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Spectral consequences of the existence of intertwining operators*

Abstract. Suppose that R and T are bounded linear operators on the Banach spaces X and Y, respectively, and that $S\colon Y \to X$ is a non-zero bounded linear operator for which $RS = ST$. The paper investigates the consequences of various assumptions on S for the fine structure of $\sigma(R) \cap \sigma(T)$. For instance, suppose that K is a component of $\sigma(T)$ but that K is not just a single point which is a pole of finite rank of T. If S is one-to-one, then $K \cap \sigma(R)$ contains a point which is not a pole of finite rank of R. If S is one-to-one and has dense range and if L is a component of $\sigma(R)$ for which $K \cap L$ is non-void, then $K \cap L$ contains a point λ which is in $\sigma_\pi(R) \cap \sigma_\delta(T)$ and which belongs to the boundary of $\sigma(R)$ or to the boundary of $\sigma(T)$. If R and T are quasi-similar and L is not just a pole of finite rank of R, then λ belongs to $\sigma_\pi(R) \cap \sigma_\delta(R) \cap \sigma_\pi(T) \cap \sigma_\delta(T)$, and neither $R-\lambda$ nor $T-\lambda$ is semi-Fredholm.

1. Introduction. Suppose that R and T are bounded linear operators on the Banach spaces X and Y, respectively, and that $S\colon Y \to X$ is a bounded non-zero linear operator which *intertwines* T and R (i.e. $RS = ST$). In this paper, we investigate the consequences of various assumptions on S for the fine structure of $\sigma(R) \cap \sigma(T)$, which is always non-void by Rosenblum's Theorem [12], Corollary 3.3, p. 265, [11], Corollary 0.13, p. 8, [4], Lemma 2.2, p. 69. We will, in particular, extend some results of Fialkow [4], [5] and of Davis and Rosenthal [2].

In Section 2, we show that if S is one-to-one, then each component of $\sigma(T)$ meets $\sigma(R)$ (for closed-open subsets of $\sigma(T)$, this is [4], Lemma 2.4, p. 60), and that each "non-trivial" component of $\sigma(T)$ contains a "non-trivial" point of $\sigma(R)$ (see Theorem (2.4) for a precise statement). We also prove the dual results for S with dense range.

In Section 3, we give a new proof, and various extensions, of the result of Davis and Rosenthal [2], Theorem 4, p. 1387, that $\sigma_\pi(R) \cap \sigma_\delta(T)$ is non-void whenever there is a non-zero S intertwining T and R (the notation is explained at the beginning of Section 3). In the major result in this section, Theorem (3.2), we show that if S is one-to-one with dense range, then any intersecting components of $\sigma(R)$ and $\sigma(T)$ contain a point which is in $\sigma_\pi(R) \cap \sigma_\delta(T)$ and which also belongs to the boundary of $\sigma(R)$ or of $\sigma(T)$.

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In Section 4, we examine $\sigma(R) \cap \sigma(T)$ when R and T are *quasi-similar*, that is, when S is one-to-one and has dense range and when there also exists a U: $X \to Y$ which is one-to-one with dense range and which satisfies $TU = UR$. Using results from Sections 2 and 3, we show that if R and T are quasi-similar, then any intersecting components of $\sigma(R)$ and $\sigma(T)$ always contain a point λ in $\sigma_\pi(R) \cap \sigma_\delta(R) \cap \sigma_\pi(T) \cap \sigma_\delta(T)$; and that "usually" neither $T-\lambda$ nor $R-\lambda$ is semi-Fredholm (see Theorem (4.2) and Corollary (4.3) for the details).

In Section 5 we compare the ascent, nullity, closed descent, and closed defect of R and T; and in Section 6 we give some conditions under which no non-zero S intertwines T and R.

Finally, in Section 7, we show that there exists a finite-rank S intertwining T and R if and only if $\sigma_p(R) \cap \sigma_d(T)$ is non-void. This seems to be the only case in which there is a simple definitive spectral characterization of the existence of some type of intertwining operator.

2. Components of the spectrum. Suppose that R and T are bounded operators on X and Y, and that S: $Y \to X$ intertwines T and R. In this section we examine the intersection of a component of $\sigma(T)$ with $\sigma(R)$ when S is one-to-one; and, dually, we study the intersection of a component of $\sigma(R)$ with $\sigma(T)$ when S has dense range. We start by showing that these intersections are non-void.

THEOREM (2.1). *Suppose that R and T are bounded linear operators on the Banach spaces X and Y, and that S: $Y \to X$ is a bounded linear operator for which $RS = ST$.*

(A) *If S is one-to-one, then each component of $\sigma(T)$ meets $\sigma(R)$.*

(B) *If S has dense range, then each component of $\sigma(R)$ meets $\sigma(T)$.*

Proof. Suppose that S is one-to-one and that K is a component of $\sigma(T)$. Since $\sigma(T)$ is a compact metric space, there is a nested sequence, $\{F_n\}$, of sets simultaneously open and closed in $\sigma(T)$ for which $K = \bigcap F_n$ ([10], Corollary 1, p. 83). Each $F_n \cap \sigma(R)$ is non-void, by [4], Lemma 2.4, p. 69; so that $\{F_n \cap \sigma(R)\}$ is a nested sequence of closed subsets of the compact set $\sigma(T)$. Therefore $K \cap \sigma(R) = \bigcap \left(F_n \cap \sigma(R)\right)$ is non-void.

