ANNALES SOCIETATIS MATHEMATICAE POLONAE
Series I: COMMENTATIONES MATHEMATICAE XXII (1980)
ROCZNIKI POLSKIEGO TOWARZYSTWA MATEMATYCZNEGO
Seria I: PRACE MATEMATYCZNE XXII (1980)

M. Kłosowska (Łódź)

Domain of operator attraction of a Gaussian measure in R^N

Let $\{X_n\}$ be a sequence of R^N-valued independent and identically distributed random variables. Consider the sums

$$(*) \qquad A_n(X_1+X_2+\ldots+X_n)+b_n,$$

where A_n are non-singular linear operators in R^N, $b_n \in R^N$. The paper aims at characterizing a class of these distributions p of random variables X_i for which at a suitable choice of norming operators A_n and vectors b_n the sequence of distribution of random variables $(*)$ is weakly convergent to some full Gaussian distribution in R^N. The detailed description of the class is given for the case of positive and diagonal operators A_n.

Let $\mathfrak{M}$ be the set of all Borel probability measures on the real Euclidean space R^N. $\mathfrak{M}$ being endowed with the topology of weak convergence of measures and the convolution as an operation, it constitues an Abelian topological semigroup. Convolution of two distributions p and q will be denoted as $p*q$. δ_x will be a probability measure concentrated at a point $x \in R^N$.

If A is a Borel mapping of R^N into R^N and $p \in \mathfrak{M}$, then Ap is a measure defined as follows

$$Ap(E) = p(A^{-1}E) \quad \text{for every Borel set } E \text{ in } R^N.$$

The characteristic function of a measure p will be denoted by $\hat{p}$. Evidently

$$A\hat{p}(y) = \hat{p}(A^*y) \quad \text{for every } y \in R^N,$$

where A^* denotes the adjoint operator.

A measure $p \in \mathfrak{M}$ is said to be *full* if its support is not contained in any $(N-1)$-dimensional hyperplane of R^N.

In [7] M. Sharpe introduced the notion of an operator-stable measure in R^N. A measure q is said to be *operator-stable* if it is a weak limit of a sequence of measures of the form

$$A_n p^{n*} * \delta_{b_n}, \tag{1}$$

where $p \in \mathfrak{M}$, $A_n \in G$, $b_n \in R^N$. Here the symbol G denotes the group of all non-singular linear operators in R^N. The general form of characteristic functions of full operator-stable measures in R^N is given in [4].

A domain of operator attraction of an operator-stable measure q is a class of these distributions p for which there are sequences $A_n \in G$, $b_n \in R^N$ such that the sequence of distributions (1) converges weakly to the distribution q.

In the paper we shall consider a class of distributions attracted in the operator sense by a full Gaussian measure in R^N, i.e. by the measure whose characteristic function is of the form

$$\hat{q}(y) = \exp[i(x_0, y) - \tfrac{1}{2}(Sy, y)] \quad \text{for every } y \in R^N,$$

where $x_0 \in R^N$, S is a linear self-adjoint positive operator acting in R^N. S is a dispersion operator of the distribution q.

LEMMA 1. *If* $\lim_{n\to\infty} A_n p^{n*} * \delta_{b_n} = q$, *where* $A_n \in G$, $b_n \in R^N$, $p, q \in \mathfrak{M}$ *and the distribution* q *is full, then the distribution* p *is also full.*

The lemma is an immediate consequence of Lemma 2 in [3].

LEMMA 2. *Let* $A_n \in G$, $b_n \in R^N$, $p, q \in \mathfrak{M}$ *and* q *be a full Gaussian distribution with a dispersion operator* S. *The condition*

$$\lim_{n\to\infty} A_n p^{n*} * \delta_{b_n} = q \tag{2}$$

is equivalent to the following

(a) $$\lim_{n\to\infty} n \int_{\|x\| \geq \varepsilon} A_n p(dx) = 0 \quad \textit{for every } \varepsilon > 0,$$

(b) $$\lim_{n\to\infty} n \Big\{ \int_{\|x\| \leq \varepsilon} (x, y)^2 A_n p(dx) - \Big[\int_{\|x\| \leq \varepsilon} (x, y) A_n p(dx) \Big]^2 \Big\} = (Sy, y)$$

for every $\varepsilon > 0$, $y \in R^N$.

