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On the asymptotic behaviour of the solutions of difference equations of second order

Abstract. The aim of the present paper is to study the asymptotic behaviour of certain classes of difference equations of second order. The principal tool we use is the discrete analogy of Bihari lemma type.

In this note we consider the non-linear difference equation of the form

$$\Delta^2 y_n + F(n, y_n, y_{n+1}) = b_n, \tag{1}$$

where $F\colon N \times R^2 \to R$ is continuous on R^2 for each fixed $n, b\colon N \to R$.

Here $y_n = y(n)$, $b_n = b(n)$, N is the set of natural numbers, by Δy_n we denote the difference $y_{n+1} - y_n$ and by $\Delta^2 y_n$ the difference $\Delta(\Delta y_n)$. Throughout we assume that the function F satisfies inequality

$$|F(n, z_1, z_2)| \leqslant B(n, |z_1|, |z_2|), \tag{2}$$

where $B(n, z_1, z_2)$ is continuous on R^2 for fixed n and such that for $n \in N$ and $z_k \geqslant 0$ $(k = 1, 2)$ hold:

(i) $\quad 0 \leqslant B(n, z_1, z_2) \leqslant B(n, \bar{z}_1, \bar{z}_2) \quad$ for $z_k \leqslant \bar{z}_k$ $(k = 1, 2)$,

(ii) $\quad B(n, a_n z_1, a_n z_2) \leqslant A(a_n) B(n, z_1, z_2) \quad$ for $a_n \geqslant \varepsilon > 0$,

where $A\colon \langle \varepsilon, \infty) \to R_+$ is non-decreasing and $\int_{\varepsilon}^{\infty} \frac{ds}{A(s)} = \infty$.

Some useful lemmas will be given prior are to our considerations.

Lemma 1. *Let* $\{F_n\}$, $\{Q_n\}$ *are non-negative sequences,* $\{Q_n\}$ *is non-decreasing for* $n \geqslant n_0$, $\lim_{n\to\infty} Q_n = \infty$ *and* $\sum_{n_0}^{\infty} \frac{F_n}{Q_n} < \infty$. *If there exists a sequence* $\{\beta_n\}$, $0 < \beta_n \leqslant n$ *such that* $\lim_{n\to\infty} \beta_n = \infty$ *and* $\lim_{n\to\infty} Q_{[\beta_n]} (Q_n)^{-1} = 0$, *then* $\lim_{n\to\infty} \frac{1}{Q_n} \sum_{k=n_0}^{n} F_k = 0$.

Here the symbol [ ] *denotes the integral part.*

Proof. Lemma 1 is an immediate consequence of the following inequality

$$\frac{1}{Q_n}\sum_{k=n_0}^{n} F_k = \frac{1}{Q_n}\sum_{k=n_0}^{[\beta_n]} Q_k\frac{F_k}{Q_k}+\frac{1}{Q_n}\sum_{k=[\beta_n]}^{n} Q_k\frac{F_k}{Q_k}$$
$$\leqslant \frac{Q_{[\beta_n]}}{Q_n}\sum_{k=n_0}^{\infty}\frac{F_k}{Q_k}+\sum_{k=[\beta_n]}^{\infty}\frac{F_k}{Q_k}.$$

LEMMA 2. *Let the function B satisfy* (i) *and* (ii): *furthemore, suppose that the non-negative sequences* $\{u_n\}$, $\{v_n\}$ *satisfy the following conditions*

(a) $$u_{n+2} \leqslant v_{n+2}\Big[c+\sum_{k=n_0}^{n} B(k, u_k, u_{k+1})\Big], \quad 0 < c = \text{const } \textit{for } n \geqslant n_0,$$

(b) $$\sum_{k=n_0}^{\infty} B(k, v_k, v_{k+1}) < \infty.$$

Then there exists a constant $M > 0$ *such that* $u_n \leqslant Mv_n$ *for* $n \geqslant n_0+2$.

