f(\bar{x}_n^{(n-1)}, y_2^n)\big) = f\big(\{f(x_i, y_2^n)\}_{i=1}^n\big). \end{aligned}$$

Similarly we can prove remaining cases of (2).

Let $Z(A)$ be a center of a binary group $\mathfrak{A} = \langle A, \cdot\rangle$. It is clear that the operation $f(x_1^n) = x_1 \cdot x_2 \cdot \ldots \cdot x_n \cdot b$ is associative if $b \in Z(A)$. Obviously, $\langle A, f\rangle$ is an n-group for every natural n.

THEOREM 5. *Let $\mathfrak{A}$ be a binary group and let* $\exp(\mathfrak{A})$ *be a divisor of $n-1$. If $f(x_1^n) = x_1 \cdot x_2 \cdot \ldots \cdot x_n \cdot b$, where $b \in Z(A)$, then an n-group $\langle A, f\rangle$ satisfies* (7). *Moreover, $\langle A, f\rangle$ is autodistributive provided that $\mathfrak{A}$ is commutative.*

Proof. From definition of the skew element we have

$$x = f(\overset{(n-1)}{x}, \bar{x}) = x^{n-1} \cdot \bar{x} \cdot b = \bar{x} \cdot b \quad \text{and} \quad \bar{x} = x \cdot b^{-1}.$$

Now, if $b \in Z(A)$, then $b^{-1} \in Z(A)$, too. Hence

$$\overline{f(x_1^n)} = f(x_1^n) \cdot b^{-1} = x_1 \cdot x_2 \cdot \ldots \cdot x_{i-1} \cdot x_i \cdot b^{-1} \cdot x_{i+1} \cdot \ldots \cdot x_n \cdot b = f(x_1^{i-1}, \bar{x}_i, x_{i+1}^n).$$

Therefore (7) holds for all $i = 1, 2, \ldots, n$ and every $x_i \in G$. The remaining part of the theorem is obvious.

The definition of the skew element implies that $\bar{x} = x^{\langle -1\rangle} = x^{\langle p-1\rangle}$ and $\bar{\bar{x}} = x^{\langle n-3\rangle}$, where $\mathrm{ord}_n(x) = p$. Hence for all 4-groups $\bar{\bar{x}} = x^{\langle 1\rangle}$, if x has a finite order. Generally:

THEOREM 6. *For all elements of an n-group $\mathcal{G}$, we have:*

(i) *if* $\mathrm{ord}_n(x) = p$ *and* $ps+r = k \geqslant 0$, *where* $0 \leqslant r < p$, *then*

$$x^{\langle k\rangle} = x^{\langle r\rangle}, \quad x^{\langle -k\rangle} = x^{\langle -r\rangle} = x^{\langle p-r\rangle},$$

(ii) $\bar{x}^{(k+2)} = (\bar{x}^{(k+1)})^{\langle -1\rangle}$,

(iii) *if* $\mathrm{ord}_n(x) = p$, *then* $\bar{x}^{(k+1)} = (\bar{x}^{(k)})^{\langle p-1\rangle} = (\bar{x}^{(k-1)})^{\langle n-3\rangle}$,

(iv) $\bar{x}^{(k)} = x^{\langle s_k\rangle}$, *where* $s_k = \dfrac{(2-n)^k-1}{n-1} = -\sum\limits_{i=0}^{k-1}(2-n)^i$.

Proof. Using condition (5) and the covering group, we shall prove (i), (ii) and (iii). We can prove condition (iv) by induction.

COROLLARY 8. *An n-ary order of x is a divisor of s_k if and only if $x = \bar{x}^{(k)}$. In particular,* $\mathrm{ord}_n(x)$ *divides* $n-3$ *if and only if* $x = \bar{\bar{x}}$.