Proof. By Lemma 1 in [2] condition (2) implies $A_n \to \theta$ which means that the distributions $A_n p$ are uniformly asymptotically negligible. On the other hand the above property of the distributions $A_n p$ follows from (a). Thus we can make use of Theorem 6.3, p. 200, in [6], and all we have to do is to show that (a) and (b) are equivalent to

(a_1) $$\lim_{n\to\infty} \int_{\|x - x_n\| \geq \varepsilon} A_n p(dx) = 0 \quad \text{for every } \varepsilon > 0,$$

(b_1) $$\lim_{n\to\infty} n \int_{\|x\| \leq \varepsilon} (x - x_n, y)^2 A_n p(dx) = (Sy, y) \quad \text{for every } \varepsilon > 0, y \in R^N,$$

where

(3) $$(x_n, y) = \int_{\|x\| \leqslant 1} (x, y) A_n p(dx).$$

Uniformly asymptotic negligibility of the distributions $A_n p$ guarantees that

(4) $$\lim_{n \to \infty} \|x_n\| = 0.$$

Thus (a) and (b) are equivalent. Simultaneously, for $\varepsilon \in (0, 1)$

$$n \Big| \int_{\|x\| \leqslant \varepsilon} (x - x_n, y)^2 A_n p(dx) - \int_{\|x\| \leqslant \varepsilon} (x, y)^2 A_n p(dx) + \Big[\int_{\|x\| \leqslant \varepsilon} (x, y) A_n p(dx) \Big]^2 \Big|$$
$$\leqslant \|y\| n \int_{\|x\| \geqslant \varepsilon} A_n p(dx) \int_{\|x\| \geqslant \varepsilon} A_n p(dx) + |(x_n, y)|^2 n \int_{\|x\| \geqslant \varepsilon} A_n p(dx).$$

Thus if (a) is satisfied, then (b) and (b_1) are equivalent.

The symbol p^y will denote a distribution induced by an element $y \in R^N$, i.e.

(5) $$p^y(Z) = p\{x \in R^N : (x, y) \in Z\} \quad \text{for every Borel set } Z \text{ in } R^1.$$

LEMMA 3. *If a distribution $p \in \mathfrak{M}$ is operator-attracted by a full Gaussian distribution $q \in \mathfrak{M}$, then for every $0 \neq y \in R^N$, the distribution p^y is attracted by a non-degenerate Gaussian distribution in R^1.*

Proof. Let $\lim_{n \to \infty} A_n p^{n*} * \delta_{b_n} = q$. In terms of characteristic functions we have

(6) $$\lim_{n \to \infty} [\hat{p}(A_n^* y)]^n e^{i(b_n, y)} = e^{-\frac{1}{2}(Sy, y)} \quad \text{for every } y \in R^N.$$

Let $0 \neq y_0 \in R^N$. Since $A_n^* \in G$, for every n there is $0 \neq y_n \in R^N$ such that $A_n^*(y_n) = y_0$, i.e.

$$\|y_n\| A_n^* \left(\frac{y_n}{\|y_n\|} \right) = y_0.$$

Let now $z_0 \neq 0$ be the limit point of the sequence $\frac{y_n}{\|y_n\|}$, i.e. $z_0 = \lim_{k \to \infty} \frac{y_{n_k}}{\|y_{n_k}\|}$ and let t be an arbitrary fixed real number. The convergence in (6) is uniform in each bounded set and thus

$$\lim_{k \to \infty} \left\{ \hat{p} \left[A_{n_k}^* \left(\frac{y_{n_k}}{\|y_{n_k}\|} \right) t \right] \right\}^{n_k} e^{it\left(b_{n_k}, \frac{y_{n_k}}{\|y_{n_k}\|}\right)} = e^{-\frac{1}{2}(Sz_0, z_0)t^2},$$

i.e.