Suppose that S has dense range. Then $T^* S^* = S^* R^*$ with S^* one-to-one ([13], Corollary (b), p. 94). Hence, by part (A), each component of $\sigma(R) = \sigma(R^*)$ meets $\sigma(T) = \sigma(T^*)$.

To simplify our detailed analysis of $K \cap \sigma(R)$, we will first consider the special case that K is a single point which is an isolated point of the spectrum of T and also an isolated point in the spectrum of R. The following definition will simplify the statement of our results.

DEFINITION (2.2). Suppose that λ is an isolated point of the spectrum of a bounded operator T, and let E be the spectral projection associated with $\{\lambda\}$. We let $\operatorname{rank}_\lambda(T)$ be the dimension of the range of E, and we let $\operatorname{ind}_\lambda(T)$ be the index of nilpotence of ET (with $\operatorname{ind}_\lambda(T) = \infty$ if ET is not nilpotent). Thus λ is a *pole* of T precisely when $\operatorname{ind}_\lambda(T) < \infty$; and λ is a *pole of finite rank* of T precisely when $\operatorname{rank}_\lambda(T) < \infty$.

LEMMA (2.3). *Suppose that R and T are bounded linear operators on the Banach spaces X and Y, that $S\colon Y \to X$ intertwines T and R, and that λ is an isolated point in $\sigma(R)$ and in $\sigma(T)$.*

(A) *If S is one-to-one and λ is a pole of R, then λ is a pole of T, with* $\operatorname{ind}_\lambda(T) \leqslant \operatorname{ind}_\lambda(R)$ *and* $\operatorname{rank}_\lambda(T) \leqslant \operatorname{rank}_\lambda(R)$.

(B) *If S has dense range and λ is a pole of T, then λ is a pole of R, with* $\operatorname{ind}_\lambda(R) \leqslant \operatorname{ind}_\lambda(T)$ *and* $\operatorname{rank}_\lambda(R) \leqslant \operatorname{rank}_\lambda(T)$.

Proof. Let E_R and E_T be the spectral projections associated with $\{\lambda\}$ for R and T, respectively.

If n is 0 or a positive integer, it follows from [4], Lemma 2.1, p. 68, that

$$E_R(R-\lambda)^n S = SE_T(T-\lambda)^n.$$

So if S is one-to-one, then $E_R(R-\lambda)^n = 0$ always implies that $(T-\lambda)^n E_T = 0$. This proves part (A) (for the rank statement, take $n = 0$). Similarly, if S has dense range and if some $(T-\lambda)^n E_T = 0$, then $E_R(R-\lambda)^n$ must be a bounded linear operator with dense null-space, so that $E_R(R-\lambda)^n = 0$. This proves (B) and completes the proof of the lemma.

We are now ready for our major result on intersections of components.

THEOREM (2.4). *Suppose that R and T are bounded linear operators on X and Y and that $S\colon Y \to X$ satisfies $RS = ST$.*

(A) *If S is one-to-one, then each component of $\sigma(T)$ which is not just a pole (a pole of finite rank) of T contains a point of $\sigma(R)$ which is not a pole (a pole of finite rank) of R.*

(B) *If S has dense range, then each component of $\sigma(R)$ which is not just a pole (a pole of finite rank) of R contains a point of $\sigma(T)$ which is not a pole (a pole of finite rank) of T.*

Proof. Part (B) will follow from part (A) by taking adjoints, so we just need to prove (A).

We first show that if F is an infinite set which is both open and closed in $\sigma(T)$, then F contains a point of $\sigma(R)$ which is not a pole of R. Let G be the set of poles of R in $F \cap \sigma(R)$. If G is an infinite subset of the compact set $F \cap \sigma(R)$, it would contain a cluster point. This cluster point would be a point which is in $F \cap \sigma(R)$ and which is not a pole of R.

Suppose that G is finite. Since each point of G is an isolated point of $\sigma(R)$, G is a closed and open subset in $\sigma(R)$. Let M be the spectral subspace of T associated with the subset $F \subseteq \sigma(T)$, and let E be the spectral projection on X associated with $(\sigma(R)\backslash G) \subseteq \sigma(R)$. Let $\hat{T}$ and $\hat{R}$ be the restrictions of T and R to M and $E(X)$, respectively; and let i be the injection of M into Y. It follows from the above definitions that

$$\hat{R}(ESi) = (ESi)\hat{T}.$$

So if $ESi \neq 0$, then Rosenblum's theorem [11], Corollary 0.13, p. 8, [4], Lemma 2.2, p. 69, implies that

$$(\sigma(R)\backslash G) \cap F = \sigma(\hat{R}) \cap \sigma(\hat{T})$$

is non-void. Then any point in $(\sigma(R)\backslash G) \cap F$ is a point of $F \cap \sigma(R)$ which is not a pole.