Proof. Define $d_n = c+\sum_{k=n_0}^{n} B(k, u_k, u_{k+1})$ and note that (a) implies that $u_{n+2} \leqslant v_{n+2}d_n$ for $n \geqslant n_0$. Differencing above equality, and by conditions on B we have

$$\Delta d_n = B(n+1, u_{n+1}, u_{n+2}) \leqslant B(n+1, v_{n+1}d_{n-1}, v_{n+2}d_n)$$
$$\leqslant B(n+1, v_{n+1}d_n, v_{n+2}d_n) \leqslant A(d_n)B(n+1, v_{n+1}, v_{n+2}) \quad \text{for } n \geqslant n_0+1.$$

Now, divide by $A(d_n)$ and use the mean value theorem to obtain

$$\int_{d_n}^{d_{n+1}} \frac{ds}{A(s)} \leqslant \frac{\Delta d_n}{A(d_n)} \leqslant B(n+1, v_{n+1}, v_{n+2}).$$

Summing the last inequality from n_0+1 to n we have

$$\int_{d_{n_0+1}}^{d_{n+1}} \frac{ds}{A(s)} \leqslant \sum_{k=n_0+1}^{n} B(k+1, v_{k+1}, v_{k+2}).$$

Thus

$$d_{n+1} \leqslant G^{-1}\Big\{G(d_{n_0+1})+\sum_{k=n_0+1}^{n} B(k+1, v_{k+1}, v_{k+2})\Big\},$$

where G^{-1} is the inverse function of G, defined by $G(z) = \int_{0<\varepsilon}^{z}\frac{ds}{A(s)}$. Since G^{-1}, as well as G, is strictly increasing, the above inequality together with condition (b) and

$$d_{n_0+1} \leqslant 2c+B(n_0, u_{n_0}, u_{n_0+1})+B(n_0+1, u_{n_0+1}, v_{n_0+2}d_{n_0}) \leqslant K$$

give for $n > n_0+1$

$$u_{n+2} \leqslant v_{n+2} G^{-1}\Big\{G(K)+\sum_{k=n_0+1}^{n-1} B(k+1, v_{k+1}, v_{k+2})\Big\} \leqslant M v_{n+2}.$$

It is easy to see that the last inequality is also valid for $n = n_0, n_0+1$ and our lemma is proved.

The following lemma is the discrete analogy of d'Hospital rule and follows by the similar argument that we omit here.

LEMMA 3. *Let* $\{v_n\}$ *be sequence such that for some* $n \geqslant n_0, \Delta v_n > 0$ *and* $\lim_{n\to\infty} v_n = \infty$. *If there exists* $\lim_{n\to\infty} \frac{\Delta u_n}{\Delta v_n} = L$, *then* $\lim_{n\to\infty} \frac{u_n}{v_n} = L$.

THEOREM 1. *Let* $\{Q_n\}$ *be non-negative sequence such that*

$$\Delta Q_n \geqslant 0, \quad \lim_{n\to\infty} Q_n = \infty \quad \text{and} \quad \lim_{n\to\infty} \frac{Q_{[\beta_n]}}{Q_n} = 0 \tag{3}$$

for some sequence $\{\beta_n\}$, *such that* $0 < \beta_n \leqslant n$, $n \in N$ *and* $\lim_{n\to\infty} \beta_n = \infty$. *If*

$$\lim_{n\to\infty} \frac{b_n}{\Delta Q_n} = L \neq 0, \quad L = \text{const}, \tag{4}$$

$$\sum^{\infty} \frac{1}{Q_n} B\big(n, nQ_n, (n+1) Q_{n+1}\big) < \infty, \tag{5}$$

then every solution $\{y_n\}$ *of* (1) *has the property*

$$\lim_{n\to\infty} \frac{\Delta y_n}{Q_n} = L. \tag{6}$$

Proof. Summing (1) twice from n_0 to n one can verify the following equality

$$y_{n+2} = y_{n_0+1} + (n-n_0+1)\Delta y_{n_0} + \sum_{k=n_0}^{n} (n+1-k) b_k - \sum_{k=n_0}^{n} (n+1-k) F(k, y_k, y_{k+1}). \tag{7}$$