$$\lim_{k \to \infty} \left[\hat{p} \left(\frac{1}{\|y_{n_k}\|} y_0 \cdot t \right) \right]^{n_k} e^{it\alpha_{n_k}} = e^{-\frac{1}{2}(Sz_0, z_0)t^2}.$$

We can take operators A_n such that $S = I$, where I is the identity operator in R^N. Then we have $(Sz_0, z_0) = (z_0, z_0) = 1$. Thus the distribution p^{y_0} is attracted by a non-degenerate normal distribution in R^1, and the numbers $\|y_n\|^{-1}$ form a sequence of norming constants.

Remark 1. The domain of operator attraction of a full Gaussian distribution in R^N coincides with the domain of operator attraction of a Gaussian distribution with the identity dispersion operator I.

LEMMA 4. *If a distribution p is operator-attracted by a full Gaussian distribution in R^N, then there are a sequence D_n of positive diagonal operators, a sequence U_n of orthogonal operators and a sequence $b_n \in R^N$ such that*

$$\lim_{n\to\infty} D_n U_n p^{n*} * \delta_{b_n} = q,$$

where q is a Gaussian distribution with the identity disporsion operator.

Proof. By the assumption we have

$$\lim_{n\to\infty} A_n p^{n*} * \delta_{c_n} = q, \quad \text{where } A_n \in G,\ c_n \in R^N. \tag{7}$$

In view of Remark 1 we may assume that the distribution q has the dispersion operator I.

The operator $A_n \in G$ may be written in the form $A_n = V_n B_n$, where V_n is an orthogonal operator and B_n is a positive self-adjoint one. On the other hand $B_n = U_n^{-1} D_n U_n$, where D_n is a positive diagonal operator and U_n is an orthogonal one. The sequence of operators $U_n V_n^{-1}$ is compact. Thus if $\lim_{k\to\infty} U_{n_k} V_{n_k}^{-1} = U_0$, then

$$\lim_{k\to\infty} D_{n_k} U_{n_k} p^{n_k^*} * \delta_{b_{n_k}} = U_0 q = q, \quad \text{where } b_n = U_n V_n^{-1} c_n.$$

Thus the sequence of distributions $D_n U_n p^{n*} * \delta_{b_n}$ is compact and all its convergent subsequences have the same limit q.

Let us assign to a full distribution $p \in \mathfrak{M}$ and to an arbitrarily fixed basis $[e_1, \ldots, e_N]$ in R^N the following correlation matrices with elements of the form

$$R_{ij}(a) = \frac{\int_{M(a)} x_i x_j p(dx) - \int_{M(a)} x_i p(dx) \int_{M(a)} x_j p(dx)}{\left\{\int_{M(a)} x_i^2 p(dx) - \left[\int_{M(a)} x_i p(dx)\right]^2\right\}^{1/2} \left\{\int_{M(a)} x_j^2 p(dx) - \left[\int_{M(a)} x_j p(dx)\right]^2\right\}^{1/2}}, \tag{8}$$

where $0 \neq a \in R^N$, $M(a) = \{x \in R^N\colon \sum_{i=1}^{N} a_i^2 x_i^2 \leqslant 1\}$, $x_i = (x, e_i)$, $a_i = (a, e_i)$ for $i = 1, 2, \ldots, N$, and functions of the form

$$g_i(\alpha) = \alpha^2 \left\{ \int_{\alpha|x| \leqslant 1} x^2 p^{e_i}(dx) - \left[\int_{\alpha|x| \leqslant 1} x p^{e_i}(dx)\right]^2 \right\}, \tag{9}$$

$$\alpha > 0,\ i = 1, 2, \ldots, N.$$

Remark 2. If the distribution p^{e_i} is attracted by a non-degenerate Gaussian distribution in R^N, then

$$\lim_{\alpha\to 0} g_i(\alpha) = 0. \tag{10}$$

THEOREM. *The full distribution $p \in \mathfrak{M}$ is operator-attracted by a full Gaussian distribution in R^N with a sequence of positive diagonal in a fixed basis norming operators if and only if for some basis $[e_1, \ldots, e_n]$ in R^N we have*