On the other hand, the assumption that $ESi = 0$ leads to a contradiction. For, since G is a finite set of poles of R, there is a polynomial $p(z)$ for which $p(R)(X) = E(X)$. Thus if $ESi = 0$, then

$$p(R)Si = (Si)(p(\hat{T})) = 0.$$

Since Si is one-to-one, we would then have that $p(\hat{T}) = 0$, and, hence, that $p(F) = \sigma(p(\hat{T})) = \{0\}$. But this contradicts the assumption that F is an infinite set. Hence every infinite closed-open subset of $\sigma(T)$ must contain a point of $\sigma(R)$ which is not a pole of R.

Suppose now that K is a component of the compact set $\sigma(T)$. There is a nested sequence $\{F_n\}$ of closed-open subsets of $\sigma(T)$ with $K = \bigcap F_n$ ([10], Corollary 1, p. 83). Since the F_n are open in $\sigma(T)$, it follows that if any F_n is a finite set, then the connected set K contains only a single point. In this case the theorem would follow from Lemma (2.3).

Suppose therefore that each F_n is an infinite set. Let $\sigma'(R)$ be the set of points in $\sigma(R)$ which are not poles of $\sigma(R)$. Each pole is an isolated point of $\sigma(R)$, so $\sigma'(R)$ is a closed subset of the plane. Hence $\{F_n \cap \sigma'(R)\}$ is a nested sequence of non-void closed subsets of the compact set $\sigma(T)$. Therefore $K \cap \sigma'(R) = \bigcap (F_n \cap \sigma'(R))$ is non-void. This completes the proof of the theorem.

We conclude this section with a pair of corollaries which illustrate how one can apply the results in this section to operators with special spectral properties.

COROLLARY (2.5). *Suppose that S is a one-to-one operator with dense range and that $RS = ST$. If $\sigma(R)$ and $\sigma(T)$ are both totally disconnected, then $\sigma(R) = \sigma(T)$.*

Proof. By Theorem (2.1), each point of $\sigma(T)$ belongs to $\sigma(R)$, and each point of $\sigma(R)$ belongs to $\sigma(T)$.

COROLLARY (2.6). *Suppose that R is a Riesz operator on X and that T is a bounded operator on Y. If there exists a one-to-one $S\colon Y \to X$ which intertwines T and R, then any component of $\sigma(T)$ which does not contain the* *origin must contain only a single point, and this point must be a pole of finite rank of both T and R.*

Proof. Let K be a component of $\sigma(T)$. If K contains more than one point which is a pole of finite rank of T, then, by Theorem (2.4), K must contain a point of $\sigma(R)$ which is not a pole of finite rank of R. But 0 is the only such point of $\sigma(R)$. On the other hand, if $K = \{\lambda\}$ with $\lambda \neq 0$, then λ belongs to $\sigma(R)$ by Theorem (2.1). Since $\lambda \neq 0$, it is a pole of finite rank of R; and it is also a pole of finite rank of T by Lemma (2.3).

3. Point and defect spectra. In this section and the next section we refine our analysis of spectral intersections by considering different kinds of spectra. If R is a bounded operator on X, we let $\sigma_p(R)$ and $\sigma_\pi(R)$ be the point spectrum and the approximate point spectrum, respectively, of R; and we let $\sigma_d(R) = \{\lambda: R-\lambda \text{ does not have dense range}\}$ and $\sigma_\delta(R) = \{\lambda: R-\lambda \text{ is not onto}\}$ be the defect spectrum and the approximate defect spectrum, respectively, of R. We also let $\partial\sigma(R)$ be the boundary of $\sigma(R)$, and recall that $\partial\sigma(R) \subseteq \sigma_\pi(R) \cap \sigma_\delta(R)$ ([1], Theorems (2.4.1) and (2.5.5), p. 28 and 31).

In Theorem (4.2), in the next section, we show that if R and T are quasi-similar operators, then any intersecting components of $\sigma(R)$ and $\sigma(T)$ contain a point which belongs to $\sigma_\pi(R) \cap \sigma_\delta(R) \cap \sigma_\pi(T) \cap \sigma_\delta(T)$ and which, in other ways as well, behaves spectrally like a point in $\partial\sigma(R) \cap \partial\sigma(T)$ (even though $\partial\sigma(R) \cap \partial\sigma(T)$ can be void [4], p. 71). In the present section we prove similar, but weaker, results when there is a one-to-one operator with dense range, or just a non-zero operator, which intertwines T and R. We start with a lemma on the set-theoretic difference of spectra.

