Domain of operator attraction of
a Gaussian measure in $\mathbb{R}^N$

Let $\{X_n\}$ be a sequence of $\mathbb{R}^N$-valued independent and identically
distributed random variables. Consider the sums

$$ (*) \quad A_n(X_1 + X_2 + \ldots + X_n) + b_n, $$

where $A_n$ are non-singular linear operators in $\mathbb{R}^N$, $b_n \in \mathbb{R}^N$. The paper aims
at characterizing a class of these distributions $p$ of random variables $X_i$
for which at a suitable choice of norming operators $A_n$ and vectors $b_n$
the sequence of distribution of random variables $(*)$ is weakly convergent
to some full Gaussian distribution in $\mathbb{R}^N$. The detailed description of the
class is given for the case of positive and diagonal operators $A_n$.

Let $\mathcal{M}$ be the set of all Borel probability measures on the real Euclidean
space $\mathbb{R}^N$. $\mathcal{M}$ being endowed with the topology of weak convergence of
measures and the convolution as an operation, it constitutes an Abelian
topological semigroup. Convolution of two distributions $p$ and $q$ will be
denoted as $p \ast q$. $\delta_x$ will be a probability measure concentrated at a point
$x \in \mathbb{R}^N$.

If $A$ is a Borel mapping of $\mathbb{R}^N$ into $\mathbb{R}^N$ and $p \in \mathcal{M}$, then $Ap$ is a measure
defined as follows

$$ Ap (E) = p(A^{-1} E) \quad \text{for every Borel set } E \text{ in } \mathbb{R}^N. $$

The characteristic function of a measure $p$ will be denoted by $\hat{p}$. Evidently

$$ Ap (y) = \hat{p}(A^* y) \quad \text{for every } y \in \mathbb{R}^N, $$

where $A^*$ denotes the adjoint operator.

A measure $p \in \mathcal{M}$ is said to be full if its support is not contained in
any $(N-1)$-dimensional hyperplane of $\mathbb{R}^N$. 
In [7] M. Sharpe introduced the notion of an operator-stable measure in $\mathbb{R}^N$. A measure $q$ is said to be operator-stable if it is a weak limit of a sequence of measures of the form

$$A_n p^\sigma \ast \delta_{b_n},$$

where $p \in \mathcal{M}$, $A_n \in G$, $b_n \in \mathbb{R}^N$. Here the symbol $G$ denotes the group of all non-singular linear operators in $\mathbb{R}^N$. The general form of characteristic functions of full operator-stable measures in $\mathbb{R}^N$ is given in [4].

A domain of operator attraction of an operator-stable measure $q$ is a class of these distributions $p$ for which there are sequences $A_n \in G$, $b_n \in \mathbb{R}^N$ such that the sequence of distributions (1) converges weakly to the distribution $q$.

In the paper we shall consider a class of distributions attracted in the operator sense by a full Gaussian measure in $\mathbb{R}^N$, i.e. by the measure whose characteristic function is of the form

$$\tilde{q}(y) = \exp \left[ i (x_0, y) - \frac{1}{2} (Sy, y) \right] \text{ for every } y \in \mathbb{R}^N,$$

where $x_0 \in \mathbb{R}^N, S$ is a linear self-adjoint positive operator acting in $\mathbb{R}^N$. $S$ is a dispersion operator of the distribution $q$.

**Lemma 1.** If $\lim_{n \to \infty} A_n p^\sigma \ast \delta_{b_n} = q$, where $A_n \in G$, $b_n \in \mathbb{R}^N$, $p, q \in \mathcal{M}$ and the distribution $q$ is full, then the distribution $p$ is also full.

The lemma is an immediate consequence of Lemma 2 in [3].

**Lemma 2.** Let $A_n \in G$, $b_n \in \mathbb{R}^N$, $p, q \in \mathcal{M}$ and $q$ be a full Gaussian distribution with a dispersion operator $S$. The condition

$$\lim_{n \to \infty} A_n p^\sigma \ast \delta_{b_n} = q$$

is equivalent to the following

(a) $\lim_{n \to \infty} n \int_{\|x\| \geq \varepsilon} A_n p(dx) = 0$ for every $\varepsilon > 0$,

(b) $\lim_{n \to \infty} \left\{ \int_{\|x\| \leq \varepsilon} (x, y)^2 A_n p(dx) - \left[ \int_{\|x\| \leq \varepsilon} (x, y) A_n p(dx) \right]^2 \right\} = (Sy, y)$ for every $\varepsilon > 0, y \in \mathbb{R}^N$.