We can also prove

COROLLARY 9. *If there exists a natural number k such that $x = \bar{x}^{(k)}$, then* $\mathrm{ord}_n(x) = \mathrm{ord}_n(\bar{x}^{(s)})$ *for all s.*

Proof. Since $\mathrm{ord}_n(\bar{x})$ divides $\mathrm{ord}_n(x)$ [6], we have

$$\mathrm{ord}_n(\bar{x}^{(k)}) \leqslant \mathrm{ord}_n(\bar{x}^{(k-1)}) \leqslant \ldots \leqslant \mathrm{ord}_n(\bar{x}) \leqslant \mathrm{ord}_n(x) = \mathrm{ord}_n(\bar{x}^{(k)}).$$

Thus $\mathrm{ord}_n(x) = \mathrm{ord}_n(\bar{x}^{(s)})$ for all s.

COROLLARY 10. *If $\mathcal{G}$ is an autodistributive n-group and p is minimal natural number such that $x = \bar{x}^{(p)}$, then* $\mathrm{ord}_n(x) = p$.

Proof. Corollary 6 implies that $p \leqslant \mathrm{ord}_n(x)$. On the other hand,

$$x^{\langle p\rangle} = f_{(p)}(\overset{(p(n-1)+1)}{x}) = f_{(p)}(\bar{x}^{(p)}, \overset{(p(n-1))}{x})$$
$$= f_{(p)}(\overset{(n-1)}{x}, \bar{x}, \overset{(n-2)}{x}, \bar{x}, \overset{(n-2)}{x}, \bar{x}, \ldots, \overset{(n-2)}{x}, \bar{x}) = x.$$

Thus $\mathrm{ord}_n(x)$ divides p. Finally, $\mathrm{ord}_n(x) = p$.

It is easily verified that in the free covering k-group of $\mathcal{G}$ (the construction—see [17]) $\mathrm{ord}_k(x)$ divides $(n-1)^2/(k-1)$. Generally, the free covering k-group of an autodistributive n-group is not autodistributive. Since an autodistributive n-group $\mathcal{G}$ is an (n, n)-ring, there exists a free covering (m, n)-ring for all m such that $n = k(m-1)+1$.

The problem of existence of autodistributive n-groups is solved by

THEOREM 7. *For every n there exists a non-reducible autodistributive n-group.*

Proof. To prove our theorem, let us consider the set $G = \{|a|: a \equiv 1 \pmod{n}\}$, where $|a| = \{x: x \equiv a \pmod{n^2}\}$ for $n \geqslant 2$. Dörnte showed in [6] that G with f defined by the formula

$$f(|a_1|, |a_2|, \ldots, |a_{n+1}|) = |a_1| + |a_2| + \ldots + |a_{n+1}|$$

is an $(n+1)$-group. This group is not reducible, i.e., it is not derived from any m-group. Indeed, if $|a|$ is the skew element to $|1|$ in $\langle G, g\rangle$, and $\langle G, f\rangle$ is derived from m-group $\langle G, g\rangle$, where $n = k(m-1)$ and $k > 1$, then

$$f(\underbrace{|1|, \overset{(m-2)}{|1|}, |a|)}_{k \text{ times}} = g_{(k)}(\underbrace{|1|, \overset{(m-2)}{|1|}, |a|)}_{k \text{ times}} = |1|.$$

We can rewrite it in the form

$$|1| + k(m-2) \cdot |1| + k \cdot |a| = |1|,$$

i.e., $k(m-2+a) \equiv 0 \pmod{n^2}$. This implies that $k(m-2+a) = rn^2$ for some r. Multiplying this equation by $m-1$ and cancellating by n, we obtain $(m-2+a) = rn(m-1)$. Since $|a| \in G$ only for $a = sn+1$, we see that $sk+1 = rk(m-1)$. Hence $1 = k(rm-r-s)$, what is impossible for $k > 1$.

This $(n+1)$-group is also autodistributive, because it is commutative and the $(n+1)$-ary order of every element in G is a divisor of n.

It is proved in [12] that the class of all n-groups forms a variety. (For the defining identities for this class see also [10].) The class of all autodistributive n-groups (n-semigroups) forms a variety, too. Observe that if $\mathcal{G}$ is an autodistributive n-group and $\mathfrak{A}$ is an invariant subgroup of $\mathcal{G}$, then $\mathcal{G}/\mathfrak{A}$ is an idempotent autodistributive n-group.