LEMMA (3.1). *Suppose that R and T are bounded operators on X and Y and that $S: Y \to X$ satisfies $RS = ST$.*

(A) *If S is one-to-one, then $\sigma(T) \backslash \operatorname{int}(\sigma_p(R)) \subseteq \sigma_\delta(T)$.*

(B) *If S has dense range, then $\sigma(R) \backslash \operatorname{int}(\sigma_d(T)) \subseteq \sigma_\pi(R)$.*

Proof. Suppose first that S is one-to-one, and let λ belong to $\sigma(T) \backslash \operatorname{int}(\sigma_p(R))$. If $\lambda \in \partial\sigma(T) \subseteq \sigma_\delta(T)$, there is nothing to prove. Also, if λ does not belong to $\sigma_p(R)$, then $\lambda \notin \sigma_p(T)$, so that $\lambda \in \sigma(T) \backslash \sigma_p(T) \subseteq \sigma_\delta(T)$. Finally consider the remaining case, that λ belongs to the interior of $\sigma(T)$ and the boundary of $\sigma_p(R)$. Then λ is a limit of a sequence $\{\lambda_n\} \subseteq \sigma(T) \backslash \sigma_p(R) \subseteq \sigma_\delta(T)$. Since $\sigma_\delta(T)$ is closed ([1], Theorem (2.5.6) (b), p. 32), $\lambda \in \sigma_\delta(T)$.

Now suppose that S has dense range, and let λ belong to $\sigma(R) \backslash \operatorname{int}(\sigma_d(T))$. It's easy to show that $\sigma_d(R) \subseteq \sigma_d(T)$, so, as above, $\lambda \in \sigma_\pi(R)$ if either $\lambda \in \partial\sigma(R)$ or $\lambda \notin \sigma_d(T)$. Suppose therefore that λ belongs to the interior of $\sigma(R)$ and the boundary of $\sigma_d(T)$. Then λ is the limit of a sequence of points in $\sigma(R) \backslash \sigma_d(T) \subseteq \sigma_\pi(R)$. Since $\sigma_\pi(R)$ is closed ([1], Theorem (2.5.6) (a), p. 32), $\lambda \in \sigma_\pi(R)$. This completes the proof.

We now consider $\sigma(R) \cap \sigma(T)$ when S is one-to-one and has dense range. In connection with the next theorem, recall from Theorem (2.1) that any component of the spectrum of one of the operators R and T intersects the spectrum of the other.

THEOREM (3.2). *Suppose that R and T are bounded operators on X and Y, that S: $Y \to X$ is a one-to-one bounded linear operator with dense range, and that $RS = ST$. If K and L are components of $\sigma(R)$ and $\sigma(T)$, respectively, and if $K \cap L$ is non-void, then $K \cap L$ contains a point in $\sigma_\pi(R) \cap \sigma_\delta(T)$ which belongs to $\partial\sigma(R)$ or to $\partial\sigma(T)$.*

Proof. First suppose that K is a subset of L. Let λ be a boundary point of K. Then $\lambda \in \partial\sigma(R) \subseteq \sigma_\pi(R)$ and $\lambda \in \sigma(T) \backslash \operatorname{int}(\sigma(R))$. So, by Lemma (3.1), $\lambda \in (\sigma_\pi(R) \cap \sigma_\delta(T)) \cap \partial\sigma(R)$.

Now suppose that K is not a subset of L. Since $K \cap L$ is not void and K is connected, K must contain a boundary point, λ, of L. Then $\lambda \in \partial\sigma(T) \subseteq \sigma_\delta(T)$ and $\lambda \in \sigma(R) \backslash \operatorname{int}(\sigma(T))$, which is a subset of $\sigma_\pi(R)$ by Lemma (3.1). This completes the proof.

There are examples of R and T with one-to-one dense intertwining operators but for which $\sigma(R) \cap \sigma(T)$ contains no point of $\sigma_\pi(T)$ ([7], p. 1437–1438, [4], p. 71). Taking adjoints then gives an example for which $\sigma(R) \cap \sigma(T)$ contains no point of $\sigma_\delta(R)$.

As a corollary of Theorem (3.2) we obtain the following result, which is proved in a different way by Davis and Rosenthal as part of [2], Theorem 4, p. 1387.

COROLLARY (3.3). *Suppose that R and T are bounded operators on X and Y. If there is a non-zero bounded operator S: $Y \to X$ for which $RS = ST$, then $\sigma_\pi(R) \cap \sigma_\delta(T)$ is non-void.*

Proof. Let $\hat{R}$ and $\hat{T}$ be the maps induced by R and T on $\overline{S(Y)}$ and $Y/N(S)$, respectively; and let $\hat{S}$: $Y/N(S) \to \overline{S(Y)}$ be the map induced by S. Since $\hat{S}$ is one-to-one with dense range, it follows from Theorem (3.2) that $\sigma_\pi(\hat{R}) \cap \sigma_\delta(\hat{T})$ is non-void. But it is easy to see that $\sigma_\pi(\hat{R}) \subseteq \sigma_\pi(R)$ and that $\sigma_\delta(\hat{T}) \subseteq \sigma_\delta(T)$. Hence $\sigma_\pi(R) \cap \sigma_\delta(T)$ is also non-void, and the proof is complete.