**Proof.** By Lemma 1 in [2] condition (2) implies $A_n \to \theta$ which means that the distributions $A_n p$ are uniformly asymptotically negligible. On the other hand the above property of the distributions $A_n p$ follows from (a). Thus we can make use of Theorem 6.3, p. 200, in [6], and all we have to do is to show that (a) and (b) are equivalent to

(a$_1$) $\lim_{n \to \infty} \int_{\|x - x_n\| \geq \varepsilon} A_n p(dx) = 0$ for every $\varepsilon > 0$,

(b$_1$) $\lim_{n \to \infty} \int_{\|x\| \leq \varepsilon} (x - x_n, y)^2 A_n p(dx) = (Sy, y)$ for every $\varepsilon > 0, y \in \mathbb{R}^N$,.
where
\[(x_n, y) = \int_{\|x\| \leq 1} (x, y) A_n p(dx).\]

Uniformly asymptotic negligibility of the distributions $A_n p$ guarantees that
\[
\lim_{n \to \infty} \|x_n\| = 0.
\]

Thus (a) and (b) are equivalent. Simultaneously, for $\varepsilon \in (0, 1)$
\[
n \int_{\|x\| \leq \varepsilon} (x - x_n, y)^2 A_n p(dx) - \int_{\|x\| \leq \varepsilon} (x, y)^2 A_n p(dx) + \int_{\|x\| \geq \varepsilon} (x_n, y)^2 A_n p(dx) \leq \|y\| n \int_{\|x\| \geq \varepsilon} A_n p(dx) \int_{\|x\| \geq \varepsilon} A_n p(dx) + \int_{\|x\| \leq \varepsilon} A_n p(dx).
\]

Thus if (a) is satisfied, then (b) and (b_1) are equivalent.

The symbol $p^y$ will denote a distribution induced by an element $y \in \mathbb{R}^N$, i.e.
\[(5) \quad p^y(Z) = p\{x \in \mathbb{R}^N : (x, y) \in Z\} \quad \text{for every Borel set } Z \text{ in } \mathbb{R}^1.
\]

**Lemma 3.** If a distribution $p \in \mathfrak{M}$ is operator-attracted by a full Gaussian distribution $q \in \mathfrak{M}$, then for every $0 \neq y \in \mathbb{R}^N$, the distribution $p^y$ is attracted by a non-degenerate Gaussian distribution in $\mathbb{R}^1$.

**Proof.** Let \(\lim_{n \to \infty} A_n p^\ast \ast \delta_{b_n} = q\). In terms of characteristic functions we have
\[(6) \quad \lim_{n \to \infty} \left[ \hat{p}(A_n^\ast y) \right]^n e^{i(b_n \cdot y)} = e^{-i(Sy, y)} \quad \text{for every } y \in \mathbb{R}^N.
\]

Let $0 \neq y_0 \in \mathbb{R}^N$. Since $A_n^\ast \in G$, for every $n$ there is $0 \neq y_n \in \mathbb{R}^N$ such that
\[A_n^\ast(y_n) = y_0, \quad \text{i.e.}\]
\[\|y_n\| A_n^\ast \left( \frac{y_n}{\|y_n\|} \right) = y_0.\]

Let now $z_0 \neq 0$ be the limit point of the sequence $\frac{y_n}{\|y_n\|}$, i.e. $z_0 = \lim_{k \to \infty} \frac{y_{nk}}{\|y_{nk}\|}$ and let $t$ be an arbitrary fixed real number. The convergence in (6) is uniform in each bounded set and thus
\[
\lim_{k \to \infty} \left\{ \hat{p} \left[ A_{nk}^\ast \left( \frac{y_{nk}}{\|y_{nk}\|} \right) t \right] \right\}^{n_k} e^{i(b_{nk} \cdot y_{nk})} = e^{-i(Sz_0, z_0)t^2},
\]
\[\text{i.e.}\]
\[
\lim_{k \to \infty} \left[ \hat{p} \left( \frac{1}{\|y_{nk}\|} y_0 \cdot t \right) \right]^{n_k} e^{iSz_{nk}} = e^{-i(Sz_0, z_0)t^2}.
\]

We can take operators $A_n$ such that $S = I$, where $I$ is the identity operator.
in $R^N$. Then we have $(Sz_0, z_0) = (z_0, z_0) = 1$. Thus the distribution $p^{y_0}$ is attracted by a non-degenerate normal distribution in $R^1$, and the numbers $\|y_n\|^{-1}$ form a sequence of norming constants.

