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On the polycaloric problem for the exterior of the ball

1. In the present paper we shall solve the limit problem for the equation

\[ P^n u(x, y, z, t) = F(r, t), \]

where

\[ P = D_{xx}^2 + D_{yy}^2 + D_{zz}^2 - D_t, \quad P^n = P(P^n - 1), \quad r = (x^2 + y^2 + z^2)^{1/2} \]

and \( n \) being arbitrary positive integer, \( F(r, t) \) is given function. We shall construct the function \( u(x, y, z, t) = v(r, t) \) satisfying equation (1) in the domain

\[ \Omega = \{(x, y, z, t): x^2 + y^2 + z^2 > a^2, t > 0\}, \]

\( a \) being the positive number. We assume that the function \( v(r, t) \) satisfies the initial conditions

\[ D_t^k v(r, 0) = f_k(r) \quad (k = 0, 1, \ldots, n-1) \]

and boundary conditions

\[ D_n L^k v(a, t) + \gamma L^k v(a, t) = g_k(t) \quad (k = 0, 1, \ldots, n-1), \]

where

\[ L = D_r^2 + \frac{2}{r} D_r - D_t, \quad L^k = L(L^k - 1), \]

\( \gamma \) is an arbitrary positive number, \( D_n \) denote the outward normal derivative on the sphere \( x^2 + y^2 + z^2 = a^2 \). The functions \( f_k(p) \) and \( g_k(s) \) are given functions defined for \( p \geq a \) and \( s \geq 0 \) respectively. We shall call problem (1), (2), (3) in the domain \( \Omega \) the \((P-C-M)\) problem.

2. Let the function \( v(r, t) \) be the solution of the \((P-C-M)\) problem. It is easy to verify that the function \( v(r, t) \) satisfies the equation
By induction we can prove

**Lemma 1.** If \( w(r, t) = rv(r, t) \), then \( L^* v(r, t) = F(r, t) \), \( L^* = D_r^2 - D_t \), \( Q^n = Q(Q^{n-1}) \).

By Lemma 1 it follows that the function \( w(r, t) \) satisfies the equation

\[ Q^n w(r, t) = F(r, t). \]

Moreover, the function \( w(r, t) \) satisfies the limit conditions

\[ D_t^k w(r, 0) = r f_k(r), \quad r > a \quad (k = 0, 1, \ldots, n - 1), \]

\[ D_r^k Q^k w(a, t) + h Q^k w(a, t) = a g_k(t), \quad t > 0, \quad h = \gamma - 1/a \quad (k = 0, 1, \ldots, n - 1). \]

3. We shall solve the \((P-C-M)\) problem using an auxiliary function \( G(X; Y) \).

Let \( X = (r, t), \ Y = (p, s), X_1 = (2a-r, t), \bar{X}_1 = (2a-r-\tau, t), \tau \geq 0 \) and let

\[ \Omega_1 = \{(r, t): r > a, \ t > 0\}, \quad \Omega_2 = \{(r, t): r \geq a, \ t \geq 0\}. \]

The function

\[ U(X; Y) = \begin{cases} (t-s)^{-1/2} \exp \left(-\frac{(r-p)^2}{4(t-s)}\right) \quad \text{for } s < t, \\ 0 \quad \text{for } s \geq t, \ X \neq Y, \end{cases} \]

is the fundamental solution of the heat equation. Let

\[ W(X; Y) = 2h \int_0^\infty e^{ht} U(\bar{X}_1; Y) \, d\tau. \]

Let us consider the function

\[ G(r, t; p, s) = G(X; Y) = U(X; Y) + U(X_1; Y) + W(X; Y), \]

where \( X \in \Omega_1, \ Y \in \Omega_2, \ 0 \leq s < t. \)

Similarly as in paper [2] we can prove

**Theorem 1.** The function \( G(X; Y) \) given by formula (8) is the Green function for the heat equation, satisfying the boundary condition of the third kind

\[ D_p G(X; Y) + hG(X; Y) = 0 \quad \text{for } p = a, \ X \neq Y. \]

4. Let

\[ \varphi_j(p) = \frac{1}{j!} \frac{1}{p f_j(p)} - \frac{1}{(j-1)!} \frac{1}{p f_{j-1}(p)} (p f_{j-1}(p))^{(2)} + \ldots + (-1)^j \frac{1}{0!} \frac{1}{p f_0(p)^{(2j)}} \]

\( (j = 0, 1, \ldots, n-1) \) and let
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(9) \( w_{1,j}(r, t) = At^j \int_a^\infty \varphi_j(p) G(r, t; p, 0) dp \quad (j = 0, 1, \ldots, n-1), \quad A = \frac{1}{2\sqrt{\pi}} \),

(10) \( w_{2,j}(r, t) = A(-1)^{j+1} a \int_0^t g_j(s) \frac{(t-s)^j}{j!} G(r, t; a, s) ds \quad (j = 0, 1, \ldots, n-1), \)