We could have extracted a bit more from Theorem (3.2) than we did in the above corollary. For instance, since the point λ that we found in $\sigma_\pi(R) \cap \sigma_\delta(T)$ actually belongs to $\sigma_\delta(\hat{T})$ we have $(T-\lambda)(T)+N(S) \neq Y$ instead of just $(T-\lambda)(Y) \neq Y$. However, in most applications that require Lemma (3.1) or Theorem (3.2) instead of just Corollary (3.3), it seems more helpful to consider the induced maps like $\hat{R}$ and $\hat{T}$ directly. (See Theorem (6.2), for instance.) The following theorem extends the Davis–Rosenthal condition [2] that we proved in Corollary (3.3).

THEOREM (3.4). *Suppose that R and T are bounded operators on X and Y.*

If there is a non-zero bounded operator $S: Y \to X$ *for which* $RS = ST$, *then* $\sigma_\pi(R) \cap \sigma_\delta(T)$ *contains a point which belongs to* $\sigma_\delta(R)$ *or to* $\sigma_\pi(T)$.

Proof. Considering intersecting components of the closed sets $\sigma_\pi(R)$ and $\sigma_\delta(T)$, as in the proof of Theorem (3.2), we see that $\sigma_\pi(R) \cap \sigma_\delta(T)$ contains a point λ which belongs to the boundary of $\sigma_\pi(R)$ or to the boundary of $\sigma_\delta(T)$. Suppose that λ belongs to the boundary of $\sigma_\pi(R)$. Then either $\lambda \in \partial\sigma(R)$, or λ is the limit of a sequence of points in $\sigma(R)\backslash\sigma_\pi(R) \subseteq \sigma_\delta(R)$. In either case $\lambda \in \sigma_\delta(R)$. Similarly, if λ belongs to the boundary of $\sigma_\delta(T)$, then it belongs to $\sigma_\pi(T)$. This completes the proof.

We conclude this section with an application of Lemma (3.1) to Banach algebra homomorphisms.

COROLLARY (3.5). *Suppose that A and B are Banach algebras with identity and that* $\varphi: A \to B$ *is a continuous identity-preserving algebra homomorphism. If x belongs to A and* $\lambda \in \partial\sigma(\varphi(x))$, *then* $(x-\lambda)A \neq A$ *and* $A(x-\lambda) \neq A$.

Proof. First suppose that φ is one-to-one. Define the bounded operators T and R on A and B, respectively, by $T(a) = xa$ and $R(b) = \varphi(x)a$. Since φ is a homomorphism $R\varphi = \varphi T$ and $\sigma(\varphi(x)) \subseteq \varphi(x)$. So

$$\lambda \in \sigma(x)\backslash \operatorname{int}\big(\sigma(\varphi(x))\big) = \sigma(T)\backslash \operatorname{int}\big(\sigma(R)\big).$$

Hence, by Lemma (3.1) (A), $\lambda \in \sigma_\delta(T)$; that is, $(x-\lambda)A \neq A$. The proof that $A(x-\lambda) \neq A$ is similar, using right multiplication operators.

If φ is not one-to-one it induces a one-to-one homomorphism from $A/N(\varphi)$ to B. Then $(x-\lambda)\big(A/N(\varphi)\big) \neq A/N(\varphi)$ so $(x-y)A \neq A$. Similarly $A(x-\lambda) \neq A$.

4. Quasi-similar operators. Recall that bounded operators R and T on the Banach spaces X and Y, respectively, are said to be *quasi-similar* if there exist one-to-one operators with dense range, $S: Y \to X$ and $U: X \to Y$, for which $RS = ST$ and $TU = UR$. In this section we apply the results and methods of the previous two sections to study spectral intersections of quasi-similar operators.

We start with a lemma which is an analogue, for quasi-similar operators, of Lemma (3.1).

LEMMA (4.1). *Suppose that R and T are bounded operators on the Banach spaces X and Y, respectively, and that R and T are quasi-similar.*

(A) *If* $\lambda \in \sigma(R)\backslash\sigma(T)$, *then* $R-\lambda$ *is one-to-one and has proper dense range.*

(B) *If* $\lambda \in \sigma(R)\backslash\operatorname{int}(\sigma(T))$, *then* $\lambda \in \sigma_\pi(R) \cap \sigma_\delta(T)$, *and either* $R-\lambda$ *is not semi-Fredholm or* λ *is a pole of finite rank of* R.

Proof. Suppose first that $\lambda \in \sigma(R)\backslash\sigma(T)$. That $R-\lambda$ is one-to-one with proper dense range is essentially given in the proof of Lemma (3.1). This implies that $\lambda \in \sigma_\pi(R) \cap \sigma_\delta(T)$ and implies also, since semi-Fredholm operators have closed range ([1], Definition (1.3.1), p. 7), that $R-\lambda$ cannot be semi-Fredholm.