Remark 1. The domain of operator attraction of a full Gaussian distribution in $R^N$ coincides with the domain of operator attraction of a Gaussian distribution with the identity dispersion operator $I$.

Lemma 4. If a distribution $p$ is operator-attracted by a full Gaussian distribution in $R^N$, then there are a sequence $D_n$ of positive diagonal operators, a sequence $U_n$ of orthogonal operators and a sequence $b_n \in R^N$ such that

$$\lim_{n \to \infty} D_n U_n p^{y_n} \ast \delta_{b_n} = q,$$

where $q$ is a Gaussian distribution with the identity dispersion operator.

Proof. By the assumption we have

$$\lim_{n \to \infty} A_n p^{y_n} \ast \delta_{c_n} = q, \quad \text{where } A_n \in G, \ c_n \in R^N.$$

In view of Remark 1 we may assume that the distribution $q$ has the dispersion operator $I$.

The operator $A_n \in G$ may be written in the form $A_n = V_n B_n$, where $V_n$ is an orthogonal operator and $B_n$ is a positive self-adjoint one. On the other hand $B_n = U_n^{-1} D_n U_n$, where $D_n$ is a positive diagonal operator and $U_n$ is an orthogonal one. The sequence of operators $U_n V_n^{-1}$ is compact. Thus if

$$\lim_{k \to \infty} U_{nk} V_{nk}^{-1} = U_0,$$

then

$$\lim_{k \to \infty} D_{nk} U_{nk} p^{y_{nk}} \ast \delta_{b_{nk}} = U_0 q = q, \quad \text{where } b_n = U_n V_n^{-1} c_n.$$

Thus the sequence of distributions $D_n U_n p^{y_n} \ast \delta_{b_n}$ is compact and all its convergent subsequences have the same limit $q$.

Let us assign to a full distribution $p \in \mathcal{M}$ and to an arbitrarily fixed basis $[e_1, \ldots, e_N]$ in $R^N$ the following correlation matrices with elements of the form

(8) $R_{ij}(a) = \int_{M(a)} x_i x_j p(dx) - \int_{M(a)} x_i p(dx) \int_{M(a)} x_j p(dx) / \int_{M(a)} x_i^2 p(dx) - \left[ \int_{M(a)} x_i p(dx) \right]^2 / 2$,  

where $0 \neq a \in R^N$, $M(a) = \{x \in R^N: \sum_{i=1}^N a_i^2 x_i^2 \leq 1\}$, $x_i = (x, e_i)$, $a_i = (a, e_i)$ for $i = 1, 2, \ldots, N$, and functions of the form

(9) $g_i(z) = \alpha^2 \left[ \int_{x|x| \leq 1} x^2 p^{x_i}(dx) - \left[ \int_{x|x| \leq 1} x p^{x_i}(dx) \right]^2 \right]$,  

$\alpha > 0$, $i = 1, 2, \ldots, N$. 


Remark 2. If the distribution \( p^{e_i} \) is attracted by a non-degenerate Gaussian distribution in \( \mathbb{R}^N \), then

\[
\lim_{x \to 0} g_i(x) = 0.
\]

Theorem. The full distribution \( p \in \mathcal{M} \) is operator-attracted by a full Gaussian distribution in \( \mathbb{R}^N \) with a sequence of positive diagonal in a fixed basis norming operators if and only if for some basis \([e_1, \ldots, e_n]\) in \( \mathbb{R}^N \) we have

\[
\lim_{x \to +\infty} \frac{X^2 p \{ x \in \mathbb{R}^N : |x_i| \geq X \}}{\int_{|x_i| \leq X} x_i^2 p(dx)} \quad \text{for } i = 1, 2, \ldots, N,
\]

i.e. the distributions \( p^{e_i}, i = 1, 2, \ldots, N, \) are attracted by a non-degenerate distribution in \( \mathbb{R}^1 \) (see [1], Theorem 1, §34);

\( 2^\circ \) for \( i, j = 1, 2, \ldots, N, i < j, \) there exists a limit of the function \( R_{ij}(a) \) as \( a \) tends to zero, so that

\[
\lim_{a \to 0} \frac{g_1(a_1)}{g_j(a_j)} = 1 \quad \text{for } j = 2, 3, \ldots, N.
\]

(See (8) and (9).)

Proof. Necessity. Let \( \lim_{n \to \infty} D_n p^{n*, \delta_{b_n}} = q \) and let \([e_1, \ldots, e_N]\) be eigenvectors and \([\lambda_n^1, \ldots, \lambda_n^N]\) eigenvalues of the positive diagonal operators \( D_n \).