We may assume without loss of generality that $Q_n > 0$ for $n > n_0$. Divide now (7) by $(n+2)Q_{n+2}$, we obtain

$$\frac{y_{n+2}}{(n+2)Q_{n+2}} = \frac{1}{Q_{n+2}}\left\{\frac{y_{n_0+1}}{n+2} + \frac{n-n_0+1}{n+2}\Delta y_{n_0}\right\} + \frac{1}{(n+2)Q_{n+2}}\sum_{k=n_0}^{n} (n+1-k) b_k - \frac{1}{(n+2)Q_{n+2}}\sum_{k=n_0}^{n} (n+1-k) F(k, y_k, y_{k+1})$$

hence

$$(8)\qquad \frac{|y_{n+2}|}{(n+2)\,Q_{n+2}} \leqslant \frac{n-n_0+1}{n+2}\,\frac{1}{Q_n}\{|y_{n_0+1}|+|\Delta y_{n_0}|\}+\left|\frac{1}{Q_{n+2}}\sum_{k=n_0}^{n} b_k\right|+$$
$$+\frac{1}{Q_{n+2}}\sum_{k=n_0}^{n} |F(k, y_k, y_{k+1})|.$$

Using Lemma 3 and by assumptions (2), (3) we observe that

$$\frac{|y_{n+2}|}{(n+2)\,Q_{n+2}} \leqslant C+\sum_{k=n_0}^{n}\frac{1}{Q_k}B(k, |y_k|, |y_{k+1}|).$$

From Lemma 2 it follows now for $n \geqslant n_1 \geqslant n_0+2$

$$(9)\qquad |y_n| \leqslant MnQ_n.$$

Summing now (1) from n_0 to n and divide by Q_{n+1} we have

$$(10)\qquad \frac{\Delta y_{n+1}}{Q_{n+1}} = \frac{\Delta y_{n_0}}{Q_{n+1}}+\frac{1}{Q_{n+1}}\sum_{k=n_0}^{n} b_k-\frac{1}{Q_{n+1}}\sum_{k=n_0}^{n} F(k, y_k, y_{k+1}).$$

First of all observe that

$$(11)\quad \lim_{n\to\infty}\frac{\Delta y_{n_0}}{Q_{n+1}} = 0, \quad \text{and by (4) and Lemma 3} \quad \lim_{n\to\infty}\frac{\sum\limits_{k=n_0}^{n} b_k}{Q_{n+1}} = L.$$

We show now, that the third term of the right-hand side of (10) approaches zero as $n \to \infty$.

From (9) we have

$$\frac{1}{Q_{n+1}}\sum_{k=n_0}^{n} |F(k, y_k, y_{k+1})|$$
$$\leqslant \frac{1}{Q_n}\sum_{k=n_0}^{n_1} |F(k, y_k, y_{k+1})|+\frac{1}{Q_n}\sum_{k=n_1}^{n} B(k, |y_k|, |y_{k+1}|)$$
$$\leqslant \frac{1}{Q_n}\sum_{k=n_0}^{n_1} |F(k, y_k, y_{k+1})|+\frac{1}{Q_n}\sum_{k=n_1}^{n} B(k, MkQ, M(k+1)\,Q_{k+1})$$
$$\leqslant \frac{1}{Q_n}\sum_{k=n_0}^{n_1} |F(k, y_k, y_{k+1})|+\frac{A(M)}{Q_n}\sum_{k=n_1}^{n} B(k, kQ_k, (k+1)\,Q_{k+1}).$$

Using now Lemma 1 to the above inequality we obtain

$$(12)\qquad \lim_{n\to\infty}\frac{1}{Q_{n+1}}\sum_{k=n_0}^{n} F(k, y_k, y_{k+1}) = 0.$$

Comparing (10), (11) and (12) we complete the proof of the theorem.

Consider the special cases of sequences for which assumption (3) holds.

COROLLARY 1. *If, under the assumptions of Theorem* 1

$$(13)\qquad Q_n = n^{\alpha} V_{n+1}, \quad \text{where } \alpha \geq 0 \text{ and } \quad \lim_{n\to\infty} n\frac{\Delta V_n}{V_{n+1}} = C$$

or

$$(14)\qquad Q_n = V_{n+1} e^{an}, \quad \text{where } a > 0 \text{ and } \quad \lim_{n\to\infty} \frac{\Delta V_n}{V_{n+1}} = 0,$$

then every solution $\{y_n\}$ *of* (1) *has the asymptotic behaviour*

$$(15)\quad \lim_{n\to\infty} \frac{\Delta^k y_n}{n^{\alpha+1-k} V_{n+k}} = \frac{L}{(\alpha+1+C)^{1-k}} \quad (k = 0, 1) \text{ in the case } (13)$$

or

$$(16)\quad \lim_{n\to\infty} \frac{\Delta^k y_n}{e^{an} V_{n+k}} = \frac{L}{(e^a-1)^{1-k}} \quad (k = 0, 1) \text{ in the case } (14).$$

Here $\Delta^0 y_n$ *denote* y_n.