Suppose now that $\lambda \in \sigma(R) \cap \partial\sigma(T)$. Then $\lambda \in \sigma_\pi(R) \cap \sigma_\delta(T)$ by Lemma (3.1). If $R-\lambda$ is semi-Fredholm, then λ cannot belong to the interior of $\sigma(R)$. For the set of semi-Fredholm operators is open in the uniform norm ([1], pp. 61–63), but λ is a limit of a sequence $\{\lambda_n\}$ for which $R-\lambda_n$ is one-to-one with dense range. Therefore, in this case $\lambda \in \partial\sigma(R)$ and $R-\lambda$ is semi-Fredholm, which implies that λ is a pole of finite rank of R ([9], Theorem 2.9, p. 205, [7], Theorem (5.4), p. 1439).

By using Lemma (4.1), in place of Lemma (3.1), we obtain the following analogue, for quasi-similar operators, of Theorem (3.2).

THEOREM (4.2). *Suppose that R and T are bounded linear operators on X and Y, and that K and L are components of $\sigma(R)$ and $\sigma(T)$, respectively. If R and T are quasi-similar, and if $K \cap L$ is non-void, then $K \cap L$ contains a point λ for which*:

(A) $\lambda \in \sigma_\pi(R) \cap \sigma_\delta(R) \cap \sigma_\pi(T) \cap \sigma_\delta(T)$;

(B) *λ belongs to $\partial\sigma(R)$ or to $\partial\sigma(T)$*;

(C) *If $R-\lambda$ is semi-Fredholm, then $K = \{\lambda\}$ and λ is a pole of finite rank of R.*

(D) *If $T-\lambda$ is semi-Fredholm, then $L = \{\lambda\}$ and λ is a pole of finite rank of T.*

Proof. As in the proof of Theorem (3.2), $K \cap L$ contains a point of $\partial\sigma(R) \cap \sigma(T)$ or of $\sigma(R) \cap \partial\sigma(T)$. This proves (B); and, since quasi-similarity is a symmetric relation, allows us to assume that $K \cap L$ contains a point $\lambda \in \sigma(R) \cap \partial\sigma(T)$. Now $\partial\sigma(T) \subseteq \sigma_\pi(T) \cap \sigma_\delta(T)$ and $\sigma(R) \cap \partial\sigma(T) \subseteq \sigma(R) \setminus \operatorname{int}(\sigma(T))$; so that parts (A) and (C) follow from Lemma (4.1).

To prove (D), we suppose that $T-\lambda$ is semi-Fredholm. Since $\lambda \in \partial\sigma(T)$, λ must be a pole of finite rank of T ([9], Theorem 2.9, p. 205, [7], Theorem (5.4), p. 1439). This completes the proof.

The following corollary is now an immediate consequence of the above theorem together with Theorem (2.4).

COROLLARY (4.3). *Suppose that R and T are quasi-similar operators on Banach spaces and that K is a component of $\sigma(R)$. If K is not just a pole of finite rank of R, then $K \cap \sigma(T)$ contains a point λ which satisfies* (4.2) (A) *and* (B), *and for which neither $R-\lambda$ nor $T-\lambda$ is semi-Fredholm.*

Since every bounded operator on an infinite-dimensional Banach space must contain in its spectrum at least one point which is not a pole of finite rank, the above corollary generalizes Fialkow's results on intersections of essential spectra ([4], Theorem 2.6, p. 71) and of left and right essential spectra ([5], Theorem 2.1).

All the results in this section which assume that $R-\lambda$ or $T-\lambda$ is semi-

Fredholm really use only the property which we isolate in condition (4.4) below.

CONDITION (4.4). *If $\lambda \in \partial\sigma(R)$, or if λ is a limit of a sequence $\{\lambda_n\}$ for which $R-\lambda_n$ is one-to-one with proper dense range, then λ is a pole of finite rank of R.*

Not only semi-Fredholm $R-\lambda$, but, more generally, any $R-\lambda$ which satisfies the hypothesis of [7], Theorem (5.4), p. 1439, must satisfy Condition (4.4), above (see [8]). Still more generally, the operators which have the property which we call "eventual topological uniform descent" in [8] satisfy Condition (4.4) with "pole of finite rank" replaced simply by "pole." Thus all the results in this section remain true when "semi-Fredholm" is replaced by "has eventual topological uniform descent", and "pole of finite rank" is replaced by "pole."