Condition \( 1^\circ \) follows directly from Lemma 3.

The distribution \( q \) is a Gaussian one with an arbitrary dispersion operator \( S \). Notice that without any loss of generality we may assume that \((Se_i, e_i) = 1, i = 1, 2, \ldots, N.\) To this aim it suffices to consider a sequence of norming operators of the form \( D_0 D_n \), where \( D_0 \) is a diagonal with the eigenvalues \( 1/\sqrt{(Se_i, e_i)}, i = 1, 2, \ldots, N. \) From Lemma 2 for any \( \varepsilon > 0 \) we have

\[
\lim_{n \to \infty} n \int_{||D_n x|| \geq \varepsilon} p(dx) = 0,
\]

\[
\lim_{n \to \infty} n (\lambda_n^i)^2 \int_{||D_n x|| \leq \varepsilon} x_i^2 p(dx) - \int_{||D_n x|| \leq \varepsilon} x_i p(dx)^2 = 1,
\]

\[
\lim_{n \to \infty} n \lambda_n^i \lambda_n^j \int_{||D_n x|| \leq \varepsilon} x_i x_j p(dx) - \int_{||D_n x|| \leq \varepsilon} x_i p(dx) \int_{||D_n x|| \leq \varepsilon} x_j p(dx) = (Se_i, e_j).
\]

Since \( ||D_n x|| = \sum_{i=1}^N (\lambda_n^i)^2 x_i^2 \), therefore using (12) and (13) we obtain

\[
\lim_{n \to \infty} R_{ij}(\lambda_n) = (Se_i, e_j), \quad \text{where } \lambda_n = \sum_{i=1}^N \lambda_n^i e_i.
\]
Since
\[
\lim_{n \to \infty} \left[ \hat{\beta} (e_t) \right]^n e^{it(b_{n, e_i})} = e^{-|t|^2} \quad \text{for every } t \in \mathbb{R}^1
\]
and \( \lim_{n \to \infty} \lambda_n^i = 0, \ i = 1, 2, \ldots, N, \) we have by Theorem 2, § 10, in [1],
\[
\lim_{n \to \infty} \frac{\lambda_n^i}{\lambda_{n+1}^i} = 1, \quad i = 1, 2, \ldots, N.
\]
Let \( \{a_n^1\}_{n=1}^\infty \) be an arbitrary sequence tending to zero. From (16) it follows that there is a subsequence \( m(n) \) such that
\[
\lim_{n \to \infty} \frac{a_n^1}{\lambda_{m(n)}^1} = 1.
\]
To show condition 2°, the sequence \( \{a_n^1\}_{n=1}^\infty \) need to be chosen so that
\[
\lim_{n \to \infty} \frac{g_1(a_n^1)}{g_j(a_n^1)} = 1, \quad j = 2, 3, \ldots, N.
\]
From (15) follows (see Theorem 2, § 26 in [1])
\[
\lim_{n \to \infty} \frac{g_1(\lambda_n^1)}{g_j(\lambda_n^1)} = 1, \quad j = 2, 3, \ldots, N.
\]
Thus in view of (19), (17) and (18)
\[
\lim_{n \to \infty} \frac{g_j(\lambda_{m(n)}^1)}{g_j(a_n^1)} = \lim_{n \to \infty} \frac{g_j(\lambda_{m(n)}^1)}{g_1(\lambda_{m(n)}^1)} \cdot \frac{g_1(\lambda_{m(n)}^1)}{g_1(a_n^1)} \cdot \frac{g_1(a_n^1)}{g_j(a_n^1)} = 1.
\]
Since \( \lim_{n \to \infty} m(n) g_j(\lambda_{m(n)}^1) = 1 \) we have
\[
\lim_{n \to \infty} m(n) g_j(a_n^1) = 1.
\]
Condition 1° and (20) imply
\[
\lim_{n \to \infty} m(n) \int_{a_n^1 \leq x \leq e} p(dx) = 0.
\]
Thus for some sequence \( \alpha_n \in \mathbb{R}^1 \) we have
\[
\lim_{n \to \infty} \left[ \hat{\beta} (a_n^1 e_t) \right]^{m(n)} e^{it} = e^{-|t|^2} \quad \text{for every } t \in \mathbb{R}^1.
\]
In view of (15) and (22)
\[
\lim_{n \to \infty} \frac{\lambda_{m(n)}^1}{a_n^1} = 1 \quad \text{for } j = 2, 3, \ldots, N.
\]
Finally by (14) and (23)
Gaussian measure

$$\lim_{n \to \infty} R_{ij}(a_n) = (Se_i, e_j), \text{ where } a_n = \sum_{i=1}^{N} a_n^i e_i.$$ 