Proof. From (6) and by Lemma 3 we have

$$\lim_{n\to\infty} \frac{y_n}{n^{\alpha+1} V_n} = \lim_{n\to\infty} \frac{\Delta y_n}{\Delta (n^{\alpha+1} V_n)} = \lim_{n\to\infty} \frac{\Delta y_n}{V_{n+1}\Delta n^{\alpha+1} + n^{\alpha+1}\Delta V_n}$$

$$= \lim_{n\to\infty} \frac{\dfrac{\Delta y_n}{n^{\alpha} V_{n+1}}}{\dfrac{\Delta n^{\alpha+1}}{n} + n\dfrac{\Delta V_n}{V_{n+1}}} = \frac{L}{\alpha+1+C} \quad \text{for the case (13),}$$

$$\lim_{n\to\infty} \frac{y_n}{V_n e^{an}} = \lim_{n\to\infty} \frac{\Delta y_n}{\Delta (V_n e^{an})} = \lim_{n\to\infty} \frac{\Delta y_n}{V_{n+1}\Delta e^{an} + e^{an}\Delta V_n}$$

$$= \lim_{n\to\infty} \frac{\dfrac{\Delta y_n}{V_{n+1} e^{an}}}{\dfrac{\Delta e^{an}}{e^{an}} + \dfrac{\Delta V_n}{V_{n+1}}} = \frac{L}{e^a-1} \quad \text{for the case (14).}$$

THEOREM 2. *Suppose that the following conditions*

$$(17)\qquad \sum^{\infty} |b_k| < \infty,$$

$$(18)\qquad \sum^{\infty} B(k, k, k+1) < \infty \quad \text{are satisfied.}$$

Then every solution $\{y_n\}$ *of* (1) *has the property*

$$(19)\qquad \lim_{n\to\infty} \frac{\Delta^k y_n}{n^{k-1}} = L = \text{const} \qquad (k = 0, 1).$$

Moreover, equation (1) *has the solution* $\{y_n\}$ *with* $L \neq 0$.

Proof. Using the same calculations as in the preceding theorem we obtain (8), where $Q_n = 1, n \geqslant n_0$:

$$\frac{|y_{n+2}|}{n+2} \leqslant \frac{|y_{n_0}|}{n+2} + \frac{n-n_0+1}{n+2}|\Delta y_{n_0}| + \sum_{k=n_0}^{n} |b_k| + \sum_{k=n_0}^{n} |F(k, y_k, y_{k+1})|.$$

Analogously as in Theorem 1, to the above we may use Lemma 2 from which and by (17), (18) we have

$$(20)\qquad |y_n| \leqslant Mn \quad \text{for } n \geqslant n_1.$$

Summing (1) from n_0 to n we obtain

$$(21)\qquad \Delta y_{n+1} = \Delta y_{n_0} + \sum_{k=n_0}^{n} b_k - \sum_{k=n_0}^{n} F(k, y_k, y_{k+1}).$$

By (17) we have

$$(22)\qquad \lim_{n\to\infty} \sum_{k=n_0}^{n} b_k = L_0.$$

We show now that there exists a finite $\lim_{n\to\infty} \sum_{k=n_0}^{n} F(k, y_k, y_{k+1})$. Observe that by (20) and preliminary conditions on functions B, F we have

$$(23)\qquad \begin{aligned} \sum_{k=n_0}^{n} |F(k, y_k, y_{k+1})| &\leqslant \sum_{k=n_0}^{n_1} |F(k, y_k, y_{k+1})| + \sum_{k=n_1}^{n} B(k, y_k, y_{k+1}) \\ &\leqslant \sum_{k=n_0}^{n_1} |F(k, y_k, y_{k+1})| + A(M) \sum_{k=n_1}^{n} B(k, k, k+1). \end{aligned}$$

Therefore comparing (21), (22) and (23) it follows that $\lim_{n\to\infty} \Delta y_n = L$. Now by Lemma 3

$$\lim_{n\to\infty} \frac{y_n}{n} = \lim_{n\to\infty} \Delta y_n = L.$$

For the rest of the proof it can be noted that we can take y_{n_0}, y_{n_0+1} such, that $L \neq 0$.