(11) \( w_1(r, t) = \sum_{j=0}^{n-1} w_{1,j}(r, t), \)

(12) \( w_2(r, t) = \sum_{j=0}^{n-1} w_{2,j}(r, t), \)

(13) \( w_3(r, t) = (-1)^n A \int_0^t \int_a^\infty pF(p, s) G(r, t; p, s) dp ds, \)

(14) \( v_i(r, t) = r^{-1} w_i(r, t) \quad (i = 1, 2, 3). \)

Let us consider the functions

(15) \( w(r, t) = \sum_{i=1}^3 w_i(r, t) \)

and

(16) \( v(r, t) = \sum_{i=1}^3 v_i(r, t). \)

5. We shall prove that the function \( v(r, t) \) defined by formula (16) is the solution of the \((P - C - M)\) problem.

Let \( H \) denote the class of the function \( v(r, t) \) continuous with the derivatives \( D_1^\alpha D_2^\beta v(r, t), \alpha + 2\beta \leq 2n \) and satisfying equation (4) in the domain \( \Omega_1 \).

We shall prove

**Lemma 2.** If the functions \( f_j^{(k)}(p), k_j \leq 2(n-j-1), g_j(s) (j = 0, 1, \ldots, n-1) \) are bounded and measurable for \( p \geq a \) and \( s \geq 0 \) respectively, the functions \( F(p, s), D_1^\alpha D_2^\beta F(p, s) \) are continuous and bounded in the set \( \Omega_2 \), then the function \( v(r, t) \) defined by formula (16) belongs to the class \( H \).

**Proof.** Indeed, integrals (9), (10), (13) and the integrals

\[
J_{1,j}(r, t) = \int_a^\infty \varphi_j(p) D_1^\alpha D_2^\beta(t^j G(r, t; p, 0)) dp,
\]

\[
J_{2,j}(r, t) = \int_0^t g_j(s) D_1^\alpha D_2^\beta((t-s)^j G(r, t; a, s)) ds,
\]

\[
J_{3,j}(r, t) = \int_0^\infty \int_a^\infty pF(p, s) D_1^\alpha D_2^\beta G(r, t; p, s) dp ds
\]


(j = 0, 1, ..., n - 1), $\alpha + 2\beta \leq 2n$, are almost uniformly convergent in the domain $\Omega_1$. Consequently the function $v(r, t)$ is continuous with its derivatives $D^k_t D^2_r v(r, t)$ in the domain $\Omega_1$. By [2], Lemma 2, follows that the function $v(r, t)$ satisfies equation (4).

**Lemma 3.** If the functions $f_j^{(k)}(p)$ ($k_j \leq 2(n-j-1), j = 0, 1, ..., n-1$) are continuous and bounded for $p \geq a$, then the function $v_1(r, t)$ satisfies the initial conditions (2).

**Proof.** We have

$$v_1(r, t) = J_1(r, t) + J_2(r, t),$$

where

$$J_1(r, t) = r^{-1} A \sum_{j=0}^{n-1} t^j \int_a^\infty \phi_j(p) U(r, t; p, 0) dp,$$

$$J_2(r, t) = r^{-1} A \sum_{j=0}^{n-1} t^j \int_a^\infty \phi_j(p) (U(2a-r, t; p, 0) + W(r, t; p, 0)) dp.$$

By formulas (6), (9), (11), (14) and by [1], the function $J_1(r, t)$ satisfies the initial conditions (2). Moreover, if $r_0 > a$ and $(r, t) \to (r_0, 0^+)$, then $|r-r_0| > r_0 - a$ for the appropriate values of the variable $r$. Consequently by the singular estimations as in [3], p. 132, we obtain the inequality

$$|D^k J_2(r, t)| \leq C_1 t^{\mu_1} \quad (k = 0, 1, ..., n-1),$$

$C_1$ and $\mu_1$ being the positive constants. Consequently

$$\lim_{(r, t) \to (r_0, 0^+)} D^k J_2(r, t) = 0 \quad as \quad (r, t) \to (r_0, 0^+), \quad r_0 > a \quad (k = 0, 1, ..., n-1).$$

**Lemma 4.** If the functions $g_j(s)$ ($j = 0, 1, ..., n-1$) are bounded and measurable for $s \geq 0$, the functions $F(p, s), D_p F(p, s)$ are continuous and bounded in the set $\Omega_2$, then

$$\lim_{(r, t) \to (r_0, 0^+)} D^k v_i(r, t) = 0 \quad as \quad (r, t) \to (r_0, 0^+), \quad r_0 > a \quad (i = 2, 3; \quad k = 0, 1, ..., n-1).$$

**Proof.** By the estimations similar to those in [3], p. 132, we obtain

$$|D^k v_i(r, t)| \leq C_2 t^{\mu_2} \quad (i = 2, 3; \quad k = 0, 1, ..., n-1),$$

$C_2$ and $\mu_2$ being the appropriate positive constants. Consequently the functions $v_i(r, t)$ ($i = 2, 3$) satisfy the initial conditions (17).