**Sufficiency.** From condition 1° it follows that there are sequence of positive numbers \(\{a_n^i\}_{n=1}^\infty, \quad i = 1, \ldots, N\), and sequence \(b_n \in R^N\) such that

$$\lim_{n \to \infty} \hat{p}(a_n^i e_j t) e^{it(b_n,e_j)} = e^{-|t|^2} \quad \text{for every } t \in R^1.$$ 

Consider the sequence of distributions

$$A_n p^e * \delta_{b_n},$$

where \(A_n\)'s are diagonal operators such that \(A_n e_j = a_n^i e_j\). Condition (24) means that sequences of one-dimensional boundary distributions of the sequence (25) induced by the elements of the basis are weakly convergent to Gaussian distributions in \(R^1\). Hence it follows that the sequence of distribution (25) is compact and that each subsequence which is convergent converges to a Gaussian distribution in \(R^N\). Thus let \(S_1\) and \(S_2\) be two dispersion operators of limit Gaussian distributions.

By (24) we have \((S_1 e_i, e_j) = (S_2 e_i, e_j) = 1, \quad i = 1, 2, \ldots, N\), which further yields the existence of the limit

$$\lim_{n \to \infty} n(a_n^i)^2 \left\{ \int_{||A_n x|| \leq \varepsilon} x_i^2 p(dx) - \int_{||A_n x|| \leq \varepsilon} x_i p(dx) \right\} = 1.$$

By (24) the sequences \(\{a_n^i\}_{n=1}^\infty, \quad i = 1, 2, \ldots, N\), satisfy the condition

$$\lim_{n \to \infty} \frac{g_1(a_n^i)}{g_j(a_n^i)} = 1.$$ 

Thus condition 2° along with (27) and (26) guarantee the existence of the limit

$$\lim_{n \to \infty} n(a_n^i)^2 \left[ \int_{||A_n x|| \leq \varepsilon} x_i x_j p(dx) - \int_{||A_n x|| \leq \varepsilon} x_i p(dx) \right]$$

for every \(\varepsilon > 0\). Thus \((S_1 e_i, e_j) = (S_2 e_i, e_j)\) for \(i, j = 1, 2, \ldots, N\), i.e. \(S_1 = S_2\).

**Corollary.** A full distribution \(p \in \mathcal{M}\) is attracted in the ordinary sense by a full Gaussian distribution in \(R^N\) if and only if for some basis \([e_1, \ldots, e_N]\) in \(R^N\)

1. \(\lim_{x \to +\infty} \frac{X^2 p \{x \in R^N: |x_j| \geq X\}}{\int_{|x_j| \leq X} x_j^2 p(dx)} = 0 \quad \text{for } j = 1, 2, \ldots, N;\)
2. there exists

$$\lim_{x \to +\infty} \frac{\int_{||x|| \leq X} x_i x_j p(dx) - \int_{||x|| \leq X} x_i p(dx) \int_{||x|| \leq X} x_j p(dx)}{\left[ \int_{||x|| \leq X} x_i^2 p(dx) - \left( \int_{||x|| \leq X} x_i p(dx) \right)^2 \right]^{1/2} \left[ \int_{||x|| \leq X} x_j^2 p(dx) - \left( \int_{||x|| \leq X} x_j p(dx) \right)^2 \right]^{1/2}}.$$
for $i, j = 1, \ldots, N, \ i < j$;

(iii) there exists

$$
\lim_{X \to \infty} \frac{\int_{|x| \leq X} x^2 p_{ei}^1 (dx) - \left[ \int_{|x| \leq X} x p_{ei}^1 (dx) \right]^2}{\int_{|x| \leq X} x^2 p_{ei}^1 (dx) - \left[ \int_{|x| \leq X} x p_{ei}^1 (dx) \right]^2} \neq 0
$$

for $i = 2, 3, \ldots, N$.

Theorem in the paper describes the domain of operator attraction of a full Gaussian distribution provided that the sequence of norming operators is a sequence of positive diagonal operators. As seen from Lemma 4, the general case reduces to considering a sequence of norming operators of the form $D_n U_n$, where $D_n$ are positive diagonal operators and $U_n$ are orthogonal ones. Unfortunately, this case is still the problem to be solved.

References