THEOREM 3. *Let*

$$(24)\qquad \sum^{\infty} n|b_n| < \infty,$$

$$(25)\qquad \sum^{\infty} nB(n, n, n+1) < \infty.$$

Then every solution $\{y_n\}$ *of* (1) *has the property*

$$(26)\qquad \Delta^k y_n = \sum_{i=k}^{1} c_i n^{1-i} + o(1), \quad c_i = \text{const } (k = 0, 1), \text{ as } n \to \infty.$$

Proof. From Theorem 2 it follows that every solution $\{y_n\}$ of (1) has a finite limit of sequence of first differences, i.e. $\lim\limits_{n\to\infty} \Delta y_n = L$. From this for large $n \geqslant n_2$ we get estimation

$$(27)\qquad |y_n| \leqslant Mn,$$

where M is a certain constant. Observe that

$$(28)\qquad \sum_{k=n_0}^{\infty} (k+1-n_0)|b_k| \leqslant \sum_{k=n_0}^{\infty} (k+1)|b_k| < \infty$$

and

$$(29)\qquad \sum_{k=n_0}^{\infty} (k+1-n_0)|F(k, y_k, y_{k+1})|$$
$$\leqslant \sum_{k=n_0}^{n_2} (k+1)|F(k, y_k, y_{k+1})| + \sum_{k=n_2}^{\infty} (k+1)B(k, k, k+1) < \infty.$$

Summing equation (1) from n to m and pass with $m \to \infty$ we have

$$(30)\qquad \Delta y_n = L - \sum_{k=n}^{\infty} b_k + \sum_{k=n}^{\infty} F(k, y_k, y_{k+1}).$$

From (28), (29) and (30) $\Delta y_n = L + o(1)$. Summing now (30) from n_2 to n we obtain

$$\begin{aligned} y_{n+1} &= y_{n_2} + (n-n_2+1)L - \sum_{i=n_2}^{n} \sum_{k=i}^{\infty} b_k + \sum_{i=n_2}^{n} \sum_{k=i}^{\infty} F(k, y_k, y_{k+1}) \\ &= y_{n_2} + (n+1)L - n_2 L - \sum_{k=n}^{\infty} (n-k)b_k - \sum_{k=n_2}^{\infty} (k-n_2+1)b_k + \\ &\quad + \sum_{k=n}^{\infty} (n-k)F(k, y_k, y_{k+1}) + \sum_{k=n_2}^{\infty} (k-n_2+1)F(k, y_k, y_{k+1}). \end{aligned}$$

Since (28) and (29) hold it follows that

$$y_n = nL + L_1 + o(1).$$

Remark. We conclude this paper by noting that by precisely the same methods as used here we can obtain similar results for some different difference equation.

Let $\Delta_a y_n = y_{n+1} - a y_n$ for arbitrary positive constant a. Consider now an equation of the form

$$\Delta_a^2 y_n + F(n, y_n, y_{n+1}) = b_n. \tag{31}$$

Putting $y_n = a^n u_n$ in (31) we obtain a following equation

$$\Delta^2 u_n + a^{-n-2} F(n, a^n u_n, a^{n+1} u_{n+1}) = a^{-n-2} b_n.$$

Taking now

$$F_1(n, u_n, u_{n+1}) = a^{-n-2} F(n, a^n u_n, a^{n+1} u_{n+1}), \qquad b_{1n} = a^{-n-2} b_n,$$

and assuming that F_1, $\{b_{1n}\}$ satisfy conditions on F, $\{b_n\}$ of our theorems we obtain (for instance for Theorem 1)

$$\lim_{n\to\infty} \frac{\Delta_a y_n}{a^{n+1} Q_n} = \lim_{n\to\infty} \frac{\Delta \dfrac{y_n}{a^n}}{Q_n} = \lim_{n\to\infty} \frac{\Delta u_n}{Q_n} = L.$$

Similar problem for differential equations was treated by many authors (see, for instance, [1], [2]).

References

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[2] P. Marušiak, *The differential equation with retarded argument asymptotically equivalent to the equation* $y^{(n)} = 0$, Matem. Časopis 23, No. 1 (1973), p. 45–54.