5. Ascent, descent, nullity, and defect. Recall that if R is a bounded operator on a Banach space X, then the *nullity* of R is the dimension of its null-space $N(R)$, and the *ascent* of R is the smallest non-negative integer n for which $N(R^n) = N(R^{n+1})$ (if no such n exists the ascent of R is ∞) [1], p. 10, [9], p. 197. Similarly we define the *closed defect* of R as the co-dimension of $\overline{R(X)}$, and the *closed descent* as the smallest integer n for which $\overline{R^n(X)} = \overline{R^{n+1}(X)}$ (again allowing ∞ if no n exists). In this section we compare the ascent, nullity, closed defect, and closed descent of two operators which have a one-to-one or dense range intertwining operator, or which are quasi-similar. Hopefully, results of the type we prove will help shed light on the open question ([5], Section 2) of whether every component of the essential spectrum of R must intersect the essential spectrum of T, when R and T are quasi-similar. The following theorem is our basic result.

THEOREM (5.1). *Suppose that R and T are bounded linear operators on the Banach spaces X and Y, and that S: $Y \to X$ is a bounded operator for which $RS = ST$.*

(A) *If S is one-to-one, then the nullity and ascent of T are less than or equal to the nullity and ascent, respectively, of R.*

(B) *If S has dense range, then the closed defect and closed descent of R are no greater than the corresponding quantities for T.*

(C) *If R and T are quasi-similar, they have the same nullity, ascent, closed defect, and closed descent.*

Proof. Suppose that S is one-to-one. Then S maps $N(T)$ one-to-one into $N(R)$, so that the nullity of T is not greater than that of R. Also, for each n, $S^{-1}(N(R^n)) = N(T^n)$. This completes the proof of (A).

Now suppose that S has dense range. Since $\overline{S(T^n(Y))}$ and $\overline{R^n(S(Y))} = R^n(X)$ have the same closure, the closed descent of R must be less than or equal to that of T. To compare the closed defects of R and T, we consider the map $\hat{S}$: $Y/\overline{T(Y)} \to X/\overline{R(X)}$, induced by S. Since S has dense range, so does $\hat{S}$. If T has finite closed defect, then $\hat{S}$ has finite-dimensional, and hence closed, range as well; so that $\hat{S}$ is onto. If T has infinite closed defect, there is nothing to prove, so this completes the proof of (B). Part (C) is an immediate consequence of parts (A) and (B).

When R and T are quasi-similar, we can improve the above theorem a bit.

COROLLARY (5.2) *Suppose that R and T are quasi-similar operators on the Banach spaces X and Y, and that S: $Y \to X$ is a one-to-one operator with dense range for which $RS = ST$.*

(A) *If either R or T have finite nullity, then $S(N(T)) = N(R)$.*

(B) *If either R or T have finite closed defect and if W is a complementary space to $\overline{T(Y)}$, then $S(W)$ is complementary to $\overline{R(X)}$.*

Proof. Since S maps $N(T)$ one-to-one in $N(R)$, part (A) follows immediately from Theorem (5.1) (A).

Since S has dense range and $S(W)$ is finite-dimensional, we have

$$X = \overline{S(Y)} = \mathrm{cl}\,[S(\overline{T(Y)})+S(W)] = \overline{S(T(Y))}+S(W)$$
$$= \overline{R(S(Y))}+S(W) = \overline{R(X)}+S(W).$$

But, by Theorem (5.1) (C), the co-dimension of $\overline{R(X)}$ equals the (finite) dimension of $S(W)$. Hence the sum $X = \overline{R(X)}+S(W)$ must be direct. This completes the proof.

Notice that all the results in this section can be applied to $R-\lambda$ and $T-\lambda$ for any complex λ. Also Corollary (5.2), and the statements about nullity and closed defect in Theorem (5.1), apply directly to R^n and T^n in place of R and T, respectively.

6. Non-existence of intertwining operators. Suppose again that R and T are bounded operators on the Banach spaces X and Y. In this section we prove two theorems which give some conditions on R and T that guarantee that no non-zero S: $Y \to X$ can intertwine T and R; and we briefly discuss a few special examples of these conditions. Other conditions can be found in [3], and [4], Section 4. For applications of the non-existence of intertwining operators, see [3], [4], Section 1.

THEOREM (6.1). *Suppose that R and T are bounded operators on X and Y, that S: $Y \to X$ is a bounded operator satisfying $RS = ST$, and that λ is a complex number. Then $S = 0$ if either of the following two conditions holds:*

(i) $\bigcup N(T-\lambda)^n$ *is dense, and $R-\lambda$ is one-to-one.*

(ii) $\bigcap (R-\lambda)^n(X) = \{0\}$, *and $T-\lambda$ has dense range.*

Proof. For simplicity we set $\lambda = 0$. If $y \in \bigcup N(T^n)$, then there is a k for which $ST^k y = R^k Sy = 0$. Hence, if R is one-to-one, then $\bigcup N(T^n)$ is a subspace on $N(S)$. This proves the theorem under hypothesis (i).

Now suppose (ii) holds. Since T has dense range, the restriction of R to $\overline{S(Y)}$ also has dense range. Therefore $\bigcap R^n(\overline{S(Y)})$ is dense in $\overline{S(Y)}$ ([14], Lemma 1.8, p. 13). So, by hypothesis, $\overline{S(Y)} = \{0\}$, and the theorem is proved.