4. Reducts and translations. In this section we consider only some binary reducts and some translations of n-groups. By a *binary reduct* of $\mathcal{G}$ with respect to a is meant the algebra $\mathrm{red}_a(\mathcal{G}) = \langle G, \circ\rangle$, where $x \circ y = f(x, \overset{(n-3)}{a}, \bar{a}, y)$. Clearly, $\mathrm{red}_a(\mathcal{G})$ is a binary group. If $\mathcal{G}$ is an autodistributive n-group, then $\mathrm{ord}_2(x)$ divides $(n-1)^2$ for all elements of $\mathrm{red}_a(\mathcal{G})$.

THEOREM 8. *All binary reducts of an autodistributive n-group are isomorphic.*

Proof. We shall prove that $\mathrm{red}_a(\mathcal{G})$ and $\mathrm{red}_c(\mathcal{G})$ are isomorphic for all $a, c \in G$.

First we note that mapping $h(x) = f(x, \overset{(n-3)}{a}, \bar{a}, \bar{c})$ is an automorphism of $\mathcal{G}$. Indeed, from axioms of an n-group, h is one-to-one and onto. The distributive law implies that h is a homomorphism, too.

Let $x \blacktriangle y = f(x, \overset{(n-3)}{c}, \bar{c}, y)$. Since $h(a) = c$ and $h(\bar{a}) = \bar{c}$, we have

$$h(x \circ y) = h\big(f(x, \overset{(n-3)}{a}, \bar{a}, y)\big) = f\big(h(x), \overset{(n-3)}{h(a)}, h(\bar{a}), h(y)\big)$$

$$= f\big(h(x), \overset{(n-3)}{c}, \bar{c}, h(y)\big) = h(x) \blacktriangle h(y).$$

Hence $\mathrm{red}_a(\mathcal{G})$ and $\mathrm{red}_c(\mathcal{G})$ are isomorphic.

By a *basic translation* [18] we mean the mapping $t(x) = f(a_1^{i-1}, x, a_{i+1}^n)$. It is always invertible. Its inverse is $t^{-1}(x) = f(b_i^{n-1}, x, c_2^i)$, where $(b_i, \ldots, b_{n-1}, a_1, \ldots, a_{i-1})$ and $(a_{i+1}, \ldots, a_n, c_2, \ldots, c_i)$ are $(n-1)$-adic identities [19]. An *elementary translation* is a composition of basic translations, and hence it is invertible. If $\mathcal{G}$ is an n-group, then the set of all elementary translations forms a transitive binary group.

By a *right* (*left*) *translation* of $\mathcal{G}$ is meant the mapping $\varphi_{s_2^n}(x) = f(x, s_2^n)$ ($\psi_{s_2^n}(x) = f(s_2^n, x)$). From [15] we know that an n-semigroup $\mathcal{G}$ is an n-group if and only if $\Phi = \{\varphi_{s_2^n}\colon s_i \in G\}$ and $\Psi = \{\psi_{s_2^n}\colon s_i \in G\}$ are binary groups. If $s_2, \ldots, s_{i-1}, s_{i+1}, \ldots, s_n$ are fixed, then Φ is denoted by $\Phi^i_{s_2^n}$. Remark that if $\mathcal{G}$ is an n-group, then the equation $s_2 = f(t_2^{i-1}, y, t_{i+1}^n, s_2)$ has a unique solution for all elements of G. Similarly, the equation $s_n = f(s_n, t_2^{i-1}, z, t_{i+1}^n)$. This implies that $\varphi_{s_2^n} = \varphi_{t_2^n}$ for all $t_i = f_{(2)}(y, t_{i+1}^n, s_2^n, t_2^{i-1}, z)$. Hence

$$\Phi^i_{s_2^n} = \Phi^j_{t_2^n} = \Phi^k_z = \{\varphi^{(k)}_{z,c}\colon \varphi^{(k)}_{z,c}(x) = f(x, \overset{(k-2)}{z}, c, \overset{(n-k)}{z})\}$$

for all $s_p, t_p, c, z \in G$ and $i, j, k = 2, 3, \ldots, n$. It is easily verified that a cancellative n-semigroup is semi-commutative if and only if for some i all $\Phi^i_{s_2^n}$ are commutative. Therefore, an n-group is Abelian if and only if, for some fixed i and z, Φ^i_z is commutative (see also [8]).