1° $$\lim_{X\to+\infty} \frac{X^2 p\{x \in R^N\colon |x_i| \geqslant X\}}{\int_{|x_i|\leqslant X} x_i^2 \, p(dx)} \quad \textit{for } i = 1, 2, \ldots, N,$$

i.e. the distributions p^{e_i}, $i = 1, 2, \ldots, N$, are attracted by a non-degenerate distribution in R^1 (see [1], Theorem 1, § 34);

2° *for $i, j = 1, 2, \ldots, N, i < j$, there exists a limit of the function $R_{ij}(a)$ as a tends to zero, so that*

$$\lim_{\substack{a_1\to 0\\ a_j\to 0}} \frac{g_1(a_1)}{g_j(a_j)} = 1 \quad \textit{for } j = 2, 3, \ldots, N.$$

(See (8) and (9).)

Proof. *Necessity.* Let $\lim_{n\to\infty} D_n p^{n^*} * \delta_{b_n} = q$ and let $[e_1, \ldots, e_N]$ be eigenvectors and $[\lambda_n^1, \ldots, \lambda_n^N]$ eigenvalues of the positive diagonal operators D_n.

Condition 1° follows directly from Lemma 3.

The distribution q is a Gaussian one with an arbitrary dispersion operator S. Notice that without any loss of generality we may assume that $(Se_i, e_i) = 1$, $i = 1, 2, \ldots, N$. To this aim it suffices to consider a sequence of norming operators of the form $D_0 D_n$, where D_0 is a diagonal with the eigenvalues $1/\sqrt{(Se_i, e_i)}$, $i = 1, 2, \ldots, N$. From Lemma 2 for any $\varepsilon > 0$ we have

$$\lim_{n\to\infty} n \int_{\|D_n x\|\geqslant\varepsilon} p(dx) = 0, \tag{11}$$

$$\lim_{n\to\infty} n(\lambda_n^i)^2 \Big\{ \int_{\|D_n x\|\leqslant\varepsilon} x_i^2 \, p(dx) - \Big[\int_{\|D_n x\|\leqslant\varepsilon} x_i \, p(dx)\Big]^2 \Big\} = 1, \tag{12}$$

$$\lim_{n\to\infty} n\lambda_n^i \lambda_n^j \Big[\int_{\|D_n x\|\leqslant\varepsilon} x_i x_j \, p(dx) - \int_{\|D_n x\|\leqslant\varepsilon} x_i \, p(dx) \int_{\|D_n x\|\leqslant\varepsilon} x_j \, p(dx)\Big] = (Se_i, e_j). \tag{13}$$

Since $\|D_n x\| = \sum_{i=1}^{N} (\lambda_n^i)^2 x_i^2$, therefore using (12) and (13) we obtain

$$\lim_{n\to\infty} R_{ij}(\lambda_n) = (Se_i, e_j), \quad \text{where } \lambda_n = \sum_{i=1}^{N} \lambda_n^i e_i. \tag{14}$$

Since

(15) $$\lim_{n\to\infty} [\hat{p}(\lambda_n^i e_i t)]^n \cdot e^{it(b_n, e_i)} = e^{-\frac{1}{2}t^2} \quad \text{for every } t \in R^1$$

and $\lim_{n\to\infty} \lambda_n^i = 0$, $i = 1, 2, \ldots, N$, we have by Theorem 2, § 10, in [1],

(16) $$\lim_{n\to\infty} \frac{\lambda_n^i}{\lambda_{n+1}^i} = 1, \quad i = 1, 2, \ldots, N.$$