The above theorem is similar to Sinclair's [14], Theorem 4.2, p. 25. In our terminology, Sinclair shows that if $T-\lambda$ is onto, instead of just having dense range as we assumed in (6.1) (ii), then $S = 0$ follows without any continuity assumption on S. While our next theorem resembles Theorem (6.1) above, its proof requires results from Section 3.

THEOREM (6.2). *Suppose that R and T are bounded operators on X and Y, that S: $Y \to X$ is a bounded operator satisfying $RS = ST$, and that λ is a complex number. Then $S = 0$ if either of the following two conditions holds*:

(i) $\bigcup N(T-\lambda)^n$ *is dense, but λ does not belong to $\sigma_\delta(T)$ or* $\operatorname{int}(\sigma_p(R))$;

(ii) $\bigcap (R-\lambda)^n(X) = \{0\}$, *but λ does not belong to $\sigma_\pi(R)$ or* $\operatorname{int}(\sigma_d(T))$.

Proof. Suppose that $\bigcup N(T-\lambda)^n$ is dense in Y, that $\lambda \notin \operatorname{int}(\sigma_p(R))$, and that $S \neq 0$. Let $\hat{T}$ be the map induced by T on $Y/N(S)$, and let $\hat{S}$: $Y/N(S) \to X$ be induced by S. Then $\hat{S}$ is one-to-one and $R\hat{S} = \hat{S}T$. Since $\bigcup N(T-\lambda)^n$ is dense, $\lambda \in \sigma(\hat{T}) \backslash \operatorname{int}(\sigma_p(R))$, and hence, by Lemma (3.1) (A), $\lambda \in \sigma_\delta(\hat{T})$. But $\sigma_\delta(\hat{T}) \subseteq \sigma_\delta(T)$, so the theorem is proved under hypothesis (i). We omit the similar proof under hypothesis (ii).

By Theorem (6.1), no backward shift can be intertwined with a one-to-one operator; and no operator with dense range can be intertwined with a forward shift. Suppose that R is a quasi-nilpotent forward shift and that T is a quasi-nilpotent backward shift. Then no non-zero S: $Y \to X$ satisfies $RS = ST$ and no non-zero U: $X \to Y$ satisfies $TU = UR$; even though $\sigma(R) = \sigma(T) = \sigma_\pi(R) = \sigma_\pi(T) = \sigma_\delta(R) = \sigma_\delta(T) = \{0\}$.

Similarly, by Theorem (6.2) (ii), no operator T with $0 \in \partial\sigma(T)$ (in particular no quasi-nilpotent T) can be intertwined with an unweighted forward shift. An analogous statement about backward shifts follows from Theorem (6.2) (ii).

7. Finite-rank intertwining operators. In Theorem (7.1) we give a simple necessary and sufficient condition for T and R to have a non-zero finite-rank intertwining operator; and in Theorem (7.2) we give a special condition under which all intertwining operators have finite rank.

THEOREM (7.1). *Suppose that R and T are bounded linear operators on the Banach spaces X and Y. There is a non-zero bounded finite rank S: $Y \to S$ for which $RS = ST$ if and only if $\sigma_p(R) \cap \sigma_d(T)$ is non-void.*

Proof. First suppose that λ belongs to $\sigma_p(R)$ and also to $\sigma_d(T) = \sigma_p(T^*)$ ([13], p. 94). Then there are non-zero x in X and f in Y^* for which

$Rx = \lambda x$ and $Tf = \lambda f$. Define the non-zero rank 1 operator $S: Y \to X$ by $Sy = f(y)x$. Then $RS = ST = \lambda S$.

Conversely suppose that there is a non-zero finite rank $S: Y \to X$ for which $RS = ST$. Then the operators $\hat{T}$ and $\hat{R}$ induced by T and R on $Y/N(S)$ and $S(Y)$, respectively, are similar operators on finite-dimensional spaces. Hence $\sigma(\hat{R}) \cap \sigma(\hat{T}) = \sigma_p(\hat{R}) \cap \sigma_d(\hat{T})$ is non-void. The theorem now follows from the observations that $\sigma_p(\hat{R}) \subseteq \sigma_p(R)$ and that $\sigma_d(\hat{T}) \subseteq \sigma_d(T)$.

THEOREM (7.2). *Suppose that R and T are bounded operators on X and Y and that $S: Y \to X$ intertwines T and R. If R is a Riesz operator and T is onto, then S has finite rank.*

Proof. Since $S(Y)$ is the range of a bounded operator and $R(S(Y)) = S(T(Y)) = S(Y)$, the fact that R is Riesz implies that $S(Y)$ is finite-dimensional ([6], Theorem 2 (ii)). This completes the proof.

If we had assumed in Theorem (7.2) that R was quasi-nilpotent, then essentially the same proof would show that $S = 0$ (cf. [7], Theorem (2.7), p. 1432). Similarly if R was decomposable at zero in the sense of [6], then S could not have dense range, by [6], Theorem 2.

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