**Lemma 5.** If the functions $g_j(s)$ ($j = 0, 1, ..., n-1$) are continuous and bounded for $s \geq 0$, then the functions $v_2(r, t)$ satisfy the boundary conditions (3).

**Proof.** Similarly as in [2] we obtain
\[ D, Q^k w_{2,j}(r, t) + h Q^k w_{2,j}(r, t) = \begin{cases} J_{j,k}(r, t) & \text{for } j \geq k, \\ 0 & \text{for } j < k, \end{cases} \]

where
\[
J_{j,k}(r, t) = A (-1)^{j+k+1} e^{t} \int_0^t g_j(s) \frac{(t-s)^{j-k-1} (a-r)}{(j-k)!} U(r, t; a, s) ds.
\]

If \( j = k \), then by [3], p. 127, we obtain
\[
\lim J_{j,k}(r, t) = a g_k(t_0) \quad \text{as } (r, t) \to (a^+, t_0).
\]

Moreover, if \( j > k \), then we obtain the estimation
\[
|J_{j,k}(r, t)| \leq C_3 (r-a) t^{\mu_3},
\]

where \( C_3 \) and \( \mu_3 \) are the positive constants. Hence
\[
\lim J_{j,k}(r, t) = 0 \quad \text{as } (r, t) \to (a^+, t_0), \quad j > k.
\]

Consequently the function \( w_2(r, t) \) satisfies the boundary conditions (7) and the function \( v_2(r, t) \) satisfies the boundary conditions (3).

**Lemma 6.** Let the functions \( f_j^{(k)}(p) \) \( (k_j \leq 2(n-j-1); \ j = 0, 1, \ldots, n-1) \) be bounded and measurable for \( p \geq a \), the function \( F(p, s) \) be bounded and measurable in the set \( \Omega_2 \); then
\[
\lim (D_r L^k v_i(r, t) + \gamma L^k v_i(r, t)) = 0 \quad \text{as } (r, t) \to (a^+, t_0)
\]
\[
(i = 1, 3; \ k = 0, 1, \ldots, n-1).
\]

**Proof.** Integrals (9) and the integrals \( J_{1,j}(r, t) \) are uniformly convergent in every set of the form
\[
\Omega_3 = \{(r, t): a \leq r \leq b, 0 < T_0 < t \leq T\},
\]

where \( b, T_0, T \) are the positive constants. Consequently we obtain
\[
D_r Q^k w_{1,j}(r, t) + h Q^k w_{1,j}(r, t) = \begin{cases} K_{j,k}(r, t) & \text{for } j \geq k, \\ 0 & \text{for } j < k, \end{cases}
\]

where
\[
K_{j,k}(r, t) = A (-1)^k \frac{t^{j-k}}{\Gamma(j-k)!} \int_a^\infty \varphi_j(p)(D_r h) G(r, t; p, 0) dp
\]
\[
(j, k = 0, 1, \ldots, n-1).
\]

By uniform convergence of the integrals \( J_{1,j}(r, t) \) the functions are continuous at the point \( (a, t_0) \), \( t_0 > 0 \). Consequently by the condition
\[
(D_r + h) G(r, t; p, 0) = 0 \quad \text{for } r = a,
\]

we obtain
lim K_{j,k}(r, t) = 0 \quad \text{as} \quad (r, t) \to (a^+, t_0), \quad t_0 > 0.

Hence

\lim (D_r L^2 v_1(r, t) + \gamma L^5 v_1(r, t)) = 0 \quad \text{as} \quad (r, t) \to (a^+, t_0).

Similarly we can prove that

\lim (D_r L^2 v_3(r, t) + \gamma L^5 v_3(r, t)) = 0 \quad \text{as} \quad (r, t) \to (a^+, t_0), \quad t_0 > 0

and we get assertion of Lemma 6.

Now we shall prove

**Theorem 2.** If the functions \( f_j^{(k_j)}(p), g_j(s), k_j \leq 2(n-j-1) \) are continuous and bounded for \( p \geq a \) and \( s \geq 0 \) respectively, the functions \( F(p, s), D_p F(p, s) \) are continuous and bounded in the set \( \Omega_2 \), then the function \( v(r, t) \) defined by formula (16) is the solution of the \((P-C-M)\) problem.

**Proof.** By Lemma 2 follows that the function \( v(r, t) \) belongs to the class \( H \). Moreover, by Lemmas 3, 4 follows that the function \( v(r, t) \) satisfies the initial conditions (2). From Lemmas 5, 6 follows that the function \( v(r, t) \) satisfies the boundary conditions (3).

**References**