Let $\{a_n^1\}_{n=1}^{\infty}$ be an arbitrary sequence tending to zero. From (16) it follows that there is a subsequence $m(n)$ such that

(17) $$\lim_{n\to\infty} \frac{a_n^1}{\lambda_{m(n)}^1} = 1.$$

To show condition 2°, the sequence $\{a_n^j\}_{n=1}^{\infty}$ need to be chosen so that

(18) $$\lim_{n\to\infty} \frac{g_1(a_n^1)}{g_j(a_n^j)} = 1, \quad j = 2, 3, \ldots, N.$$

From (15) follows (see Theorem 2, § 26 in [1])

(19) $$\lim_{n\to\infty} \frac{g_1(\lambda_n^1)}{g_j(\lambda_n^j)} = 1, \quad j = 2, 3, \ldots, N.$$

Thus in view of (19), (17) and (18)

$$\lim_{n\to\infty} \frac{g_j(\lambda_{m(n)}^j)}{g_j(a_n^j)} = \lim_{n\to\infty} \frac{g_j(\lambda_{m(n)}^j)}{g_1(\lambda_{m(n)}^1)} \cdot \frac{g_1(\lambda_{m(n)}^1)}{g_1(a_n^1)} \cdot \frac{g_1(a_n^1)}{g_j(a_n^j)} = 1.$$

Since $\lim_{n\to\infty} m(n) g_j(\lambda_{m(n)}^j) = 1$ we have

(20) $$\lim_{n\to\infty} m(n) g_j(a_n^j) = 1.$$

Condition 1° and (20) imply

(21) $$\lim_{n\to\infty} m(n) \int_{a_n^j |x_j| \geq \varepsilon} p(dx) = 0.$$

Thus for some sequence $\alpha_n \in R^1$ we have

(22) $$\lim_{n\to\infty} [\hat{p}(a_n^j e_j t)]^{m(n)} e^{it\alpha_n} = e^{-\frac{1}{2}t^2} \quad \text{for every } t \in R^1.$$

In view of (15) and (22)

(23) $$\lim_{n\to\infty} \frac{\lambda_{m(n)}^j}{a_n^j} = 1 \quad \text{for } j = 2, 3, \ldots, N.$$

Finally by (14) and (23)

$$\lim_{n\to\infty} R_{ij}(a_n) = (Se_i, e_j), \quad \text{where } a_n = \sum_{i=1}^{N} a_n^i e_i.$$

Sufficiency. From condition 1° it follows that there are sequence of positive numbers $\{a_n^i\}_{n=1}^{\infty}$, $i = 1, \ldots, N$, and sequence $b_n \in R^N$ such that

$$\lim_{n\to\infty} \hat{p}(a_n^j e_j t)^n e^{it(b_n, e_j)} = e^{-\frac{1}{2}t^2} \quad \text{for every } t \in R^1. \tag{24}$$

Consider the sequence of distributions

$$A_n p^{n^*} * \delta_{b_n}, \tag{25}$$

where A_n's are diagonal operators such that $A_n e_j = a_n^j e_j$. Condition (24) means that sequences of one-dimensional boundary distributions of the sequence (25) induced by the elements of the basis are weakly convergent to Gaussian distributions in R^1. Hence it follows that the sequence of distribution (25) is compact and that each subsequence which is convergent converges to a Gaussian distribution in R^N. Thus let S_1 and S_2 be two dispersion operators of limit Gaussian distributions.