The remarks above yield to

COROLLARY 11. *If $\mathcal{G}$ is an n-group, then $\Phi^i_{s_2^n}$ is a transitive group and has* card(G) *elements. Moreover, $\Phi^i_{s_2^n} = \Phi^j_z = \Phi$ for all $z, s_p \in G$ and $i, j = 2, 3, \ldots, n$.*

Analogous results hold for left translations.

In the same manner as Theorem 2 in [10] we can prove

THEOREM 9. *An n-semigroup $\mathcal{G}$ is an n-group if and only if for some $i, j = 2, 3, \ldots, n$ and all $a \in G$ there exist $b, c \in G$ such that $\varphi^{(i)}_{a,b}(x) = \psi^{(j)}_{a,c}(x) = x$, where $\psi^{(j)}_{a,c}(x) = f(\overset{(n-j)}{a}, c, \overset{(j-2)}{a}, x)$.*

Obviously, if $\mathcal{G}$ is an autodistributive n-semigroup (an n-group), then all translations are homomorphisms (automorphisms). If $\mathcal{G}$ is commutative, then $\mathcal{G}$ is autodistributive whenever all right (all elementary) translations are homomorphisms.

5. Final remarks. If $\mathcal{G}$ is an n-semigroup, then a non-empty subset $B \subset G$ such that $B^{\langle k\rangle} \subset G \setminus B^{\langle s\rangle}$ is called an (k, s)-*mutant* in $\mathcal{G}$.

J. B. Kim [14] proved that any binary semigroup has no decompositions into a finite number of disjoint (2, 1)-mutants. We generalized this result in [9]. This result is not true for n-semigroup where $n > 2$.

THEOREM 10. *Each non-idempotent autodistributive n-group has a decomposition into some number of disjoint* $(n-2, n-3)$-*mutants.*

Proof. If $B_i = \{a_i\}$, then $B_i^{\langle n-2\rangle} = \{\bar{a}_i\}$ and $B_i^{\langle n-3\rangle} = \{\bar{\bar{a}}_i\}$. Hence B_i are $(n-2, n-3)$-mutants.

Added in proof (February, 1983). As it is well known (see [8], [10], [12]) an n-group may be defined as a special n-semigroup (G, f) with a unary operation $-: x \to \bar{x}$, i.e., as a some universal algebra $(G; f, -)$ of type $(n, 1)$. If (G, f) is an n-group, then this unary operation is uniquely defined as a solution $\bar{x}$ of the equation $f(x, x, \ldots, x, \bar{x}) = x$. If (G, f) is an n-semigroup (but it is not an n-group), then the solution $\bar{x}$ (if it exists) is not unique. For example, in an n-semigroup (G, f) with f defined as $f(x_1^n) = x_1$ every unary operation g satisfies $f(x, x, \ldots, x, g(x)) = x$.

An n-semigroup (G, f) with a unary operation g satisfying the last equation is called *distributive* (or *weakly distributive*) if $f(x_1^{i-1}, g(x_i), x_{i+1}^n) = g(f(x_1^n))$ for all $x_1, x_2, \ldots, x_n \in G$ and $i = 1, 2, \ldots, n$. Hence an n-group $(G; f, -)$ is distributive if and only if it satisfies (7). Observe that an n-semigroup (G, f) with a unary operation g defined as above is distributive if $(G; f, g)$ is an $(n, 1)$-semiring. Moreover, these n-semigroups are special cases of (f/g)-algebras in sense of Hoehnke (comp. J. H. Hoehnke, *On the principle of distributivity*, Preprint of the Math. Inst. Hungarian Acad. Sci., Budapest 1980).

The class of all distributive n-groups forms a variety. The class of all autodistributive n-groups is a proper subvariety of a variety of all distributive n-groups. Free distributive (autodistributive) n-groups have n-ary exponent equal to $n-1$. Therefore free n-groups in these varieties are set-theoretic unions of disjoint cyclic autodistributive n-groups without proper subgroups.

All the above results will be proved in my paper *On distributive n-groups.*

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