By (24) we have $(S_1 e_i, e_i) = (S_2 e_i, e_i) = 1$, $i = 1, 2, \ldots, N$, which further yields the existence of the limit

$$\lim_{n\to\infty} n(a_n^i)^2 \{ \int_{\|A_n x\| \leqslant \varepsilon} x_i^2 p(dx) - |\int_{\|A_n x\| \leqslant \varepsilon} x_i p(dx)|^2 \} = 1. \tag{26}$$

By (24) the sequences $\{a_n^i\}_{n=1}^{\infty}$, $i = 1, 2, \ldots, N$, satisfy the condition

$$\lim_{n\to\infty} \frac{g_1(a_n^1)}{g_j(a_n^j)} = 1. \tag{27}$$

Thus condition 2° along with (27) and (26) guarantee the existence of the limit

$$\lim_{n\to\infty} n a_n^i a_n^j [\int_{\|A_n x\| \leqslant \varepsilon} x_i x_j p(dx) - \int_{\|A_n x\| \leqslant \varepsilon} x_i p(dx) \int_{\|A_n x\| \leqslant \varepsilon} x_j p(dx)]$$

for every $\varepsilon > 0$. Thus $(S_1 e_i, e_j) = (S_2 e_i, e_j)$ for $i, j = 1, 2, \ldots, N$, i.e. $S_1 = S_2$.

COROLLARY. *A full distribution $p \in \mathfrak{M}$ is attracted in the ordinary sense by a full Gaussian distribution in R^N if and only if for some basis $[e_1, \ldots, e_N]$ in R^N*

(i) $$\lim_{X\to+\infty} \frac{X^2 p\{x \in R^N : |x_j| \geqslant X\}}{\int_{|x_j| \leqslant X} x_j^2 p(dx)} = 0 \quad \textit{for } j = 1, 2, \ldots, N;$$

(ii) *there exists*

$$\lim_{X\to+\infty} \frac{\int_{\|x\| \leqslant X} x_i x_j p(dx) - \int_{\|x\| \leqslant X} x_i p(dx) \int_{\|x\| \leqslant X} x_j p(dx)}{\{\int_{\|x\| \leqslant X} x_i^2 p(dx) - [\int_{\|x\| \leqslant X} x_i p(dx)]^2\}^{\frac{1}{2}} \{\int_{\|x\| \leqslant X} x_j^2 p(dx) - [\int_{\|x\| \leqslant X} x_i p(dx)]^2\}^{\frac{1}{2}}}$$

for $i, j = 1, \ldots, N$, $i < j$;

(iii) *there exists*

$$\lim_{X \to \infty} \frac{\int\limits_{|x| \leqslant X} x^2 p^{e_1}(dx) - [\int\limits_{|x| \leqslant X} x p^{e_1}(dx)]^2}{\int\limits_{|x| \leqslant X} x^2 p^{e_i}(dx) - [\int\limits_{|x| \leqslant X} x p^{e_i}(dx)]^2} \neq 0$$

for $i = 2, 3, \ldots, N$.

Theorem in the paper describes the domain of operator attraction of a full Gaussian distribution provided that the sequence of norming operators is a sequence of positive diagonal operators. As seen from Lemma 4, the general case reduces to considering a sequence of norming operators of the form $D_n U_n$, where D_n are positive diagonal operators and U_n are orthogonal ones. Unfortunately, this case is still the problem to be solved.

References

[1] B. V. Gnedenko, A. N. Kolmogorov, *Limit distributions for sums of independent random variables*, Cambridge 1954.

[2] R. Jajte, *Semi-stable probability measures on* R^N, Studia Math. 61 (1977), p. 29–39.

[3] J. Kucharczak, *On operator-stable probability measures*, Bull. Acad. Polon. Sci. 23 (5) (1975), p. 571–576.

[4] — *Remarks an operator-stable measures*, Colloq. Math. 34 (1) (1976), p. 109–119.

[5] P. Levy, *Théorie de l'addition des variables aléatoires*, Paris 1937.

[6] K. P. Parthasarathy, *Probability measures on metric spaces*, Academic Press, New York–London 1967.

[7] M. Sharpe, *Operator-stable probability distributions on vector groups*, Trans. Amer. Math. Soc. (136) (1969), p. 51–65.

INSTITUTE OF MATHEMATICS
UNIVERSITY, ŁÓDŹ